UNIVERSITY CARLOS III

Master in Economics | **Master in Industrial Economics and Markets**

Game Theory

MIDTERM EXAM–November 6th, 2018

NAME:

Problem 1: Consider the following normal form game:

	X	Y	Z
A	0, 1	20, 1	15, 2
B	1, 0	30, 0	10 , 1
C	10, 2	2 , -1	0, 1
D	5, 1	5, 4	-1 , 5

(a) **(5 points)** What are the strategies that survive the iterated elimination of strictly dominated strategies?

Solution: *Strategy Y is strictly dominated by strategy Z for player 2. We obtain*

	X	Z
A	*0, 1*	*15, 2*
B	*1, 0*	*10 , 1*
C	*10, 2*	*0, 1*
D	*5, 1*	*-1 , 5*

Now, strategy D is strictly dominated by strategy C for player 1. We obtain

	X	Z
A	*0, 1*	*15, 2*
B	*1, 0*	*10 , 1*
C	*10, 2*	*0, 1*

The set of rationalizable strategies is $\{A, B, C\} \times \{X, Z\}$.

(b) **(5 points)** Find all pure strategy Nash equilibria and the payoffs of these equilibria.

Solution: *The best replies of the players are*

	X	Z
A	*0, 1*	$\underline{15}$, $\underline{2}$
B	*1, 0*	*10 ,* $\underline{1}$
C	$\underline{10}$, $\underline{2}$	*0, 1*

We obtain the NE (A, Z) with payoffs $u_1 = 15$, $u_2 = 2$ and (C, X) with payoffs $u_1 = 10$, $u_2 = 2$.

(c) **(15 points)** Compute the mixed strategy Nash equilibria and the expected payoffs of these equilibria.

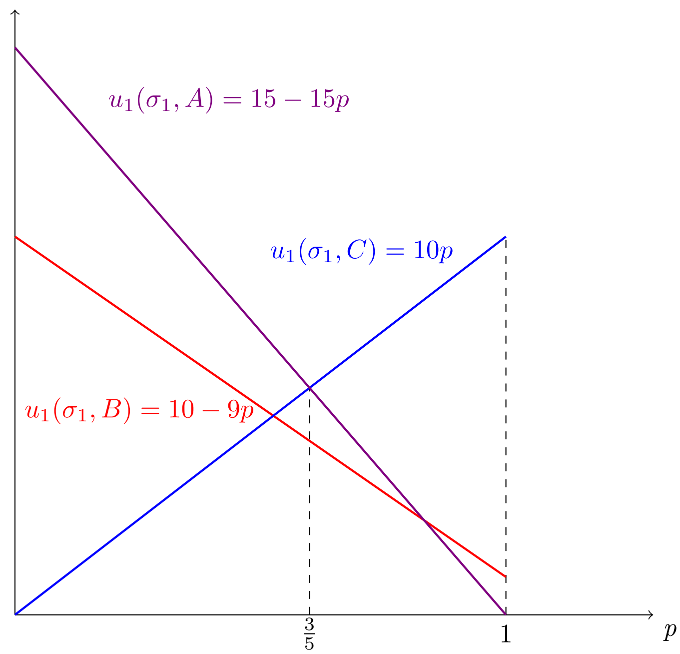
Solution: *Let us look for a NE of the form*

$$\sigma_1 = xA + yB + (1 - x - y)C$$
$$\sigma_2 = pX + (1 - p)Z$$

The expected utilities of player 1 are

$$u_1(A, \sigma_2) = 0 \times p + 15(1 - p) = 15 - 15p$$
$$u_1(B, \sigma_2) = 1 \times p + 10(1 - p) = 10 - 9p$$
$$u_1(C, \sigma_2) = 10p + 0 \times (1 - p) = 10p$$

Graphically,



Thus,

$$\text{BR}_1(\sigma_2) = \begin{cases} x = 1, y = 0 & \textit{if} \quad p < \frac{3}{5} \\ x \in [0,1], y = 0 & \textit{if} \quad p = \frac{3}{5} \\ x = 0, y = 0 & \textit{if} \quad p > \frac{3}{5} \end{cases}$$

