

Game Theory

MIDTERM EXAM–November 3rd, 2021

NAME:

Problem 1: Consider the following normal form game:

	X	Y	Z	T
A	20, 0	8, 4	2, 2	15, 1
B	-1, -1	6, 1	4, 2	7, 1
C	6, 9	5, 6	3, 4	2, 10
D	1, 10	3, 7	0, -4	1, 4

(a) (5 points) What are the strategies that survive the iterated elimination of strictly dominated strategies?

Solution: Strategy D is dominated by C for player 1:

	X	Y	Z	T
A	20, 0	8, 4	2, 2	15, 1
B	-1, -1	6, 1	4, 2	7, 1
C	6, 9	5, 6	3, 4	2, 10

Now strategy X is dominated by strategy T for player 2,

	Y	Z	T
A	8, 4	2, 2	15, 1
B	6, 1	4, 2	7, 1
C	5, 6	3, 4	2, 10

In the above game, strategy C is dominated by strategy B for player 1.

	Y	Z	T
A	8, 4	2, 2	15, 1
B	6, 1	4, 2	7, 1

Finally, strategy T is dominated by strategy Z for player 2.

	Y	Z
A	8, 4	2, 2
B	6, 1	4, 2

The set of rationalizable strategies are $\{A, B\} \times \{Y, Z\}$.

(b) (5 points) Find all pure strategy Nash equilibria and the payoffs of these equilibria.

Solution: The best replies of the players are

	Y	Z
A	8, 4	2, 2
B	6, 1	4, 2

The NE in pure strategies are (A, Y) with payoffs $u_1 = 8, u_2 = 4$ and (B, Z) with payoffs $u_1 = 4, u_2 = 2$.

- (c) **(15 points)** Draw the best reply functions of the players. Compute the mixed strategy Nash equilibria and the expected payoffs of these equilibria.

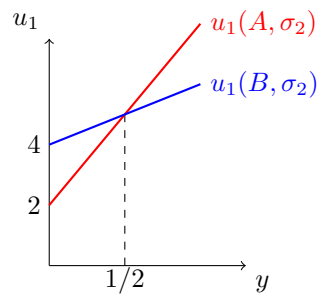
Solution: We look for a NE of the form

$$\begin{aligned}\sigma_1 &= xA + (1-x)B \\ \sigma_2 &= yY + (1-y)Z\end{aligned}$$

The expected utilities of player 1 are

$$\begin{aligned}u_1(A, \sigma_2) &= 8y + 2(1-y) = 2 + 6y \\ u_1(B, \sigma_2) &= 6y + 4(1-y) = 4 + 2y\end{aligned}$$

Graphically,



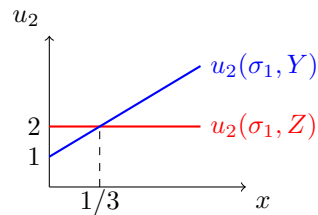
We see that

$$\text{BR}_1(y) = \begin{cases} B & (x = 0) & \text{if } y < \frac{1}{2} \\ \{A, B\} & (x \in [0, 1]) & \text{if } y = \frac{1}{2} \\ A & (x = 1) & \text{if } y > \frac{1}{2} \end{cases}$$

The expected utilities of player 2 are

$$\begin{aligned}u_2(\sigma_1, Y) &= 4x + 1 - x = 1 + 3x \\ u_2(\sigma_1, Z) &= 2\end{aligned}$$

