LAST NAME: NAME:

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**UNIVERSITY CARLOS III**

**Master in Economics** **Master in Industrial Economics and Markets**

**Game Theory**

**MIDTERM EXAM–October 28th, 2022**

**Instructions:** Total time. 2 hours. This is NOT an open book exam. Calculators are allowed. Write your work on this booklet. Unsupported unswers will receive little credit.

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**Problem 1:** Agents 1 (kettledrums) and 2 (trumpet) like to play music together. For this, they have to practice separately, so that when they meet each one knows well his part. If agent 1 spends $x_1$ hours practicing his part and agent 2 spends $x_2$ hours practicing her part, their utilities are

$$\begin{aligned} u_1(x_1, x_2) &= \sqrt{x_1 x_2} - x_1^2 \\ u_2(x_1, x_2) &= \sqrt{x_1 x_2} - x_2^2 \end{aligned}$$

(a) **(10 points)** What is the most favorable agreement that maximizes the welfare of both agents? What are the utilities of each agent in this agreement? (You may assume the solution is symmetric $x_1 = x_2$)

**Solution:** *The players maximize the sum of the utilities*

$$\max_{x_1, x_2} u_1 + u_2 = 2\sqrt{x_1 x_2} - x_1^2 - x_2^2$$

*The first order conditions are*

$$\begin{aligned} 0 &= \frac{x_2}{\sqrt{x_1 x_2}} - 2x_1 \\ 0 &= \frac{x_1}{\sqrt{x_1 x_2}} - 2x_2 \end{aligned}$$

*We obtain solution* $x_1 = x_2 = \frac{1}{2}$. *The utilities of the players are*

$$u_1\left(\frac{1}{4}, \frac{1}{4}\right) = u_2\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{1}{4}$$

(b) **(10 points)** Suppose from now on, that after the meeting both agents decide the time they practice $x_1$ and $x_2$ simultaneously and without knowing what the other one is doing. Find the best response function of each agent, $\mathrm{BR}_1(x_2)$ and $\mathrm{BR}_2(x_1)$.

**Solution:** *Player 1 maximizes* $u_1(x_1, x_2) = \sqrt{x_1 x_2} - x_1^2$ *The first order condition is*

$$\frac{x_2}{2\sqrt{x_1 x_2}} - 2x_1 = 0$$

*The best reply of player 1 is*

$$x_1(x_2) = \left(\frac{x_2}{16}\right)^{\frac{1}{3}}$$

*Similarly, the best reply of player 2 is*

$$x_2(x_1) = \left(\frac{x_1}{16}\right)^{\frac{1}{3}}$$

(c) **(5 points)** Why the agreement reached in part (a) will not be implemented if each agent takes its decision simultaneous and independently?

**Solution:** *Because*

$$\mathrm{BR}_1\left(\frac{1}{2}\right) = \left(\frac{1}{32}\right)^{\frac{1}{3}} \neq \frac{1}{2}$$

$$\mathrm{BR}_2\left(\frac{1}{2}\right) = \left(\frac{1}{32}\right)^{\frac{1}{3}} \neq \frac{1}{2}$$

(d) **(10 points)** Find the Nash equilibrium $(x_1^*, x_2^*)$ if the agents take their decisions simultaneous and independently. Compute the utilities of the agents in the NE. (You may assume the solution is symmetric $x_1 = x_2$)

**Solution:** *The NE is the solution of*

$$x_1 = \left(\frac{x_2}{16}\right)^{\frac{1}{3}}, \quad x_2 = \left(\frac{x_1}{16}\right)^{\frac{1}{3}}$$

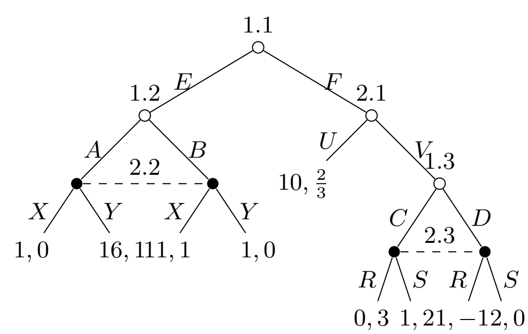
*That is*

$$x_1^* = x_2^* = \frac{1}{4}$$

*The utilities of the players are*

$$u_1\left(\frac{1}{2}, \frac{1}{2}\right) = u_2\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{3}{16}$$

**Problem 2:** Consider the following dynamic form game.




(a) **(20 points)** Write the normal form of the game that starts at node 1.2 and find all (that is, in pure and mixed strategies) NE of this game.

