UNIVERSITY CARLOS III
Master in Industrial Economics and Markets
Game Theory

MIDTERM EXAM–November 2017

NAME:

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(1) *Consider the following game in normal form.*

	X	Y	Z
A	4,2	1,1	5,0
B	1, 1	2,3	6,2
C	3, −1	0,1	3,1

(a) **(7 points)** *Which are the strategies that survive the iterated deletion of strictly dominated strategies?*

Solution: Strategy C is dominated by B. After we eliminate it we obtain the game

	X	Y	Z
A	4,2	1,1	5,0
B	1,1	2,3	6,2

Now, we see that strategy Z is dominated by Y. We obtain the new game

	X	Y
A	4,2	1,1
B	1, 1	2,3

There are no other strategies which are strictly dominated. The strategies that survive are $\{A, B\} \times \{X, Y\}$.

(b) **(8 points)** *Find all the Nash equilibria in pure strategies and the payoffs of the players.*

Solution: The best response of the players to pure strategies are represented in the following table,

	X	Y
A	<u>4</u>,<u>2</u>	1,1
B	1, 1	<u>2</u>,<u>3</u>

We see that the NE in pure strategies are (A, X) and (B, Y).

(c) **(10 points)** *Find all the Nash equilibria in mixed strategies and the payoffs of the players.*

Solution: Let us look for a NE in which players follow the mixed strategy

$$\begin{aligned}\sigma_1 &= pA + (1-p)B \\ \sigma_2 &= xX + (1-q)Y\end{aligned}$$

The expected payoffs of player 1 are

$$\begin{aligned}u_1(A, \sigma_1) &= 4q + 1 - q = 1 + 3q \\ u_1(B, \sigma_1) &= q + 2(1-q) = 2 - q\end{aligned}$$

And player 1 is indifferent between the two strategies if and only if $1 + 3q = 2 - q$, that is if and only if $q = \frac{1}{4}$ and the payoff of player 1 is $\frac{7}{4}$.

The expected payoffs of player 2 are

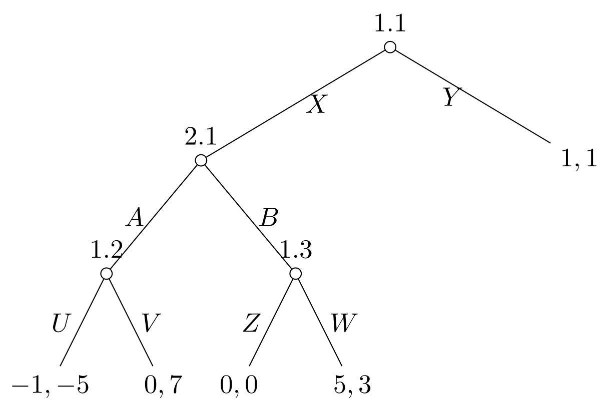
$$
\begin{aligned}
u_1(A, \sigma_1) &= 2p + 1 - p = 1 + p \\
u_1(B, \sigma_1) &= p + 3(1 - p) = 3 - 2p
\end{aligned}
$$

And player 2 is indifferent between the two strategies if and only if $1 + p = 3 - 2p$, that is if and only if $p = \frac{2}{3}$ and the payoff of player 2 is $\frac{5}{3}$.
We conclude that there is a mixed strategy NE,

$$\left(\frac{2}{3}A + \frac{1}{3}B, \frac{1}{4}X + \frac{3}{4}Y\right)$$

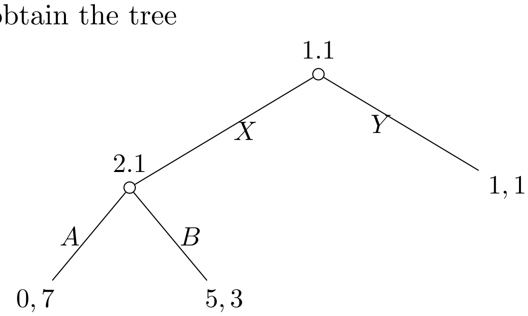
with payoffs $u_1 = \frac{7}{4}$ and $u_2 = \frac{5}{3}$.

(2) *Consider the following game in extensive form*

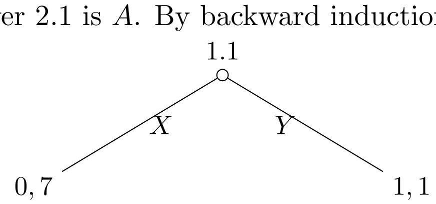


(a) *Find the sub-game perfect Nash equilibria in pure strategies.*
(10 points)

Solution: The best response of player 1.2 is V and the best response of player 1.3 is W. By backward induction, we obtain the tree



and the best response of player 2.1 is A. By backward induction, we obtain the tree



and the best response of player 1.1 is Y. We obtain the SPNE

$$((Y, V, W), A)$$

(b) *Write the normal form of the whole game. Find all the Nash equilibria of this sub-game, in pure strategies. Compute the payoffs of the players.*
(10 points)

Solution: The normal form is summarized in the following table

	A	B
UZ	-1,-5	0, 0
UW	-1,-5	5, 3
VZ	0,7	0,0
VW	0,7	5 ,3

The NE are (UW, B), (VZ, A) and (VW, A).