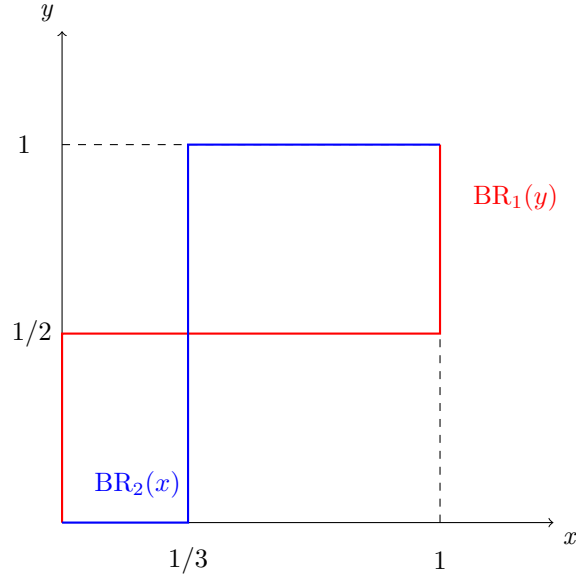
Graphically,



We see that

$$\text{BR}_2(x) = \begin{cases} Z & (y = 0) & \text{if } x < \frac{1}{3} \\ \{Y, Z\} & (y \in [0, 1]) & \text{if } x = \frac{1}{3} \\ Y & (y = 1) & \text{if } x > \frac{1}{3} \end{cases}$$

Graphically,



And we obtain three NE

- (a) (B, Z) with payoffs $u_1 = 4, u_2 = 2$;
- (b) (A, Y) with payoffs $u_1 = 8, u_2 = 4$; and
- (c) $(\frac{1}{3}A + \frac{2}{3}B, \frac{1}{2}Y + \frac{1}{2}Z)$, with payoffs $u_1 = 5, u_2 = 2$.

Problem 2: Two firms operate in a market with a unique homogenous good. The inverse demand function is $P = 17 - 2Q$, where $Q = q_1 + q_2$ and q_i is the amount produced by firm $i = 1, 2$. The cost of the firms is $c_1(q_1) = q_1^2$, $c_2(q_2) = 3q_2$. Both firms decide simultaneously the quantities produced q_1, q_2 .

- (a) (10 points) Write the benefit of each of the firms, as a function of the output (q_1, q_2) .

Solution: The profits are

$$\begin{aligned} \Pi_1(q_1, q_2) &= (17 - 2q_1 - 2q_2)q_1 - q_1^2 = 17q_1 - 3q_1^2 - 2q_1q_2 \\ \Pi_2(q_1, q_2) &= (17 - 2q_1 - 2q_2)q_2 - 3q_2 = (14 - 2q_1 - 2q_2)q_2 \end{aligned}$$

- (b) (10 points) Compute the best reply of each of the firms when it considers the production of the other firm as given.

Solution: Player 1 maximizes $\Pi_1(q_1, q_2) = 17q_1 - 3q_1^2 - 2q_1q_2$. The first order condition is

$$17 - 6q_1 - 2q_2 = 0$$

Solving for q_1 we obtain

$$q_1 = \frac{17 - 2q_2}{6}$$

The best reply of player 1 is

$$BR_1(q_2) = \max \left\{ \frac{17 - 2q_2}{6}, 0 \right\}$$

Player 2 maximizes $\Pi_2(q_1, q_2) = (14 - 2q_1 - 2q_2)q_2 = 14q_2 - 2q_2^2 - 2q_1q_2$ The first order condition is