**Solution:** *The normal form game is*

| | $X$ | $Y$ |
|---|---|---|
| $A$ | *1 , 0* | *16 , 1* |
| $B$ | *11 , 1* | *1 , 0* |

*In problem 1 we have computed the NE of this game.*

*(a)* $(A, Y)$ *with payoffs* $u_1 = 16$, $u_2 = 1$ *;*

*(b)* $(B, X)$ *with payoffs* $u_1 = 11$, $u_2 = 1$*; and*

*(c)* $\left(\frac{1}{2}A + \frac{1}{2}B, \frac{3}{5}X + \frac{3}{5}Y\right)$*, with payoffs* $u_1 = 7$, $u_2 = \frac{1}{2}$.

(b) **(10 points)** Write the normal form of the game that starts at node 1.3 and find all (that is, in pure and mixed strategies) NE of this game.

**Solution:** *The normal form game is*

| | R | S |
|---|---|---|
| C | 0 , 3 | 1 , 2 |
| D | 1 , -1 | 2 , 0 |

*Strategy* $C$ *is (strictly) dominated by* $D$,

| | R | S |
|---|---|---|
| D | 1 , -1 | 2 , 0 |

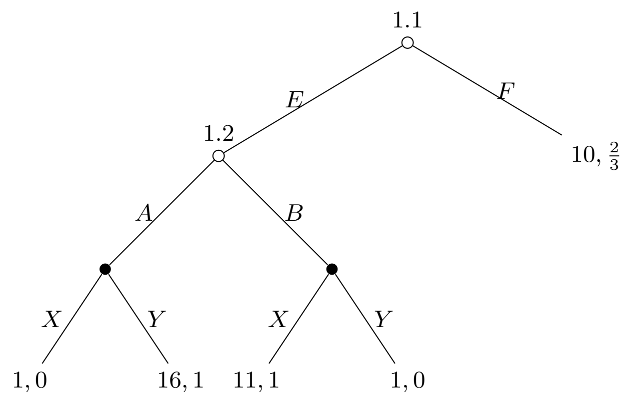
*Now, strategy* $S$ *is dominated by* $S$,

| | S |
|---|---|
| D | 2 , 0 |

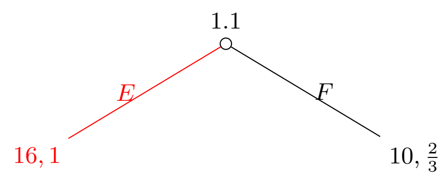
*There is a unique equilibrium with payoffs* $u_1 = 2$, $u_2 = 0$.

(c) **(20 points)** Find the subgame perfect Nash equilibria of the complete game. Compute the utilities attained by the players in each SPNE.

**Solution:** *At node* 2.1 , *player 2 chooses* $U$. *The payoffs at the sub-game starting at node* 2.1 *are* $u_1 = 10$, $u_2 = \frac{2}{3}$.

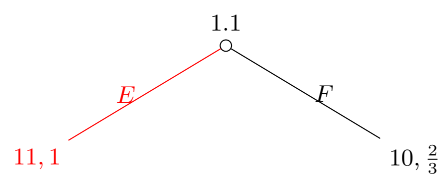



(a) *Let us look for a SPNE in which at node* 1.2 *the NE* $(A, Y)$ *is played.*



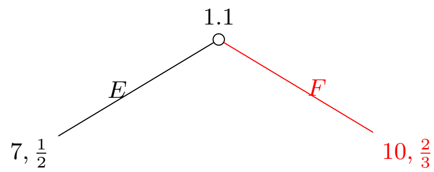

*At node* 1.1 *player 1 chooses* $E$. *We have the SPNE:* $(E, (A, Y), U, (D, S))$, *with payoffs* $u_1 = 16$, $u_2 = 1$.

(b) *Let us look for a SPNE in which at node* 1.2 *the NE* $(B, X)$ *is played.*




*At node* 1.2 *player 1 chooses* $E$. *We have the SPNE* $(E, (B, X), U, (D, S))$, *with payoffs* $u_1 = 11$, $u_2 = 1$.