And we see that player 1 will not use strategy B. Hence we may assume $y = 0$. The expected utilities of player 2 are

$$\begin{aligned} u_2(\sigma_1, X) &= 1 \times x + 0 \times y + 2(1 - x - y) = 2 - x - 2y \\ u_2(\sigma_1, Y) &= 2x + 1 \times y + 1 \times (1 - x - y) = 1 + x \end{aligned}$$

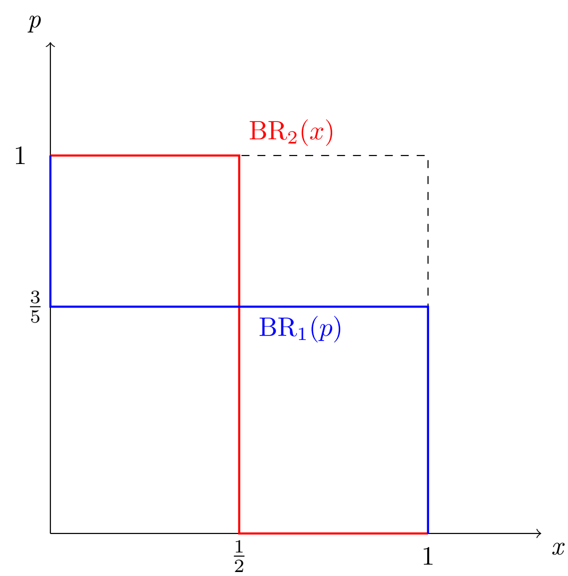
In the NE, we must have that $y = 0$. Hence, we may assume

$$\begin{aligned} u_2(\sigma_1, X) &= 2 - x \\ u_2(\sigma_1, Y) &= 1 + x \end{aligned}$$

Hence,

$$\text{BR}_2(\sigma_1) = \begin{cases} p = 1 & \textit{if} \quad x < \frac{1}{2} \\ [0,1] & \textit{if} \quad x = \frac{1}{2} \\ p = 0 & \textit{if} \quad x > \frac{1}{2} \end{cases}$$

Graphically,



And we obtain three NE

(a) $x = 0$, $y = 0$, $p = 1$. *That is,* (C, X) *with payoffs* $u_1 = 10$, $u_2 = 2$.

(b) $x = 1$, $y = 0$, $p = 0$. *That is,* (A, Z) *with payoffs* $u_1 = 15$, $u_2 = 2$.

(c) $x = \frac{1}{2}$, $y = 0$, $p = \frac{3}{5}$. *That is,*

$$\left(\frac{1}{2}A + \frac{1}{2}C, \frac{3}{5}X + \frac{2}{5}Z\right)$$

with payoffs $u_1 = 6$, $u_2 = \frac{3}{2}$.

Problem 2: Two agents, 1 and 2, produce heterogeneous products. If the two agents set, respectively, prices p_1 and p_2 the quantities demanded of the products of the two agents will be,

$$\begin{aligned} x_1(p_1, p_2) &= \max\left\{0, 90 - p_1 + \frac{p_2}{2}\right\} \\ x_2(p_1, p_2) &= \max\left\{0, 90 - p_2 + \frac{p_1}{2}\right\} \end{aligned}$$

Assume that the agents have constant marginal costs $\text{MC}_1 = \text{MC}_2 = 30$. Assume also that the characteristics of the market are such that the agents have to set simultaneously the prices.

(a) **(5 points)** Write down the agents' profits, $\pi_1(p_1, p_2)$ and $\pi_2(p_1, p_2)$, as functions of the prices they set.