$$14 - 4q_2 - 2q_1 = 0$$

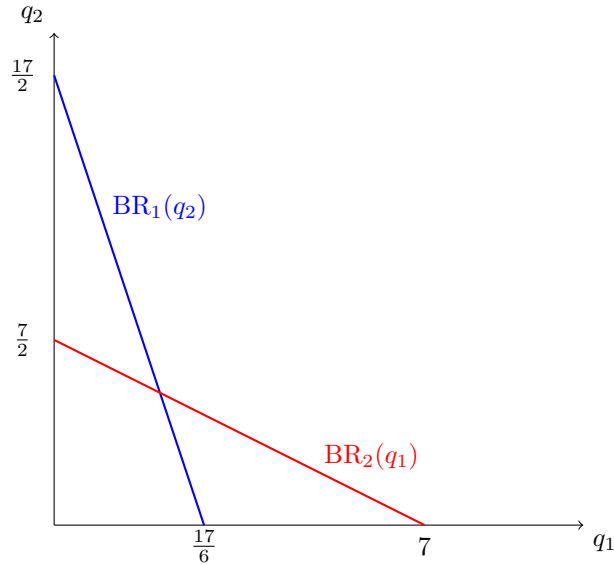
Solving for q_2 we obtain

$$q_2 = \frac{14 - 2q_1}{4} = \frac{7 - q_1}{2}$$

The best reply of player 1 is

$$BR_2(q_1) = \max \left\{ \frac{7 - q_1}{2}, 0 \right\}$$

Graphically,



Note that

$$\frac{\partial^2 \Pi_1}{(\partial q_1)^2} = -6 < 0, \quad \frac{\partial^2 \Pi_2}{(\partial q_2)^2} = -4 < 0$$

So, the second order conditions for optimization are fulfilled.

- (c) **(10 points)** Compute the Nash equilibrium and the profits of each firm in the equilibrium.

Solution: The NE is the solution of the system of equations

$$\begin{aligned} q_1 &= \frac{17 - 2q_2}{6} \\ q_2 &= \frac{7 - q_1}{2} \end{aligned}$$

We obtain

$$q_1^* = 2, \quad q_2 = \frac{5}{2}$$

The profits are

$$\pi_1^* = 12 \quad \pi_2^* = \frac{25}{2}$$

and the equilibrium price is $p^* = 8$.

- (d) **(10 points)** Suppose now that firm 1 and firm 2 merge and behave like a monopolist. Which of the two technologies available to the original firms will the merger adopt?

Solution: The merger behaves as a monopolist. If the monopolist adopts the technology $c(q) = q^2$, it maximizes

$$\Pi_1(q) = (17 - 2q)q - q^2 = 17q - 3q^2$$

The first order condition is

$$17 - 6q = 0$$

and the monopoly quantity is $\hat{q} = \frac{17}{6}$. The monopoly price is $\hat{p} = \frac{34}{3}$ and the monopoly profit is $\hat{\Pi} = \frac{289}{12} = 24.0833$.
 If the monopolist adopts the technology $c(q) = 3q$, it maximizes

$$\Pi_1(q) = (17 - 2q)q - 3q = 14q - 2q^2$$

The first order condition is

$$14 - 4q = 0$$

and the monopoly quantity is $\hat{q} = \frac{7}{2}$. The monopoly price is $\hat{p} = 10$ and the monopoly profit is $\hat{\Pi} = \frac{49}{2} = 24.5$.

Thus the monopolist adopts the technology $c(q) = 3q$.

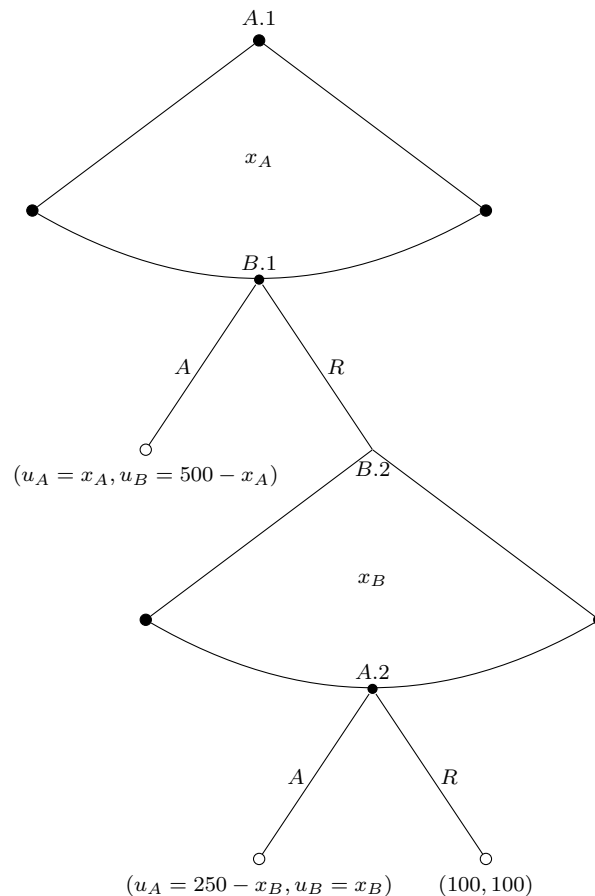
Problem 3: Two managers are negotiating how to share a Market, They have two chances to reach an agreement. And the procedure is the following: In the first period the market generates a profit of 500 million. In this first period manager A proposes to manager B a way in which they share the 500 million. If manager B accepts, the agreement is reached and the bill is split as A has proposed.

