UNIVERSITY CARLOS III

Master in Economics **Master in Industrial Economics and Markets**

Game Theory

MIDTERM EXAM–November 10th, 2020

NAME:

Problem 1: Consider the following normal form game:

	X	Y	Z
A	10, 2	4, 6	1,3
B	0, 1	8, 4	2,2
C	0,1	6 , 1	4,2
D	6,3	5, 7	3,4

(a) **(5 points)** What are the strategies that survive the iterated elimination of strictly dominated strategies?

Solution: *Strategy X is dominated by Z for player 2,*

	Y	Z
A	*4,6*	*1, 3*
B	*8,4*	*2,2*
C	*6,1*	*4,2*
D	*5,7*	*3,4*

Now strategy A (resp. D) is dominated by strategy B (resp. C) for player 1

	Y	Z
B	*8,4*	*2,2*
C	*6,1*	*4,2*

The set of rationalizable strategies are $\{B, C\} \times \{Y, Z\}$.

(b) **(5 points)** Find all pure strategy Nash equilibria and the payoffs of these equilibria.

Solution: *The best replies of the players are*

	Y	Z
B	*8, 4*	*2,2*
C	*6,1 ,*	*4, 2*

The NE in pure strategies are (B, Y) with payoffs $u_1 = 8$, $u_2 = 4$ and (C, Z) with payoffs $u_1 = 4$, $u_2 = 2$.

(c) **(15 points)** Draw the best reply functions of the players. Compute the mixed strategy Nash equilibria and the expected payoffs of these equilibria.

Solution: *We look for a NE of the form*

$$\sigma_1 = xB + (1-x)C$$
$$\sigma_2 = yY + (1-y)Z$$

The expected utilities of player 1 are

$$u_1(B, \sigma_2) = 8y + 2(1-y) = 2 + 6y$$
$$u_1(C, \sigma_2) = 6y + 4(1-y) = 4 + 2y$$

We see that

$$\text{BR}_1(y) = \begin{cases} C \quad (x=0) & \text{if } y < \frac{1}{2} \\ \{B, C\} \quad (x \in [0,1]) & \text{if } y = \frac{1}{2} \\ B \quad (x=1) & \text{if } y > \frac{1}{2} \end{cases}$$

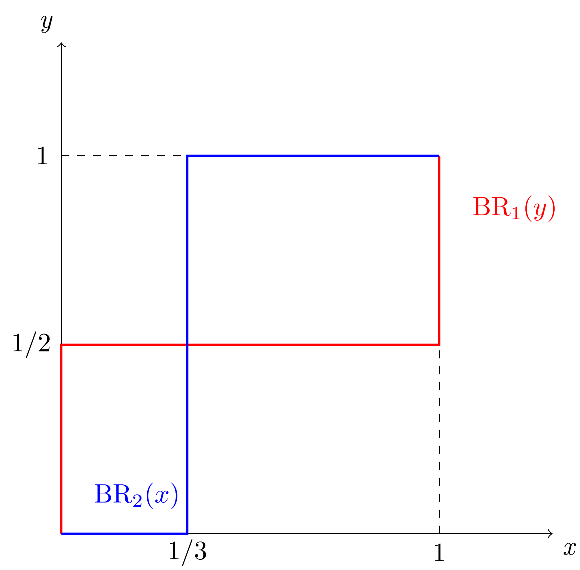
The expected utilities of player 2 are

$$\begin{aligned} u_2(Y, \sigma_2) &= 4x + 1 - x = 1 + 3x \\ u_2(Z, \sigma_2) &= 2 \end{aligned}$$

We see that

$$\text{BR}_2(x) = \begin{cases} Z \quad (y=0) & \text{if } x < \frac{1}{3} \\ \{Y, Z\} \quad (y \in [0,1]) & \text{if } x = \frac{1}{3} \\ Y \quad (y=1) & \text{if } x > \frac{1}{3} \end{cases}$$

Graphically,



And we obtain three NE

(a) (C, Z) *with payoffs* $u_1 = 4$, $u_2 = 2$;

(b) (B, Y) *with payoffs* $u_1 = 8$, $u_2 = 4$; *and*

(c) $\left(\frac{1}{3}B + \frac{2}{3}C, \frac{1}{2}Y + \frac{1}{2}Z\right)$, *with payoffs* $u_1 = 5$, $u_2 = 2$.

Problem 2: Two firms operate in a market with a unique homogenous good. The inverse demand function is $P = 17 - 2Q$, where $Q = q_1 + q_2$ and q_i is the amount produced by firm $i = 1, 2$. The firms have constant marginal cost $c_1 = 1$, $c_2 = 3$. Both firms decide simultaneously the quantities produced q_1, q_2.

(a) **(10 points)** Write the benefit of each of the firms, as a function of the output (q_1, q_2).