(c) *Let us look for a SPNE in which at node* 1.2 *the NE* $\left(\frac{1}{2}A + \frac{1}{2}B, \frac{3}{5}X + \frac{3}{5}Y\right)$ *is played.*




*At node* 1.2 *player 1 chooses* $F$. *We have the SPNE* $\left(F, \left(\frac{1}{2}A + \frac{1}{2}B, \frac{3}{5}X + \frac{3}{5}Y\right), U, (D, S)\right)$, *with payoffs* $u_1 = 10$, $u_2 = \frac{2}{3}$.

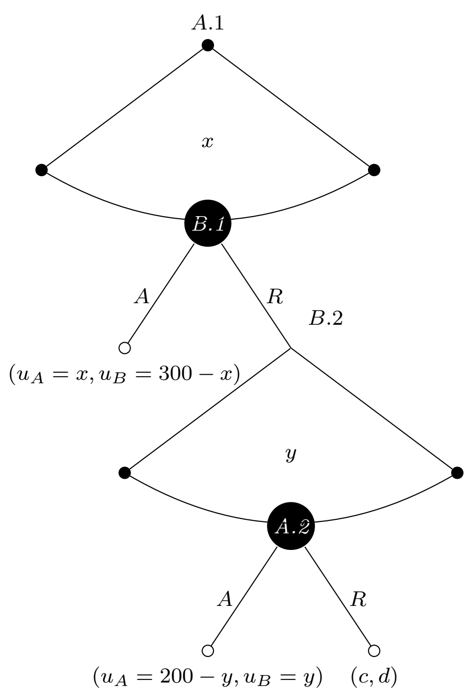
**Problem 3:** Two agents engage in a sequential bargaining process as follows:

1. In **stage** 1, agent $A$ proposes $0 \leq x \leq 300$.
   (a) If agent $B$ accepts, agent $A$ gets $x$, agent $B$ gets $300 - x$.
   (b) If agent $B$ refuses, they go into the stage 2.
2. In **stage** 2, agent $B$ proposes $0 \leq y \leq 200$.
   (a) If agent $A$ accepts, agent $A$ gets $200 - y$, agent $B$ gets $y$.
   (b) If agent $A$ refuses, agent $A$ gets the amount $c$ and agent $B$ gets the amount $d$.

Assume $c + d < 100$.

(a) **(10 points)** Describe the situation as an extensive form game. Draw the tree that represents the game.

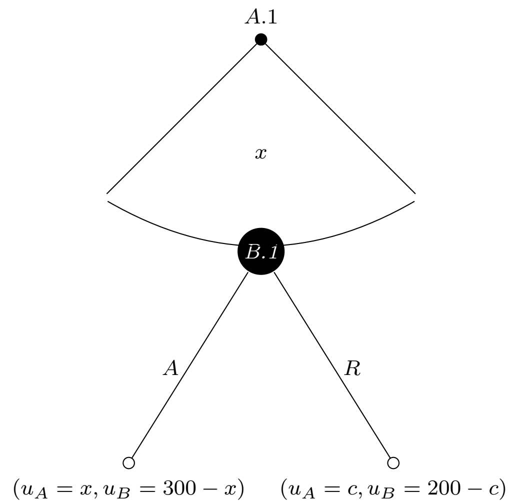
**Solution:**




*There are infinitely many subgames.*

(b) **(10 points)** Compute the subgame perfect Nash equilibria of this game. Write the strategies of the players at each node. Write the payoffs of the players in each of the SPNE.

**Solution:** *At node A.2, agent A accepts iff* $200 - y \geq c$. *That is agent A accepts iff* $y \leq 200 - c$. *Given the best reply of agent A at node A.2, the best response of agent B at node B.2 is to offer* $y = 200 - c$. *Thus, we may assume that if we ever reach node B.2, agent B will offer* $y = 200 - c$ *and agent A accepts. The payoffs will be* $u_A = c$, $u_B = 200 - c > d$. *We replace this payoffs at node A.2*




*Now, player B at node B.1 accepts iff* $300 - x \geq 200 - c$. *That, is at node B.1 player B accepts iff* $x \leq 100 + c$. *The best response now for player A is to offer* $x = 100 + c$ *at node A.1. The SPNE is the following.*

- *Node A.1:* $x = 100 + c$.
- *Node B.1: accept iff* $x =\leq 100 + c$.
- *Node B.2:* $y = 200 - c$.
- *Node A.2: accept iff* $y \leq 200 - c$.

*The payoffs are* $u_A = 100 + c$, $u_B = 200 - c$.