If manager B rejects the offer, they have to wait until the second period, in which the profit of the market has reduced to 250 million. In this period it is manager B who proposes to manager A how to share the 250 million. If manager A accepts, the agreement is reached and the bill is split as B has proposed. Otherwise, they do not reach an agreement and the market will generate a profit of 100 million each.

Assume that, in case of indifference, the offer is accepted.

(a) (10 points) Describe the situation as an extensive form game. Draw a diagram. How many subgames does it have?

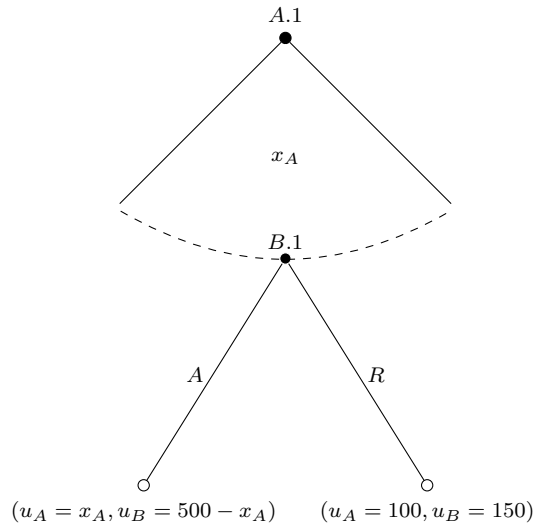
Solution: The extensive game form representation is the following.



There are infinitely many subgames.

- (b) **(10 points)** Compute the subgame perfect Nash equilibria of this game. Write the strategies of the players at each node. Write the payoffs of the players in each of the SPNE.

Solution: At node A.2, agent A accepts iff $250 - x_B \geq 100$. That is agent A accepts iff $x_B \leq 150$. Given the best reply of agent A at node A.2, the best response of agent B at node B.2 is to propose $x_B = 150$ for himself. Thus, we may assume that if we ever reach node B.2, agent B will propose to keep $x_B = 150$ for himself and agent A accepts. The payoffs will be $u_A = 100, u_B = 150$. We replace this payoffs at node B.2



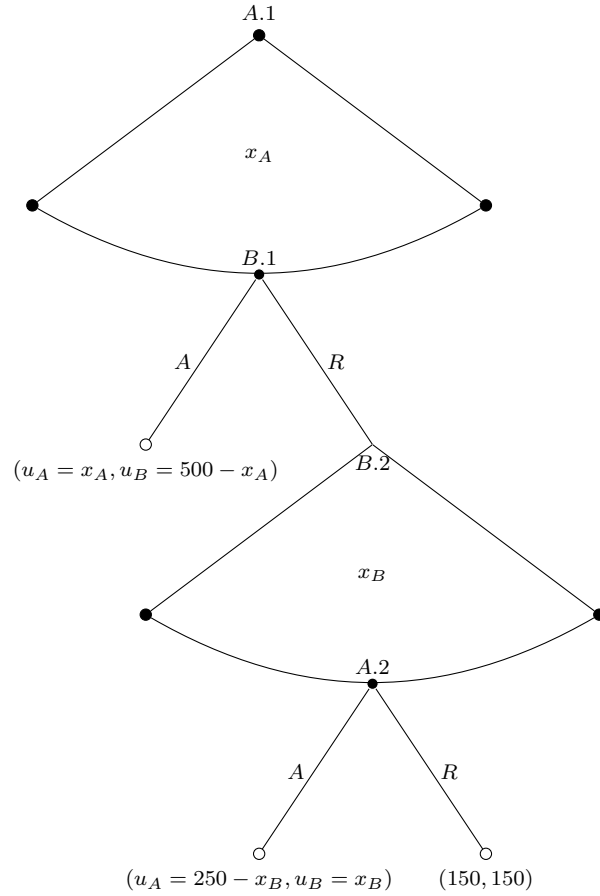
Now, player B at node B.1 accepts iff $500 - x_A \geq 150$. That, is at node B.1 player B accepts iff $x_A \leq 350$. The best response now for player A is to keep $x_A = 350$ for himself at node A.1. The SPNE is the following.

