UNIVERSITY CARLOS III

Master in Economics **Master in Industrial Economics and Markets**

Game Theory

QUIZ 2. December 16th, 2020

NAME:

Exercise 1: *Consider the following normal form game:*

		Player 2		
		X	Y	Z
	A	$7,7$	$0,12$	$0,6$
Player 1	B	$0,4$	$5,5$	$0,3$
	C	$12,0$	$3,0$	$1,1$

The stage game G

(a) *(5 points) Find all pure strategy Nash equilibria and the payoffs of these equilibria.*

Solution: The NE are (B, Y), with payoffs $u_1 = u_2 = 7$ and (C, Z), with payoffs $u_1 = u_2 = 1$.

(b) *(10 points) Assume that the above stage game is played two times. After the first round, players observe the moves made by the other player. The total payoffs of the repeated game are the sum of the payoffs obtained in each round. Is there a subgame perfect Nash equilibrium in pure strategies in which (A, X) is played in the first round?*

Solution: Let us consider the following strategy profile S: player at

- $t = 1$ plays (A, X);
- $t = 2$ plays (B, Y) if (A, X) was played at $t = 1$ and (C, Z) otherwise.

Is S a SPNE of the repeated game? At $t = 2$ the strategy profile S suggests the players to play a NE. Hence, no player has an incentive to deviate at $t = 2$. Anticipating the payoffs at $t = 2$ the players anticipate the following payoffs at $t = 1$.

		Player 2		
		X	Y	Z
	A	$12,12$	$1,13$	$1,7$
Player 1	B	$1,5$	$6,6$	$1,4$
	C	$13,1$	$4,1$	$2,2$

and we see that playing A is not a best reply for player 1, when player 2 chooses X. Hence playing (A, X) at $t = 1$ is not part of a SPNE of the repeated game.

(c) *(15 points) Assume that the above stage game is played three times. After the first round, players observe the moves made by the other player. The total payoffs of the repeated game are the sum of the payoffs obtained in each round. Is there a subgame perfect Nash equilibrium in pure strategies in which (A, X) is played in the first round?*

Solution

Let us consider the following strategy profile S: at

- $t = 1$ play (A, X);
- $t = 2$ play (B, Y) if (A, X) was played at $t = 1$. Otherwise, play (C, Z);
- $t = 3$, play (B, Y) if (A, X) was played at $t = 1$ and (B, Y) was played at $t = 2$. Otherwise, play (C, Z).

Let us check that the above strategy profile S is a SPNE. At stage $t = 3$, it dictates players to play a NE of the stage game. Thus S is a NE of in each subgame that starts at $t = 3$.

Let us check check that there is no profitable deviation at $t = 2$. Suppose player i deviates at $t = 2$ and player $j \neq i$ follows the strategy S. At $t = 2$ player i cannot gain from deviating, because player j is playing a NE. And at $t = 3$ player j is playing the NE (C, Z). Thus, the maximum player i can get at $t = 3$ is 1. However, by following the strategy S the minimum payoff player i would obtain at $t = 3$ is 1. Thus, player i cannot improve his payoff by deviating at $t = 2$.

Let us check that it is also a NE of the subgame that starts at $t = 1$. The payoff obtained by the players in the strategy profile S es $u_1^S = u_2^S = 7 + 5 + 5 = 17$. If any of the players deviates (and the other player does not deviate) at $t = 1$ from S, the highest payoff it can attain is $u_1^D = u_2^D = 12 + 1 + 1 = 14$ Hence, no player has incentives to deviate and S is a SPNE.

Exercise 2: *Consider the following normal form game.*

		Player 2 X	Y
Player 1	A	11, 0	1, 1
	B	6, 6	0, 11

The stage game G

1. *Find the all the Nash equilibria of the game G. (5 points)* **Solution:**

   There is. unique NE in dominant strategies, (A, Y) with payofss $u_1 = u_2 = 1$.

2. *(15 points) Assume that the above stage game is played infinitely many times. After each round, players observe the moves done by the other player. The total payoffs of the repeated game are the discounted (with discount factor δ) sums of the payoffs obtained in each round. For what values of the discount factor δ is there a subgame perfect Nash equilibrium in pure strategies in which (B, X) is played in every round?*

   Solution:

   Let us consider trigger strategies: at

   - $t = 1$ play (B, X);
   - $t > 1$ play (B, X) if (B, X) was played at $t = 1, \dots, t - 1$. Otherwise, play (A, Y).

   Let us check that it is a SPNE of the repeated game. We first check that it is a NE of the repeated game. The payoffs obtained by both players with the trigger strategy are,

   $$u^c = 6 + 6\delta + \cdots + 6\delta^t + 6\delta^{t+1} + 6\delta^{t+2} + 6\delta^{t+3} + \cdots$$

   If one player deviates at stage t and the other player follows the trigger strategy, the payoff of the player which deviates is

   $$u^d = 6 + 6\delta + \cdots + 6\delta^t + 11\delta^{t+1} + \delta^{t+2} + \delta^{t+3} + \cdots$$

   The trigger strategy is NE iff $u^c \geq u^d$. That is,

   $$6\delta^{t+1} + 6\delta^{t+2} + 6\delta^{t+3} + \cdots \geq 11\delta^{t+1} + \delta^{t+2} + \delta^{t+3} + \cdots$$

   That is

   $$6\delta^{t+1}(1 + \delta + \delta^2 + \cdots) \geq \delta^{t+1}(11 + \delta + \delta^2 + \cdots) =$$

   which is the same as

   $$\frac{6}{1-\delta} \geq 11 + \frac{\delta}{1-\delta}$$

   Thus, the trigger strategy is a NE of the repeated game iff

   $$\delta \geq \frac{1}{2}$$

   Now, the standard argument shows that it is also a NE in every subgame: There are two types of subgames starting at a stage t.

- Subgames in which at every stage $1, 2, \ldots, t-1$ it was played (B, X). Then, the situation in the subgame that starts at this node is exactly as above, except that the payoffs are multiplied by δ^{t-1}. The above argument shows that the trigger strategy is also a NE of that subgame.
- Subgames in which at some stage $1, 2, \ldots, t-1$ the strategy profile (B, X) was not played. In these subgames the trigger strategy prescribes that (A, Y), a NE of the stage game G, is played. But, this is a SPNE of this subgame.

Therefore, if $\delta \geq \frac{1}{2}$, the trigger strategy is a SPNE of the repeated game. Note that in this strategy profile, the players cooperate at every stage of the repeated game.