UNIVERSITY CARLOS III

Master in Economics **Master in Industrial Economics and Markets**

Game Theory

MIDTERM EXAM–November 6th, 2019

NAME:

Problem 1: Consider the following normal form game:

	X	Y	Z
A	0, 1	20, 1	15, 2
B	6, 0	30, 0	-1 , 1
C	10, 2	2 , -1	0, 1
D	8, 1	5, 4	-2 , 5

(a) **(5 points)** What are the strategies that survive the iterated elimination of strictly dominated strategies?

Solution: *Strategy Y is dominated by Z for player 2,*

	X	Z
A	*0, 1*	*15, 2*
B	*6, 0*	*-1 , 1*
C	*10, 2*	*0, 1*
D	*8, 1*	*-2 , 5*

Now strategies B and D are dominated by strategy C for player 1

	X	Z
A	*0, 1*	*15, 2*
C	*10, 2*	*0, 1*

The set of rationalizable strategies are $\{A, C\} \times \{X, Y\}$.

(b) **(5 points)** Find all pure strategy Nash equilibria and the payoffs of these equilibria.

Solution: *The best replies of the players are*

	X	Z
A	*0, 1*	*15, 2*
C	*10, 2*	*0, 1*

The NE in pure strategies are (A, Z) with payoffs $u_1 = 15$, $u_2 = 2$ and (C, X) with payoffs $u_1 = 10$, $u_2 = 2$.

(c) **(15 points)** Draw the best reply functions of the players. Compute the mixed strategy Nash equilibria and the expected payoffs of these equilibria.

Solution: *We look for a NE of the form*

$$\begin{aligned} \sigma_1 &= xA + (1-x)C \\ \sigma_2 &= yX + (1-y)Z \end{aligned}$$

The expected utilities of player 1 are

$$\begin{aligned} u_1(A, \sigma_2) &= 15(1-y) = 15 - 15y \\ u_1(C, \sigma_2) &= 10y \end{aligned}$$

We see that

$$\mathrm{BR}_1(y) = \begin{cases} A & \text{if } y < \frac{3}{5} \\ \{A, C\} & \text{if } y = \frac{3}{5} \\ C & \text{if } y > \frac{3}{5} \end{cases}$$

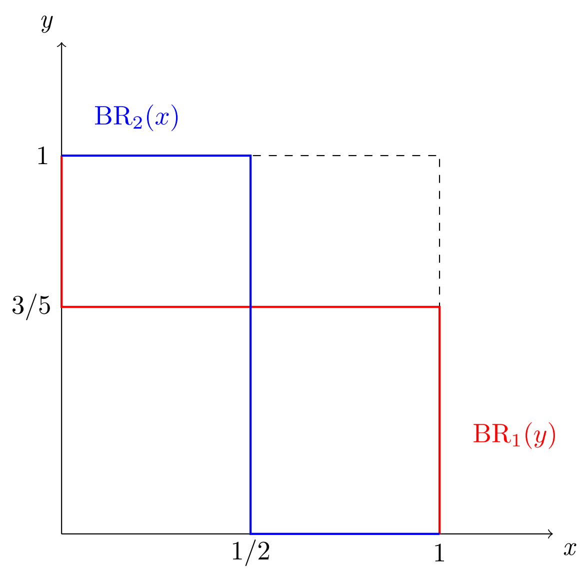
The expected utilities of player 2 are

$$\begin{aligned} u_2(X, \sigma_2) &= x + 2(1-x) = 2 - x \\ u_2(Z, \sigma_2) &= 2x + 1 - x = 1 + x \end{aligned}$$

We see that

$$\mathrm{BR}_2(x) = \begin{cases} X & \text{if } x < \frac{1}{2} \\ \{X, Z\} & \text{if } x = \frac{1}{2} \\ Z & \text{if } x > \frac{1}{2} \end{cases}$$

Graphically,



And we obtain three NE

(a) (A, Z) *with payoffs* $u_1 = 15$, $u_2 = 2$ *;*

(b) (C, X) *with payoffs* $u_1 = 10$, $u_2 = 2$*; and*

(c) $\left(\frac{1}{2}A + \frac{1}{2}C, \frac{3}{5}X + \frac{3}{5}Z\right)$, *with payoffs* $u_1 = 6$, $u_2 = 3/2$.