Solution: *The profits are*

$$\begin{aligned} \Pi_1(q_1, q_2) &= (17 - 2q_1 - 2q_2)q_1 - q_1 = (16 - 2q_1 - 2q_2)q_1 \\ \Pi_2(q_1, q_2) &= (17 - 2q_1 - 2q_2)q_1 - 3q_1 = (14 - 2q_1 - 2q_2)q_2 \end{aligned}$$

(b) **(10 points)** Compute the best reply of each of the firms when it considers the production of the other firm as given.

Solution: *Player 1 maximizes* $\Pi_1(q_1, q_2) = (16 - 2q_1 - 2q_2)q_1 = 16q_1 - 2q_1^2 - 2q_1q_2$ *The first order condition is*

$$16 - 4q_1 - 2q_2 = 0$$

Solving for q_1 *we obtain*

$$q_1 = \frac{16 - 2q_2}{4} = \frac{8 - q_2}{2}$$

The best reply of player 1 is

$$\mathrm{BR}_1(q_2) = \max\left\{\frac{8 - q_2}{2}, 0\right\}$$

Player 2 maximizes $\Pi_2(q_1, q_2) = (14 - 2q_1 - 2q_2)q_2 = 14q_2 - 2q_2^2 - 2q_1q_2$ *The first order condition is*

$$14 - 4q_2 - 2q_1 = 0$$

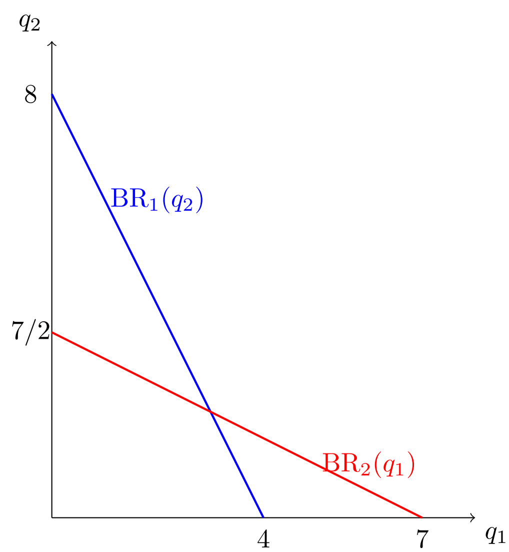
Solving for q_2 *we obtain*

$$q_2 = \frac{14 - 2q_1}{4} = \frac{7 - q_1}{2}$$

The best reply of player 1 is

$$\mathrm{BR}_2(q_1) = \max\left\{\frac{7 - q_1}{2}, 0\right\}$$

Graphically,



Note that

$$\frac{\partial^2 \Pi_1}{(\partial q_1)^2} = \frac{\partial^2 \Pi_2}{(\partial q_2)^2} = -4 < 0$$

So, the second order conditions for optimization are fulfilled.

(c) **(10 points)** Compute the Nash equilibrium and the profits of each firm in the equilibrium.

Solution: *The NE is the solution of the system of equations*

$$\begin{aligned} q_1 &= \frac{8 - q_2}{2} \\ q_2 &= \frac{7 - q_1}{2} \end{aligned}$$

We obtain

$$q_1^* = 3, \quad q_2 = 2$$

The profits are

$$\pi_1^* = 18 \quad \pi_2^* = 8$$

(d) **(10 points)** Suppose now that firm 1 can buy firm 2 and adopts the most efficient technology. In this case, firm 1 behaves a monopolist. What is the maximum amount firm 1 would be willing to pay to acquire firm 2?

Solution: *Firm 1 behaves as a monopolist. Thus it maximizes*

$$\Pi_1(q_1) = (17 - 2q_1)q_1 - q_1 = 16q_1 - 2q_1^2$$

The first order condition is

$$16 - 4q_1 = 0$$

and the monopoly quantity is $\hat{q}_1 = 4$*. The monopoly price is* $\hat{p} = 9$ *and the monopoly profit is* $\hat{\Pi}_1 = 32$*. Thus, the maximum amount that firm 1 is willing to pay to acquire firm 2 is* $32 - 18 = 14$.