- Node A.1: $x_A = 350$.
- Node B.1: accept iff $x_A \leq 350$.
- Node B.2: $x_B = 150$.
- Node A.2: accept iff $x_B \leq 150$.

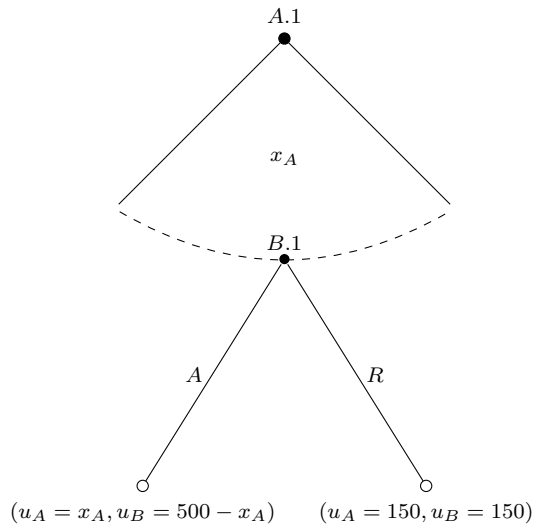
The payoffs are $u_A = 350, u_B = 150$.

- (c) **(10 points)** Imagine now that, if no agreement is reached in any of the two periods, in the last period the profit generated by the market is 150 for each firm. With respect to the previous situation which player is now better off?

Solution: The new extensive game form representation is the following.



Note that at node A.2, accept iff $x_B \leq 100$. But this yields $u_B = x_B \leq 100 < 150$. So at node B.2 agent B offers any $x_B \geq 150$ and at node A.2, agent A rejects. The payoffs are $u_A = 150, u_B = 150$. We replace this payoffs at node B.2



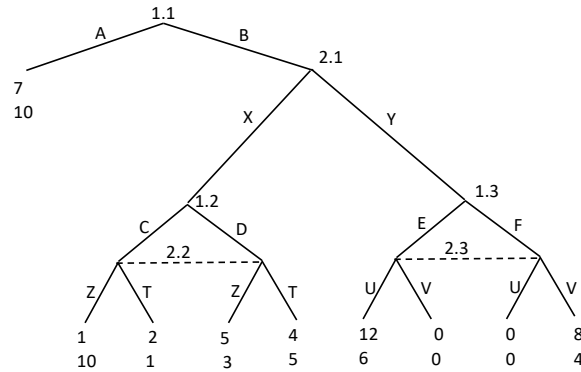
The new SPNE is

- Node A.1: $x_A = 350$.

- *Node B.1: accept iff $x_A \leq 350$.*
- *Node B.2: $x_B \geq 150$.*
- *Node A.2: accept iff $x_B \leq 150$.*

The payoffs are $u_A = 350$, $u_B = 150$. The situation is the same.

Problem 4: Consider the following extensive form game.



- (a) **(10 points)** Write the normal form of the game that starts at node 1.2 and find all (that is in pure and mixed strategies) NE of this game.

