UNIVERSITY CARLOS III

Master in Economics **Master in Industrial Economics and Markets**

Game Theory. Final Exam

January 24th, 2018

TIME: 2 hours. Write your answers in this booklet.

NAME:

Problem 1:

Problem 2: *Consider the following normal form game.*

	A	B	C
A	7,7	1,10	1,1
B	10, 1	4,4	1,1
C	1, 1	1, 1	2, 2

(1) *Find the Nash equilibria in pure strategies.*

Solution: There are two NE in pure strategies (B, B) and (C, C).

(2) *Suppose that the above game is played two times. Can you find a sub-game perfect NE of the repeated game with the following properties?*
(a) *no mixed strategies of the stage game are used.*
(b) (A, A) *is played in the first stage.*

Solution: We try the trigger strategy. Each player $i = 1, 2$ plays the following
- At $t = 1$ play A.
- At $t = 2$ play B if (A, A) was played at stage $t = 1$. Otherwise, play C.

If both players follow the grim strategy their payoffs are $u_1^* = u_2^* = 7 + 4 = 11$. We check if the above strategy constitutes a NE of the whole game. No player has incentives to deviate at $t = 2$, since the strategy proposed is a NE of the stage game.

If say player $i = 1, 2$ deviates at $t = 1$ and the other player $j \neq i$ follows the grim strategy, the payoff for player i is $u_i = 10 + 2 = 12$. Hence, both players have incentive to deviate. Therefore, the grim strategy is not a NE of the repeated game. The answer to the question is no.

(3) *Suppose that the above game is played three times. Can you find a sub-game perfect NE of the repeated game in which* (A, A) *is played in the first stage?*

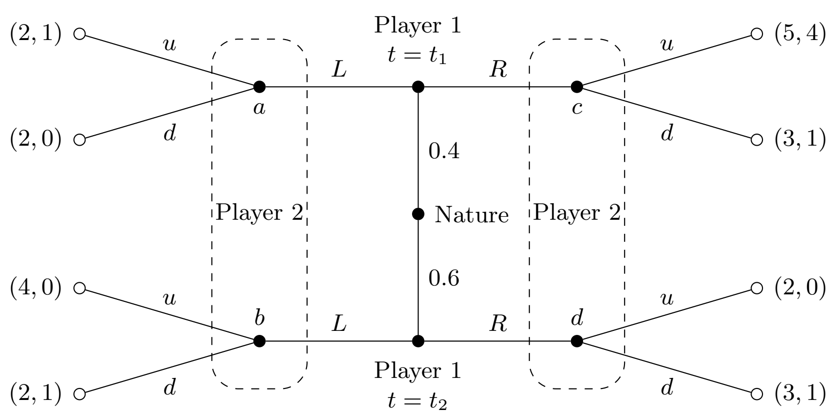
Solution: We try the trigger strategy. Each player $i = 1, 2$ plays the following
- At $t = 1$ play A.
- At $t = 2$ play B if (A, A) was played at stage $t = 1$. Otherwise, play C.
- At $t = 3$ play B if (A, A) was played at stage $t = 1$ and (B, B) was played at stage $t = 2$. Otherwise, play C.

If both players follow the grim strategy their payoffs are $u_1^* = u_2^* = 7+4+4 = 15$. We check if the above strategy constitutes a NE of the whole game. No player has incentives to deviate at $t = 1, 2$, since the strategy proposed plays a NE of the stage game in each period.

If say player $i = 1, 2$ deviates at $t = 1$ and the other player $j \neq i$ follows the grim strategy, the payoff for player i is at most $u_i = 10 + 2 + 2 = 14$. Hence, no player has incentives to deviate. Therefore, the grim strategy is a SPNE of the repeated game. The answer to the question is yes.

Problem 3:

Problem 4: *Consider the following signalling game. There are two types of player 1, $t = t_1$ and $t = t_2$.*



(1) *One of the types of player 1 has a dominating strategy. Which one?*

Solution: If player t_1 plays L his maximum payoff would be 2. If he plays R his minimum payoff is 3. Hence, R is a dominant strategy for player t_1. We will assume that in any PBNE , $\sigma(t_1) = R$.

(2) *Compute all the separating perfect Bayesian Nash equilibria (in pure strategies) of the above game. Write the separating PBNE, including the beliefs of player 2. Justify your answer.*

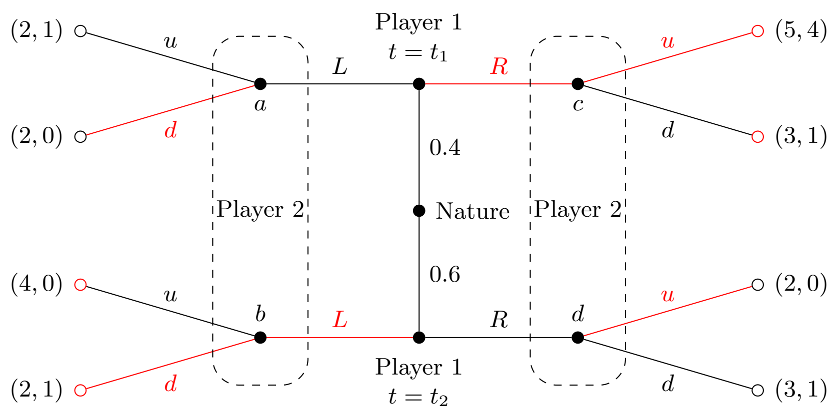
Solution: By part (1) the only candidate for a separating PBNE is $\sigma(t_1) = R$, $\sigma(t_2) = L$. In this equilibrium the beliefs of player 2 are

$$\mu_2(c|R) = 1, \mu_2(b|L) = 1$$

Given these beliefs, the best reply of player 2 is

$$\mathrm{BR}_2(R|\mu_2) = \{u\}, \quad \mathrm{BR}_2(L|\mu_2) = \{d\}$$

Graphically,



Now, anticipating that player 2 will play $\sigma_2(R) = \{u\}$ and $\sigma_2(L) = \{d\}$, the optimal strategy for player 1 is to choose $\sigma(t_1) = R$, $\sigma(t_2) = \{L, R\}$. We conclude that the following is a PBNE.