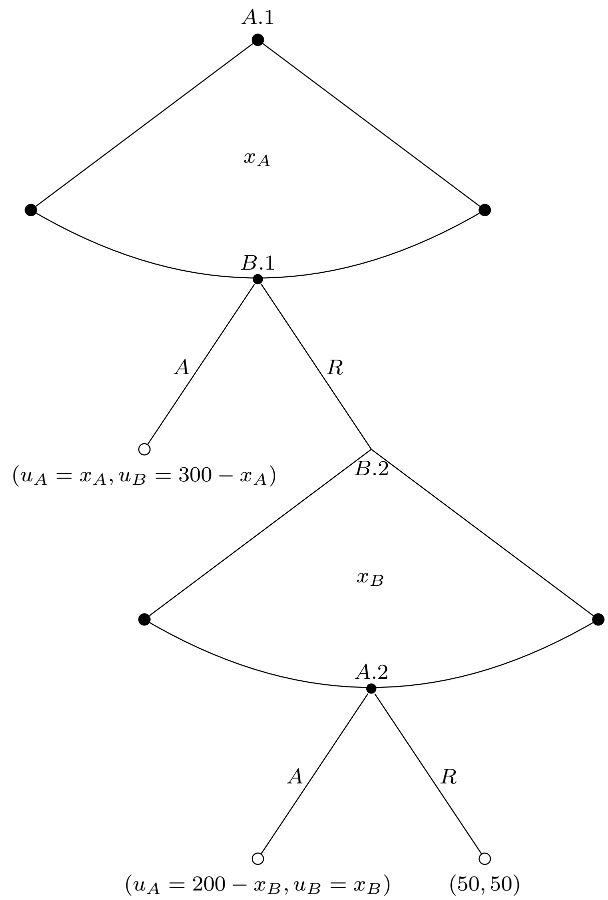
Problem 3: Two managers are negotiating how to share a Market, They have two chances to reach an agreement. And the procedure is the following: In the first period the market generates a profit of 300 million. In this first period manager A proposes to manager B a way in which they share the 300 million. If manager B accepts, the agreement is reached and the bill is split as A has proposed.

If manager B rejects the offer, they have to wait until the second period, in which the profit of the market has reduced to 200 million. In this period it is manager B who proposes to manager A how to share the 200 million. If manager A accepts, the agreement is reached and the bill is split as B has proposed. Otherwise, they do not reach an agreement and the market will generate a profit of 50 million each.

Assume that, in case of indifference, the offer is accepted.

(a) **(10 points)** Describe the situation as an extensive form game. Draw a diagram. How many subgames does it have?

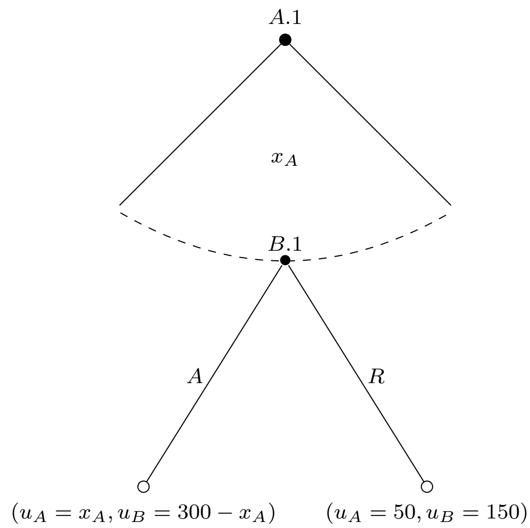
Solution: *The extensive game form representation is the following.*



There are infinitely many subgames.

(b) **(10 points)** Compute the subgame perfect Nash equilibria of this game. Write the strategies of the players at each node. Write the payoffs of the players in each of the SPNE.

Solution: *At node A.2, agent A accepts iff* $200 - x_B \geq 50$. *That is agent A accepts iff* $x_B \leq 150$. *Given the best reply of agent A at node A.2, the best response of agent B at node B.2 is to propose* $x_B = 150$ *for himself. Thus, we may assume that if we ever reach node B.2, agent B will propose to keep* $x_B = 150$ *for himself and agent A accepts. The payoffs will be* $u_A = 50$, $u_B = 150$. *We replace this payoffs at node A.2*



Now, player B at node B.1 accepts iff $300 - x_A \geq 150$*. That, is at node B.1 player B accepts iff* $x_A \leq 150$*. The best response now for player A is to keep* $x_A = 150$ *for himself at node A.1. The SPNE is the following.*

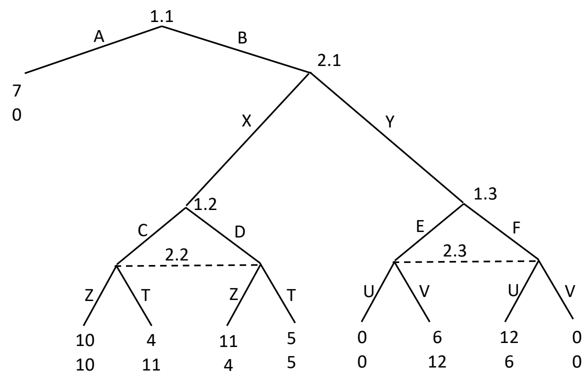
- *Node A.1:* $x_A = 150$.
- *Node B.1: accept iff* $x_A \leq 150$.
- *Node B.2:* $x_B = 150$.
- *Node A.2: accept iff* $x_B \leq 150$.