(c) **(5 points)** Is there a player who wins/looses if $c$ increases? Which one(s)?

**Solution:** *Since, the payoffs are* $u_A = 100 + c$, $u_B = 200 - c$, *agent A is better off and agent B is worse off.*

(d) **(5 points)** Is there a player who wins/looses if $d$ increases? Which one(s)?

**Solution:** *Since, the payoffs are* $u_A = 100 + c$, $u_B = 200 - c$, *the value of d plays no role.*

(e) Imagine now that, if, in the second stage agent $A$ refuses, they go into a third stage:

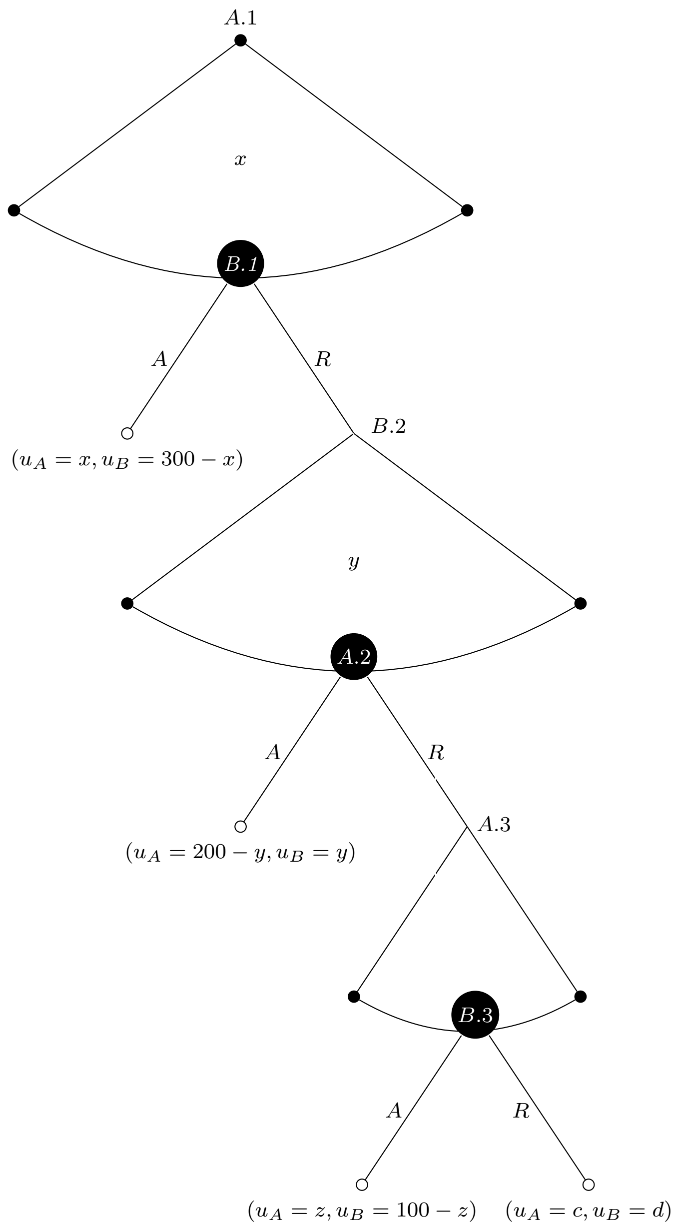
3. In **stage** 3, agent $A$ proposes $0 \leq z \leq 100$. If agent $B$ accepts, agent $A$ gets $z$, agent $B$ gets $100 - z$. If agent $B$ refuses, agent $A$ gets the amount $c$ and agent $B$ gets the amount $d$.

(a) **(20 points)** Describe the new dynamic game and compute the SPNE.

**Solution:** *The SPNE is the following.*

- *Node A.1:* $x = 200 - d$.
- *Node B.1: accept iff* $x =\leq 200 - d$.
- *Node B.2:* $y = 100 + d$.
- *Node A.2: accept iff* $y \leq 100 + d$.
- *Node A.3:* $z = 100 - d$.
- *Node B.3: accept iff* $z \leq 100 - d$.