Problem 2: The planet is divided into two countries: 1 and 2. Politicians from both countries meet at the most fashionable and expensive resort to negotiate how to deal with climate change. If country 1 invests x_1 and country 2 invests x_2 monetary units in reducing climate change, their utilities will be the following

$$\begin{aligned} u_1(x_1, x_2) &= (3 + x_2)x_1 - 2(x_1)^2 \\ u_2(x_1, x_2) &= (3 + x_1)x_2 - 2(x_2)^2 \end{aligned}$$

(a) **(10 points)** What is the most favorable agreement that maximizes the welfare of both countries? What are the utilities of each country in this agreement?

Solution: *The players maximize the sum of the utilities*

$$\max_{x_1, x_2} u_1 + u_2$$

The first order conditions are

$$\begin{aligned} 0 &= 3 + x_2 - 4x_1 + x_2 \\ 0 &= x_1 + 3 + x_1 - 4x_2 \end{aligned}$$

That is

$$\begin{aligned} 3 &= 4x_1 - 2x_2 \\ 3 &= x_2 - 2x_1 \end{aligned}$$

The solution is

$$x_1 = x_2 = \frac{3}{2}$$

(b) **(10 points)** Suppose from now on, that after the meeting both countries decide their investments x_1 and x_2 simultaneously and without knowing what the other one is doing. Find the best response function of each country, $\mathrm{BR}_1(x_2)$ and $\mathrm{BR}_2(x_1)$. Draw the best response function in the $x_1 - x_2$ plane (x_1 on the horizontal axis and x_2 on the vertical axis).

Solution: *Player 1 maximizes* $u_1(x_1, x_2) = (3 + x_2)x_1 - 2(x_1)^2$ *The first order condition is*

$$3 + x_2 - 4x_1 = 0$$

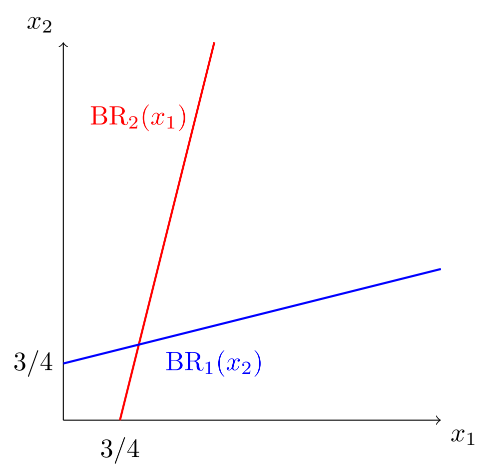
The best reply of player 1 is

$$q_1(q_2) = \frac{3 + x_2}{4}$$

Similarly, the reply of player 2 is

$$q_2(q_1) = \frac{3 + x_1}{4}$$

Graphically,



(c) **(10 points)** Why the agreement reached in part (a) will not be implemented if each country takes its decision simultaneous and independently?

Solution: *Because*

$$\mathrm{BR}_1\left(\frac{3}{2}\right) = \frac{3 + \frac{3}{2}}{4} = \frac{9}{8}$$

$$\mathrm{BR}_2\left(\frac{3}{2}\right) = \frac{3 + \frac{3}{2}}{4} = \frac{9}{8}$$

(d) **(10 points)** Find the Nash equilibrium (x_1^*, x_2^*) if the countries take their decisions simultaneous and independently. Compute the utilities of the countries in equilibrium.

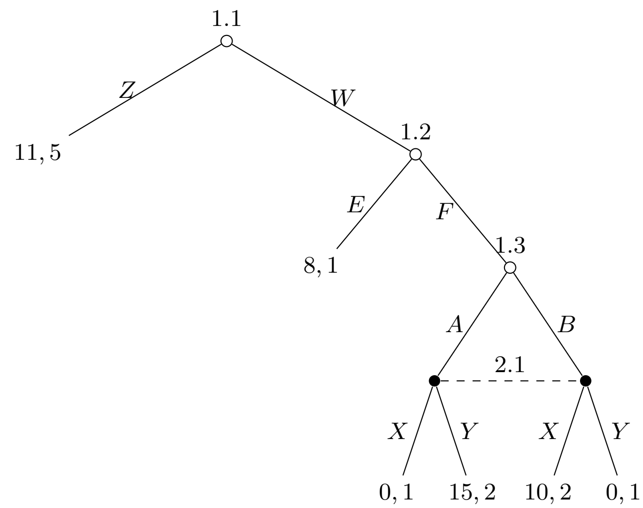
Solution: *The NE is the solution of*

$$q_1 = \frac{3+x_2}{4}, \quad q_2 = \frac{3+x_1}{4}$$

That is

$$q_1^* = q_2^* = 1$$

Problem 3: Consider the following extensive form game. Consider the following game.