Solution: *Assuming that the demand is positive, the profits of the agents are*

$$\begin{aligned}\pi_1(p_1,p_2) &= \left(90 - p_1 + \frac{p_2}{2}\right)(p_1 - 30) = 120p_1 - p_1^2 - 15p_2 + \frac{p_1p_2}{2} - 2700 \\ \pi_2(p_1,p_2) &= \left(90 - p_2 + \frac{p_1}{2}\right)(p_2 - 30) = 120p_2 - p_2^2 - 15p_1 + \frac{p_1p_2}{2} - 2700\end{aligned}$$

(b) **(10 points)** Find the best response of both agents: $\text{BR}_1(p_2)$ and $\text{BR}_2(p_1)$. Draw the best response function and the Nash equilibrium in the $p_1 - p_2$ plane (p_1 on the horizontal axis and p_2 on the vertical axis).

Solution: *The first order condition for agent 1 is*

$$\frac{\partial \pi_1}{\partial p_1} = 120 - 2p_1 + \frac{1}{2}p_2 = 0$$

The solution is the best reply of player 1

$$\text{BR}_1(p_2) = \frac{1}{4}(p_2 + 240)$$

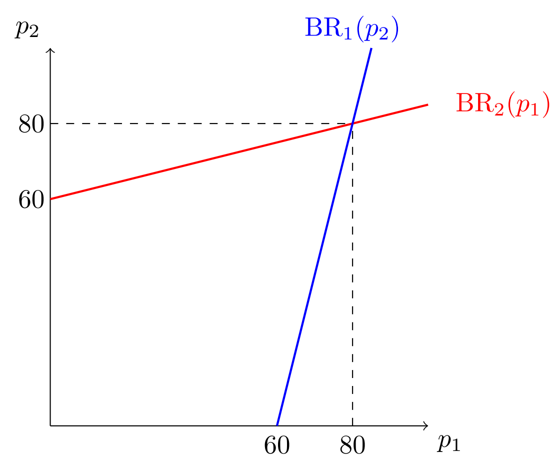
Note that the second order condition is fulfilled.

$$\frac{\partial^2 \pi_1}{\partial p_1^2} = -2 < 0$$

Similarly, best reply of player 2 is

$$\text{BR}_2(p_1) = \frac{1}{4}(p_1 + 240)$$

Graphically,



(c) **(10 points)** Find the Nash equilibrium (p_1^*, p_2^*) and the equilibrium profits $\pi_1(p_1,^* p_2^*)$ and $\pi_2(p_1^*, p_2^*)$.

Solution: *The NE is the solution to the equations*

$$p_1 = \text{BR}_1(p_2), \quad p_2 = \text{BR}_2(p_1),$$

That is

$$\begin{aligned}p_1 &= \frac{1}{4}(p_2 + 240) \\ p_2 &= \frac{1}{4}(p_1 + 240)\end{aligned}$$

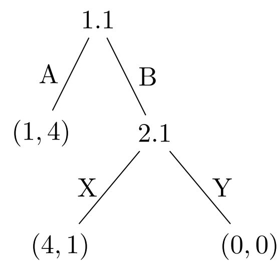
The NE is

$$p_1^* = p_2^* = 80$$

The profits are

$$\pi_1(p_1^*, p_2^*) = \pi_2(p_1^*, p_2^*) = 2500$$

Problem 3: Consider the following extensive form game.



(a) **(5 points)** Write the set strategies for each player. Write the normal form of the game.

Solution: *The set of strategies for the players are* $S_1 = \{A, B\}$, $S_2 = \{X, Y\}$. *The game may be represented by the table*

		Player 2 X	*Player 2* Y
Player 1	A	1, 4	1, 4
Player 1	B	4, 1	0, 0

(b) **(10 points)** Find all Nash equilibria of this game.