$$\begin{aligned} \sigma(t_1) &= R, \quad \sigma(t_2) = L \\ \sigma_2(R) &= u, \quad \sigma_2(L) = d \\ \mu_2(a|L) &= 0, \quad \mu_2(b|L) = 1 \\ \mu_2(c|R) &= 1, \quad \mu_2(d|R) = 0 \end{aligned}$$

(3) *Compute all the pooling perfect Bayesian Nash equilibria (in pure strategies) of the above game. Write the pooling PBNE, including the beliefs of player 2. Justify your answer.*

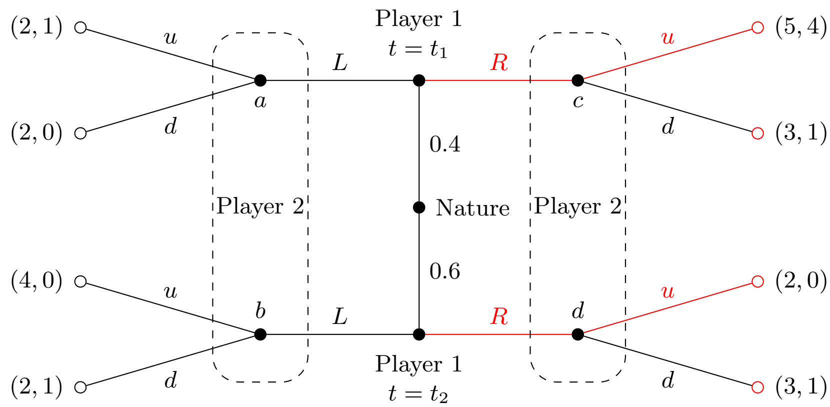
Solution: By part (1) the only candidate for a pooling PBNE is $\sigma(t_1) = R$, $\sigma(t_2) = R$. In this equilibrium the beliefs of player 2 are

$$\begin{aligned} \mu_2(a|L) &= x, \quad \mu_2(b|L) = 1 - x \\ \mu_2(c|R) &= 0.4, \quad \mu_2(d|R) = 0.6 \end{aligned}$$

Given these beliefs, the expected utilities of player 2, given R, are

$$\begin{aligned} u_2(u|R) &= 0.4 \times 4 + 0.6 \times 0 = 1.6 \\ u_2(d|R) &= 0.4 \times 1 + 0.6 \times 1 = 1 \end{aligned}$$

Hence, $\text{BR}_2(R|\sigma_1, \mu_2) = \{u\}$. Graphically,



On the other hand, the expected utilities of player 2 , given L, are

$$\begin{aligned} u_2(u|L) &= x \times 1 + (1-x) \times 0 = x \\ u_2(d|R) &= x \times 0 + (1-x) \times 1 = 1-x \end{aligned}$$

Hence,

$$\mathrm{BR}_2(R) = \begin{cases} u & \text{if } x > 1/2 \\ u, d & \text{if } x = 1/2 \\ d & \text{if } x < 1/2 \end{cases}$$

Suppose $x > 1/2$. Then $\mathrm{BR}_2(R) = u$ and player t_2 would like to deviate from the strategy $\sigma(t_2) = R$ to $\sigma(t_2) = L$. We conclude that there there is no pooling equilibrium with $x > 1/2$.

On the other hand, if $x \leq 1/2$, then d is a best reply for player 2 and player 1 has no incentives to deviate from the strategy $\sigma(t_1) = R$, $\sigma(t_2) = R$. We see that the following is are PBNE for each value $0 \leq x \leq 1/2$,

$$\begin{aligned} \sigma(t_1) &= R, \sigma(t_2) = R \\ \sigma_2(R) &= u, \sigma_2(L) = d \\ \mu_2(a|L) &= x, \mu_2(b|L) = 1-x \\ \mu_2(c|R) &= 0.4, \mu_2(d|R) = 0.6 \end{aligned}$$

(4) *Imagine player $t = t_1$ follows the mixed stragey $xL + (1-x)R$ and player $t = t_2$ follows the mixed strategy $yL + (1-y)R$ with $0 < x, y < 1$. What are the consistent beliefs of player 2?*

Solution: Note that

$$p(a) = 0.4x \quad p(b) = 0.6y, \quad p(c) = 0.4(1-x), \quad p(d) = 0.6(1-y)$$

Therefore

$$\mu_2(a|L) = \frac{p(a)}{p(a)+p(b)} = \frac{2x}{2x+3y} \quad \mu_2(b|L) = \frac{p(p)}{p(a)+p(b)} = \frac{3y}{2x+3y}$$

$$\mu_2(c|R) = \frac{p(c)}{p(c)+p(d)} = \frac{2(1-x)}{5-2x-3y} \quad \mu_2(d|R) = \frac{p(d)}{p(c)+p(d)} = \frac{3(1-y)}{5-2x-3y}$$