(a) **(5 points)** Write the normal form of the game that starts at node 1.3 and find all (that is in pure and mixed strategies) NE of this game.

Solution: *The normal form game is*

	X	*Y*
A	*0, 1*	*15, 2*
B	*10, 2*	*0, 1*

In problem 1 we have computed the NE of this game.

(a) (A, Y) *with payoffs* $u_1 = 15$, $u_2 = 2$;

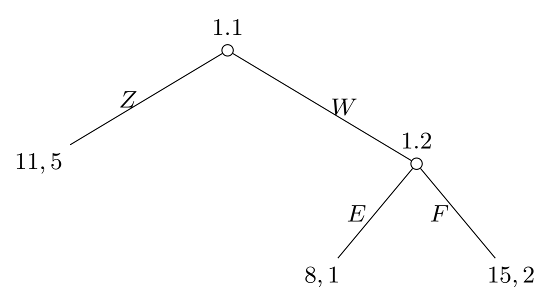
(b) (B, X) *with payoffs* $u_1 = 10$, $u_2 = 2$; *and*

(c) $\left(\frac{1}{2}A + \frac{1}{2}B, \frac{3}{5}X + \frac{3}{5}Y\right)$, *with payoffs* $u_1 = 6$, $u_2 = 3/2$.

(b) **(10 points)** Find the subgame perfect Nash equilibria of the complete game. Compute the utilities attained by the players in each SPNE.

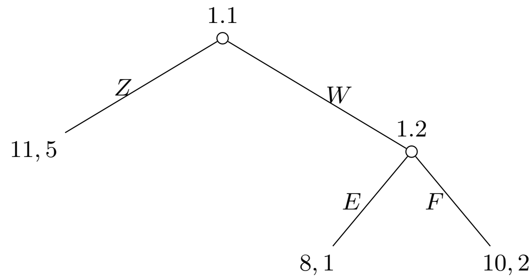
Solution:

(a) Let us look for a SPNE in which at node 1.3 *the NE* (A, Y) *is played.*



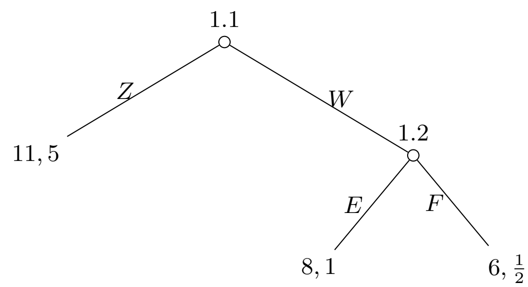
At node 1.2 *player 1 chooses* F. *Given this, at node* 1.2 *player 1 chooses* W. *We have the SPNE* $((W, F, A), Y)$, *with payoffs* $u_1 = 15$, $u_2 = 2$.

(b) Let us look for a SPNE in which at node 1.3 *the NE* (B, X) *is played.*



At node 1.2 *player 1 chooses* F. *Given this, at node* 1.2 *player 1 chooses* Z. *We have the SPNE* $((Z, F, B), X)$, *with payoffs* $u_1 = 11$, $u_2 = 5$. .

(c) Let us look for a SPNE in which at node 1.3 *the NE* $\left(\frac{1}{2}A + \frac{1}{2}B, \frac{3}{5}X + \frac{3}{5}Y\right)$ *is played.*