Solution: *Let us look for a NE of the form*

$$\begin{aligned} \sigma_1 &= pA + (1-p)B \\ \sigma_2 &= qX + (1-q)Y \end{aligned}$$

We compute the expected utilities of the players

$$\begin{aligned} u_1(A, \sigma_2) &= 1 \\ u_1(B, \sigma_2) &= 4q \\ u_2(\sigma_1, X) &= 1 + 3q \\ u_2(\sigma_1, Y) &= 4q \end{aligned}$$

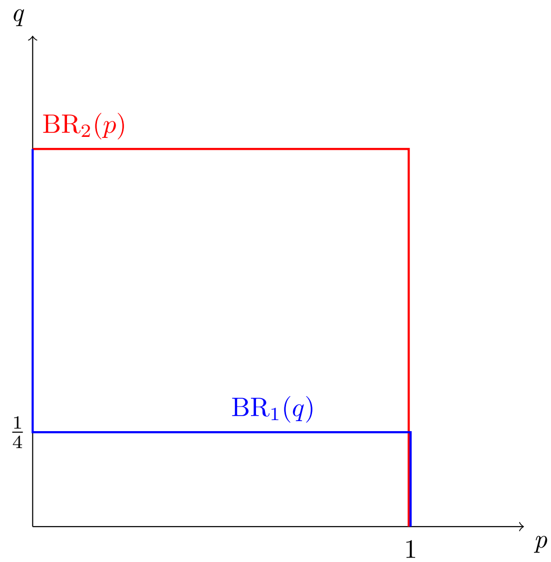
Thus, we have that best reply of player 1 is

$$\mathrm{BR}_1(\sigma_2) = \begin{cases} p = 1 & \textit{if} \quad 0 \le q < \frac{1}{4} \\ [0,1] & \textit{if} \quad q = \frac{1}{4} \\ p = 0 & \textit{if} \quad \frac{1}{4} < q \le 1 \end{cases}$$

And the best reply of player 2 is

$$\mathrm{BR}_2(\sigma_1) = \begin{cases} q = 1 & \textit{if} \quad 0 \le p < 1 \\ [0,1] & \textit{if} \quad q = 1 \end{cases}$$

Graphically,



We see that there are two NE, (A, Y), with payoffs $u_1 = 1$, $u_2 = 4$ and (B, X), with payoffs $u_1 = 4$, $u_2 = 1$, in pure strategies and the following NE

$$(A, qX + (1-q)Y) \quad 0 \leq q \leq \frac{1}{4}$$

with payoffs $u_1 = 1$, $u_2 = 4$, in mixed strategies.

(c) **(5 points)** Find the subgame perfect Nash equilibrium of the game.

Solution: *There are two subgames. The NE of the subgame that starts in node 1.2 is X. Thus, the SPNE is (B, X) with payoffs $u_1 = 4$, $u_2 = 1$.*

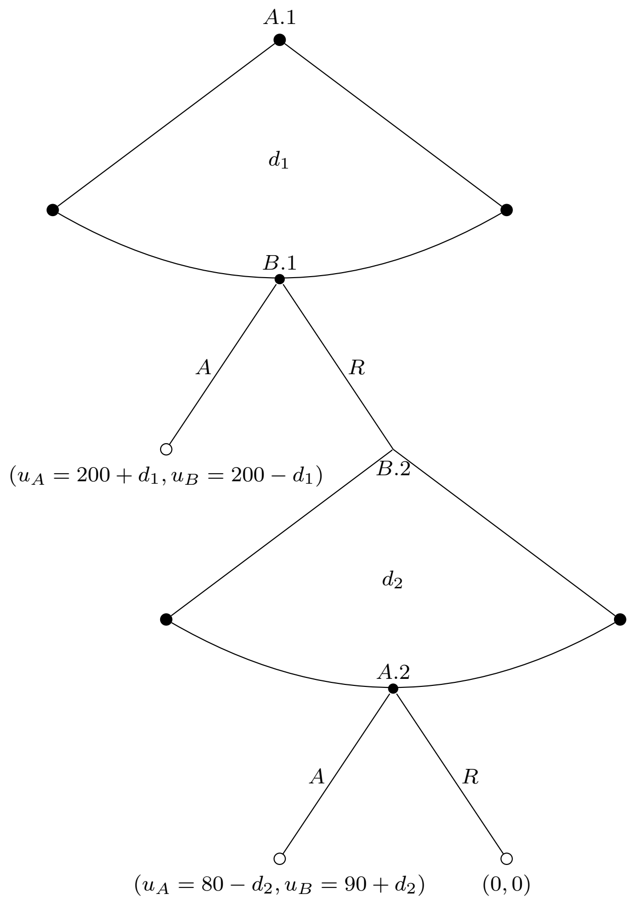
(d) **(5 points)** Can you identify a non-credible threat?