Solution: *The normal form game is*

	Z	T
C	1, 10	2, 1
D	5, 3	4, 5

Strategy C is dominated by strategy D. Therefore, the unique NE is (D, T) with payoffs $u_1 = 4, u_2 = 5$.

- (b) **(10 points)** Write the normal form of the game that starts at node 1.3 and find all (that is in pure and mixed strategies) NE of this game.

Solution: *The normal form game is*

	U	V
E	12, 6	0, 0
F	0, 0	8, 4

There are two NE in pure strategies,

- (a) *(E, U) with payoffs $u_1 = 12, u_2 = 6$; and*
 (b) *(F, V) with payoffs $u_1 = 8, u_2 = 4$.*

Let us look for a NE in mixed strategies of the form

$$\begin{aligned}\sigma_1 &= pE + (1-p)F \\ \sigma_2 &= qu + (1-q)V\end{aligned}$$

We must have

$$\begin{aligned}12q &= 8 - 8q \\ 6p &= 4 - 4p\end{aligned}$$

so,

$$q = p = \frac{2}{5}$$

The NE in mixed strategies is

$$\left(\frac{2}{5}E + \frac{3}{5}F, \frac{2}{5}U + \frac{3}{5}V\right)$$

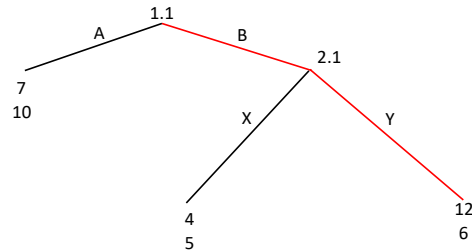
with payoffs

$$u_1 = \frac{24}{5}, \quad u_2 = \frac{12}{5}$$

- (c) (10 points) Find the subgame perfect Nash equilibria of the complete game. Compute the utilities attained by the players in each SPNE.

Solution: The subgame that starts at node 1.2 has a unique NE. The subgame that starts at node 1.3 has 3 NE. Correspondingly, we obtain 3 SPNE. We use the notation $((1.1, 1.2, 1.3), (2.1, 2.2, 2.3))$.

- (a) At node 1.3 the NE (E, U) is played, with payoffs $u_1 = 12, u_2 = 6$.

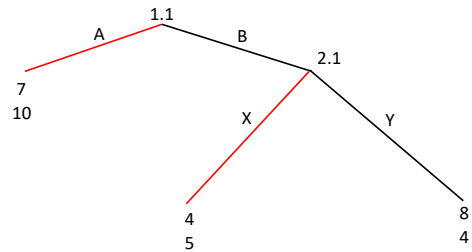


We obtain the SPN

$$((B, D, E), (Y, T, U))$$

with payoffs $u_1 = 12, u_2 = 6$.

- (b) At node 1.3 the NE (F, V) is played, with payoffs $u_1 = 8, u_2 = 4$.

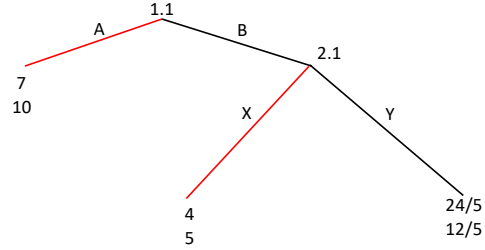


We obtain the SPN

$$((A, D, F), (X, T, V))$$

with payoffs $u_1 = 7, u_2 = 10$.

(c) At node 1.3 the NE $(\frac{2}{5}E + \frac{3}{5}F, \frac{2}{5}U + \frac{3}{5}V)$ is played, with payoffs $u_1 = \frac{24}{5}, u_2 = \frac{12}{5}$.



We obtain the SPN

$$\left(\left(A, D, \frac{2}{5}E + \frac{3}{5}F \right), \left(X, T, \frac{2}{5}U + \frac{3}{5}V \right) \right)$$

with payoffs $u_1 = 7, u_2 = 10$.