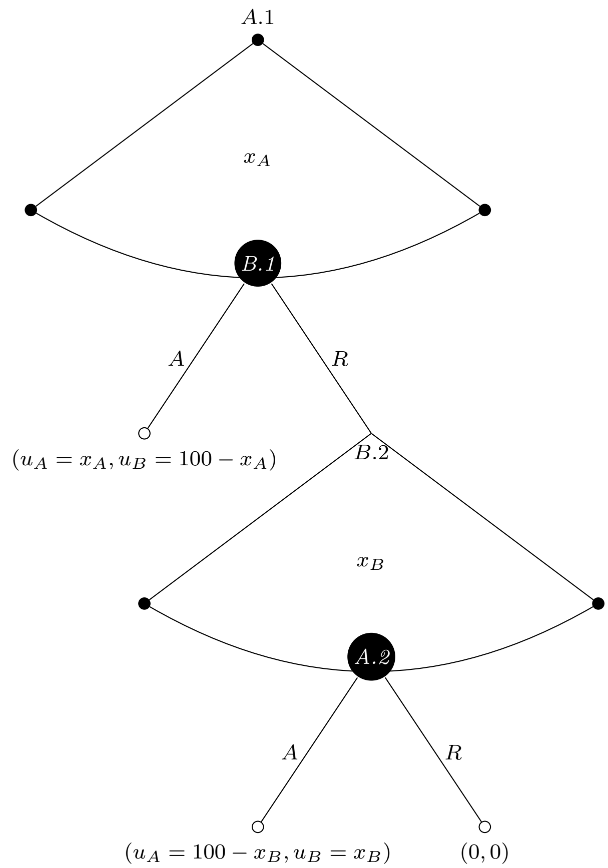
At node 1.2 *player 1 chooses* E. *Given this, at node* 1.2 *player 1 chooses* Z. *We have the SPNE* $\left(\left(Z, E, \frac{1}{2}A + \frac{1}{2}B\right), \frac{3}{5}X + \frac{3}{5}Y\right)$, *with payoffs* $u_1 = 11$, $u_2 = 5$.

Problem 4: Two students, who live in the same apartment, find a 100 euros bill at the entrance of the building. To decide how to share it, they consult their spiritual leader who tells them to divide the bill according to the procedure below. They will have two chances to find an agreement. Otherwise, God commands that the spiritual leader keeps the bill and uses it to finance her TV show.

The procedure is the following: In the first period agent A demands a payment $x_A \in [0, 100]$ of agent B in order to reach the agreement. If agent B accepts, the agreement is reached and the bill is split as A has proposed. If agent B rejects the offer, it will make an offer to agent A in the second period with the offer consisting again of a demand of a payment $x_B \in [0, 100]$ in order to sign the agreement.

(a) **(15 points)** Describe the situation as an extensive form game. How many subgames does it have?

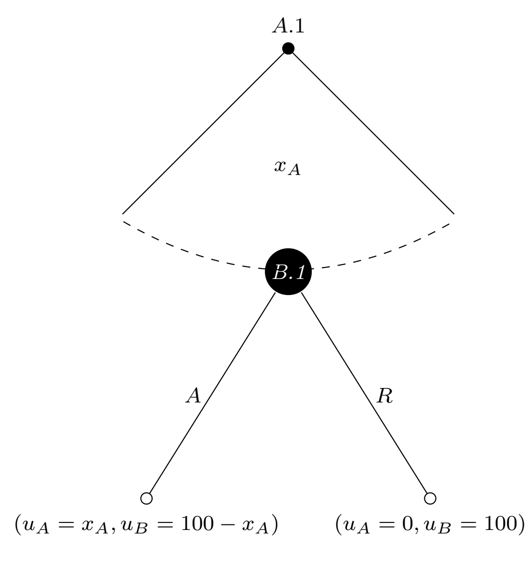
Solution:



There are infinitely many subgames.

(b) **(10 points)** Compute the subgame perfect Nash equilibria of this game. Write the strategies of the players at each node. Write the payoffs of the players in each of the SPNE.

Solution: *At node A.2, agent A accepts iff* $100 - x_B \geq 0$*. That is agent A accepts iff* $x_B \leq 100$*. Given the best reply of agent A at node A.2, the best response of agent B at node B.2 is to offer* $x_B = 100$*. Thus, we may assume that if we ever reach node B.2, agent B will offer* $x_B = 100$ *and agent A accepts. The payoffs will be* $u_A = 0$*,* $u_B = 100$*. We replace this payoffs at node A.2*



Now, player B at node B.1 accepts iff $100 - x_A \geq 100$. *That, is at node B.1 player B accepts iff* $x_A = 0$. *The best response now for player A is to offer any* x_A *at node A.1. The SPNE is the following.*

- *Node A.1:* $x_A \geq 0$.
- *Node B.1: accept iff* $x_A = 0$.
- *Node B.2:* $x_B = 100$.
- *Node A.2: accept any* $x_B \leq 100$.

The payoffs are $u_A = 0$, $u_B = 100$.