Solution: *The NE (A, Y) is a non-credible threat.*

Problem 4: Suppose that two agents are negotiating an agreement. If the two parties reach an agreement immediately, each will have a gain of 200. If the two parties don't achieve an agreement immediately they will have a second and last chance of achieving an agreement in a month. If the agreement is reached in a month, agent A will have a gain of 80 and agent B will have a gain of 90. Negotiation works in the following way: In the first period agent A demands a payment of agent B in order to sign the agreement. If agent B accepts, the agreement is reached, the payoff to agent A will be 200 plus the payment that agent B has accepted to make. And the payoff to agent B will be 200 minus the payment it has accepted to make. If agent B rejects the offer, it will make an offer to agent A in the second period with the offer consisting again of a demand of a payment in order to sign the agreement.

(a) **(15 points)** Describe the situation as an extensive form game. How many subgames does it have?

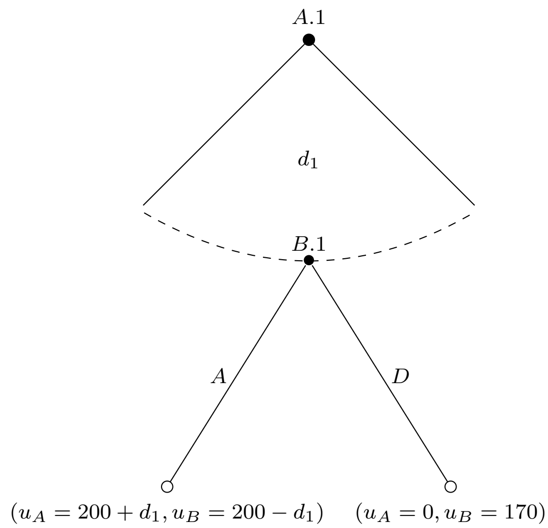
Solution: *The extensive game form representation is the following.*



- *There is one subgame that starts at* $A.1$*.*
- *There are infinitely many subgames that start at* $B.1$
- *There is one subgame that starts at* $B.2$*.*
- *There are infinitely many subgames that start at* $A.2$*.*

(b) **(10 points)** Conpute the subgame perfect Nash equilibrium of this game. Write the strategies of the players at each node. Write the payoffs of the players in the SPNE.

Solution: *At node* $A.2$*, firm A accepts iff* $80 - d_2 \geq 0$*. That is, firm A accepts iff* $d_2 \leq 80$*. Given the best reply of firm A at node* $A.2$*, the best response of firm B at node* $B.2$ *is to offer* $d_2 = 80$*. Thus, we may assume that if we ever reach node* $B.2$*, firm B will offer* $d_2 = 80$ *and firm A accepts. The payoffs will be* $u_A = 0$*,* $u_B = 170$*. We replace this payoffs at node* $B.2$



Now, player B at node $B.1$ accepts iff $200 - d_1 \geq 170$. That is, at node $B.1$ player B accepts iff $d_1 \leq 30$. The best response now for player A is to offer $d_1 = 30$ at node $A.1$ and player B accepts. The SPNE is the following.

- *Node $A.1$: $d_1 = 30$.*
- *Node $B.1$: accept iff $d_1 \leq 30$.*
- *Node $B.2$: $d_2 = 80$.*
- *Node $A.2$: accept iff $d_2 \leq 80$.*

The payoffs are $u_A = 230$, $u_B = 170$.