The payoffs are $u_A = 150$, $u_B = 150$.

(c) **(10 points)** Imagine now that, if no agreement is reached in any of the two periods, in the second period the profit generated by the market is X for each firm, with $50 < X \leq 100$. With respect to the previous situation which player is now better off?

Solution: *Since* $X > 50$*, player A has more bargaining power in the last period (his outside option in case of not accepting is higher) and this translates into a higher payoff in the equilibrium.*

Problem 4: Consider the following extensive form game.



(a) **(10 points)** Write the normal form of the game that starts at node 1.2 and find all (that is in pure and mixed strategies) NE of this game.

Solution: *The normal form game is*

	Z	T
C	*10, 10*	*4, 11*
D	*11, 4*	*5, 5*

Strategy C (resp. Z) is dominated by strategy D (resp. Z). Therefore, the unique NE is (D, T) *with payoffs* $u_1 = u_2 = 5$.

(b) **(10 points)** Write the normal form of the game that starts at node 1.3 and find all (that is in pure and mixed strategies) NE of this game.

Solution: *The normal form game is*

	U	V
E	*0, 0*	*6, 12*
F	*12, 6*	*0, 0*

There are two NE in pure strategies,

(a) (E, V) *with payoffs* $u_1 = 6$, $u_2 = 12$*; and*

(b) (F, U) *with payoffs* $u_1 = 12$, $u_2 = 6$.

Let us look for a NE in mixed strategies of the form

$$\sigma_1 = pE + (1-p)F$$
$$\sigma_2 = qu + (1-q)V$$

We must have

$$6 - 6q = 12q$$
$$6 - 6p = 12p$$

so,

$$q = p = \frac{1}{3}$$

The NE in mixed estrategies is

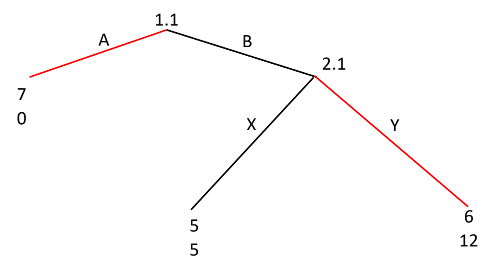
$$\left(\frac{1}{3}3+\frac{2}{3}F, \frac{1}{3}U+\frac{2}{3}V\right)$$

with payoffs $u_1 = u_2 = 4$.

(c) **(10 points)** Find the subgame perfect Nash equilibria of the complete game. Compute the utilities attained by the players in each SPNE.

Solution: *The subgame that starts at node* 1.2 *has a unique NE. The subgame that starts at node* 1.3 *has* 3 *NE. Correspondingly, we obtain* 3 *SPNE. We use the notation* $((1.1, 1.2, 1.3), (2.1, 2.2, 2.3))$.

(a) *At node* 1.3 *the NE* (E, V) *with payoffs* $u_1 = 6$, $u_2 = 12$ *is played.*

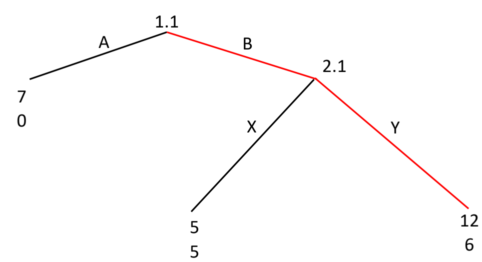


We obtain the SPN

$$((A, D, E), (Y, T, V))$$

with payoffs $u_1 = 7$, $u_2 = 0$.

(b) *At node* 1.3 *the NE* (F, U) *with payoffs* $u_1 = 12$, $u_2 = 6$ *is played.*

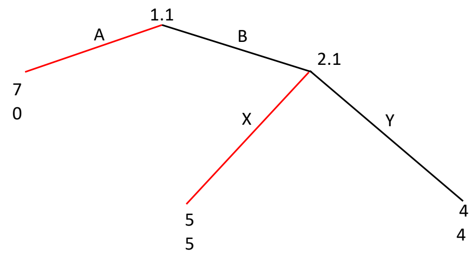


We obtain the SPN

$$((B, D, F), (Y, T, U))$$

with payoffs $u_1 = 12$, $u_2 = 6$.

(c) *At node* 1.3 *the NE* $\left(\frac{1}{3}3 + \frac{2}{3}F, \frac{1}{3}U + \frac{2}{3}V\right)$ *with payoffs* $u_1 = u_2 = 4$ *is played.*



We obtain the SPN

$$\left(\left(A, D, \frac{1}{3}3 + \frac{2}{3}F\right), \left(X, T, \frac{1}{3}U + \frac{2}{3}V\right)\right)$$

with payoffs $u_1 = 7$, $u_2 = 0$.