(c) *Can you identify a non-credible threat among the NE of the previous part?*
(5 points)

Solution: The NE (UW, B) is a non-credible threat. Agent 1 threatens with U if player 2 chooses A. But, U is not a best response in the sub-game that starts in 1.2.

(3) *If a teacher (agent 1) invests x_1 units of time into preparing the lecture and a student (agent 2) invests x_2 units of time into studying for the lecture, their utilities are the following*

$$\begin{aligned} u_1(x_1, x_2) &= (42 + x_2)x_1 - 4(x_1)^2 \\ u_2(x_1, x_2) &= (42 + x_1)x_2 - 4(x_2)^2 \end{aligned}$$

That is each one exerts benefits on the other one by preparing for the lecture, but preparation is costly for both.

(a) *Assuming that both take their decisions simultaneously and independently without knowing the decision of the other agent, compute the NE.*
(10 points)

Solution: Let us compute the best response of player 1. This player maximizes his utility, $\max_{x_1} u_1$. The first order condition is $42 + x_2 - 8x_1 = 0$. So, the best response of player 1 is

$$\mathrm{BR}_1(x_2) = \frac{42 + x_2}{8}$$

Similarly, the best response of player 2 is

$$\mathrm{BR}_2(x_1) = \frac{42 + x_1}{8}$$

The NE satisfies the equations

$$\begin{aligned} x_1 &= \frac{42 + x_2}{8} \\ x_2 &= \frac{42 + x_1}{8} \end{aligned}$$

The solution is $x_1 = x_2 = 6$. The utilities of the players are $u_1(6, 6) = u_2(6, 6) = 144$.

(b) *Could the teacher and the student achieve a higher utility if they could agree on the number of hours they spend on preparing the lectures?*
(10 points)

Solution: They could achieve a higher utility maximizing $\max_{x_1,x_2} u_1(x_1,x_2) + u_2(x_1,x_2)$. The first order conditions for this problem are

$$\begin{aligned} 0 &= 42 + x_2 - 8x_1 + x_2 \\ 0 &= 42 + x_1 - 8x_1 + x_1 \end{aligned}$$

The solution is $x_1 = x_2 = 7$. With this allocation the utilities of the student and professor are $u_1(7,7) = u_2(7,7) = 147$.

(c) *Why the solution of the previous part is not reasonable if the teacher and the student cannot commit to their agreement?*
(5 points)

Solution: The allocation $x_1 = x_2 = 7$ is not a NE. If, for example, player 1 knows that $x_2 = 7$ then he we would choose $x_1 = \mathrm{BR}_1(7) = \frac{42+7}{8} \neq 7$. The same applies to player 2.

(4) *Two politicians are competing for office. Their objective is to win the election. Assume that their payoff is 1 if they win the election and 0 if they do not. Politicians propose one of the following tax rates: 10%, 20%, 30%, 40% and 50% and voters vote for the politician who propose the tax rate closest to their ideal point. There are*

- *4 agents whose preferred tax rate is 10%,*
- *4 agents whose preferred tax rate is 20%,*
- *6 agents whose preferred tax rate is 30%,*
- *6 agents whose preferred tax rate is 30%; and*
- *2 agents whose preferred tax rate is 50%.*

(a) *Describe the above situation as a normal form (static) game. Assume that if a voter is indifferent between the two politicians votes for each with probability* 1/2.
(10 points)

Solution: The normal form is summarized in the following table

	10	20	30	40	50
10	1/2,1/2	0,1	0,1	0,1	1/2, 1/2
20	1,0	1/2,1/2	0,1	1/2,1/2	1,0
30	1,0	1,0	1/2,1/2	1,0	1,0
40	1,0	1/2,1/2	0,1	1/2,1/2	1,0
50.	1/2,1/2	0,1	0,1	0,1	1/2, 1/2

(b) *Find the pure strategy Nash equilibria.*
(10 points)
Solution:

	10	20	30	40	50
10	1/2,1/2	0,1	0,1	0, 1	1/2, 1/2
20	1,0	1/2,1/2	0,1	1/2,1/2	1,0
30	1,0	1,0	1/2,1/2	1,0	1,0
40	1,0	1/2,1/2	0,1	1/2,1/2	1,0
50.	1/2,1/2	0,1	0,1	0,1	1/2, 1/2

The NE is $(30,30)$.

(c) *Can you give an interpretation of the result?*
(5 points)

Solution: It's the median voter Theorem: The agent in the median is indifferent between the two proposals.