*The payoffs are $u_A = 200 - d$, $u_B = 100 + d$.*




(b) **(5 points)** Is there a player who wins/looses if $c$ increases? Which one(s)?
**Solution:** *Since, the payoffs are $u_A = 200 - d$, $u_B = 100 + d$, the value of $c$ plays no role.*

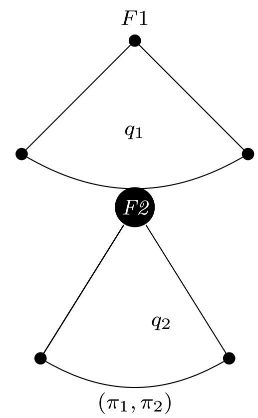
(c) **(5 points)** Is there a player who wins/looses if $d$ increases? Which one(s)?
**Solution:** *Since, the payoffs are $u_A = 200 - d$, $u_B = 100 + d$, agent $B$ is better off and agent $A$ is worse off.*

(d) **(10 points)** Depending on the parameters $c, d$ who prefers the 2-stage bargaining procedure versus the 3-stage bargaining procedure?
**Solution:**
*Since, $c+d < 100$, we have that $100+c < 200-d$ and $200-c > 100+d$. Therefore, $A$ prefers the two-stage game and agent $B$ prefers the three-stage game.*

**Problem 4:** Consider a market with two firms and a homogeneous product. The inverse demand function is $p = 150 - 3(q_1 + q_2)$, where $q_1$ is the output of firm 1 and $q_2$ is the output of firm 2. The firms have constant marginal cost $c = 6$. Firm 1 (the leader) chooses its output level $q_1$ first. Then, firm 2 (the follower) chooses its output level $q_2$, after observing $q_1$.

(a) **(10 points)** Describe the situation as an extensive form game. Write the payoffs of the firms.

**Solution:** *The set of player is* $N = \{1, 2\}$. *The strategy set for player 1 is* $S_1 = [0, \infty)$. *The strategy set for player 2 consists of functions* $q_2(q_1) : [0, \infty) \to [0, \infty)$. *The associated tree is*




*The profits of the firms are*

$$\begin{aligned} \pi_1(q_1, q_2) &= (150 - 3q_1 - 3q_2)\, q_1 - 6q_1 = 144q_1 - 3q_1^2 - 3q_1q_2 \\ \pi_2(q_1, q_2) &= (150 - 3q_1 - 3q_2)\, q_2 - 6q_2 = 144q_2 - 3q_1q_2 - 3q_2^2 \end{aligned}$$

(b) **(20 points)** Compute the subgame perfect Nash equilibria of this game. Write the strategies of the players at each node. Write the payoffs of the players in each of the SPNE.

**Solution:** *Firm 2 maximizes its profit function. The first order condition is*

$$-3q_2 - 3(-48 + q_1 + q_2) = 0$$

*Firm 2's best response to a price set by firm 1 is*

$$\mathrm{BR}_2(q_1) = \max\left\{0, \frac{48 - q_1}{2}\right\}$$

*Now, in taking its decision, Firm 1 anticipates the reaction of Firm 2. Firm 1 maximizes*

$$\pi_1\left(q_1, \frac{48 - q_1}{2}\right) = 144q_1 - 3q_1^2 - 3q_1\left(\frac{48 - q_1}{2}\right) = 72q_1 - \frac{3q_1^2}{2}$$

*The first order condition is*

$$72 - 3q_1 = 0$$

*and we obtain* $q_1^S = 24$, $q_2^S = \mathrm{BR}_2(24) = 12$. *The profits are* $\pi_1^S(24, 12) = 864$ *and* $\pi_2^S(24, 12) = 432$.

(c) **(20 points)** Suppose now that firms decide simultanous and independently the quantities $q_1$ and $q_2$. That is, assume that now firm 2 does not observe $q_1$ when it decides $q_2$. Compute the Cournot-Nash equilibrium of the resulting game. As compared with the previous situation, is there firm which is better/worse off? Which one(s)?

In doing your computations, you may assume the Cournot-Nash is symmetric.

**Solution:**

$$\mathrm{BR}_1(q_2) = \max\left\{0, \frac{48 - q_2}{2}\right\} \quad \mathrm{BR}_2(q_1) = \max\left\{0, \frac{48 - q_1}{2}\right\}$$

*The Nash-Cournot equilibrium is the solution of the following system of equations*

$$q_1 = \frac{48 - q_2}{2}, \quad q_2 = \frac{48 - q_1}{2}$$

*The solution is* $q_1^* = q_2^* = 16$. *The profits of the firms are* $\pi_1^*(16, 16) = \pi_2^*(16, 16) = 768$. *Firm 1 is better off with the Stackelberg setting. Firm 2 is better off with the Cournot setting.*