**UNIVERSITY CARLOS III**
**Master in Industrial Economics and Markets**
**Game Theory**
**Final Exam** **January 27, 2017**
**Solutions**

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**Problem 1:** Consider the following game in normal form.

| | $A_2$ | $B_2$ | $C_2$ | $D_2$ |
|---|---|---|---|---|
| $A_1$ | 6,-1 | 3,3 | 4,0 | 2,2 |
| $B1$ | 2, 301 | 2,-10 | 2,15 | 1,2 |
| $C_1$ | 5,-1 | 5,0 | 3,1 | 4,1 |

(1) **(10 points)** Which are the strategies that survive the iterated deletion of strictly dominated strategies?

**Solution:** *The stratey $B1$ is strictly dominated by $A_1$ (and $C_1$). After we eliminate this strategy we obtain the game*

| | $A_2$ | $B_2$ | $C_2$ | $D_2$ |
|---|---|---|---|---|
| $A_1$ | *6,-1* | *3,3* | *4,0* | *2,2* |
| $C_1$ | *5,-1* | *5,0* | *3,1* | *4,1* |

*Now, the strategy $A_2$ e is strictly dominated by the strategies $B_2$, $C_2$ and $D_2$. DAfter we eliminate this strategy we obtain the following game*

| | $B_2$ | $C_2$ | $D_2$ |
|---|---|---|---|
| $A_1$ | *3,3* | *4,0* | *2,2* |
| $C_1$ | *5,0* | *3,1* | *4,1* |

*Hence, the strategies that survive the iterated elimination of dominated strategies are $S_1 = \{A_1, C_1\}$, $S_2 = \{B_2, C_2, D_2\}$.*

(2) **(5 points)** Find all the Nash equilibria in pure strategies and the payoffs of the players.

**Solution:** *The best responses of the players are*

| | $B_2$ | $C_2$ | $D_2$ |
|---|---|---|---|
| $A_1$ | $3,\underline{3}$ | $\underline{4},0$ | $2,2$ |
| $C_1$ | $\underline{5},0$ | $3,\underline{1}$ | $\underline{4},\underline{1}$ |

*The unique Nash equilibria in pure strategies is $(C_1, D_2)$. The payoffs are $u_1(C_1, D_2) = 4$, $u_2(C_1, D_2) = 1$.*

(3) **(10 points)** Find all the Nash equilibria in mixed strategies and the payoffs of the players.

**Solution:** *Suppose that player 1 follows the mixed strategy $s_1 = pA_1 + (1-p)C_1$ and that player 2 follows the mixed strategy $s_2 = q_1B_2 + q_2C_2 + (1-q_1-q_2)D_2$. The payoffs for player 1 are*

$$\begin{aligned} u_1(A_1, s_2) &= 3q_1 + 4q_2 + 2(1-q_1-q_2) = 2 + q_1 + 2q_2 \\ u_1(C_1, s_2) &= 5q_1 + 3q_2 + 4(1-q_1-q_2) = 4 + q_1 - q_2 \end{aligned}$$

*and the payoffs for player 2 are*

$$\begin{aligned} u_2(s_1, B_2) &= 3p \\ u_2(s_1, C_2) &= 1 - p \\ u_2(s_1, D_2) &= 1 + p \end{aligned}$$

*Note first that* $2 + q_1 + 2q_2 \geq 4 + q_1 - q_2$ *iff* $q_2 \geq \frac{2}{3}$. *It follows that*

$$\mathrm{BR}_1\left(q_1 B_2 + q_2 C_2 + (1 - q_1 - q_2) D_2\right) = \begin{cases} C_1, \text{if } q_2 < \frac{2}{3} \\ \{A_1, C_1\}, \text{if } q_2 = \frac{2}{3} \\ A_1, \text{if } q_2 > \frac{2}{3} \end{cases}$$

*In particular, it follows that player 1 follows a mixed strategy only if* $q_2 = \frac{2}{3}$. *On the other hand, since* $1 + p > 1 - p$ *for* $p > 0$ *we see that player 2 will not use strategy* $C_2$ *unless* $p = 0$.

*Can player 2 follow a mixed strategy with* $q_2 = 0$ *and* $0 < q_1 < 1$? *In such a case, we must have that* $u_2(s_1, B_2) = u_2(s_1, D_2)$ *and we obtain that* $p = \frac{1}{2}$. *This means that player 1 is using a completely mixed strategy. But, as we have proved above this is possible only if* $q_2 = \frac{2}{3}$, *which contradicts that* $q_2 = 0$. *Hence,* **there are no NE with** $q_2 = 0$ **and** $0 < q_1 < 1$**.**

*Can player 2 follow a mixed strategy with* $q_1 = 0$ *and* $0 < q_2 < 1$? *In such a case, we must have that* $u_2(s_1, C_2) = u_2(s_1, D_2)$. *That is,* $1 - p = 1 + p$ *which implies* $p = 0$. *This means player 1 best response is* $C_1$, *so* $q_2 \leq \frac{2}{3}$. *Thus, we find that* **there are infinitely many NE of the form**

$$(C_1, q_2 C_2 + (1 - q_2) D_2)$$

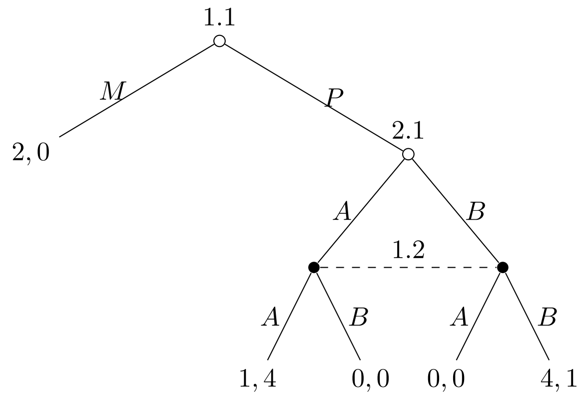
**with** $0 \leq q_2 \leq \frac{2}{3}$**.** *The payoffs are* $u_1 = 4 - q_2$, $u_2 = 1$.

**Problem 2:** Agent 001 has to decide between completing his mission (M) and escorting the president of the country, or to pretend (P) that he is ill and go to the town $T$ where he would play cards with agent 002. If agent 001 fulfills his duty, then agent 002 watches agent 001 on TV protecting the president and would know that they are not going to play cards. In this situation the utilities of the agents would be $u_1 = 2$ and $u_2 = 0$.

If agent 001 pretends to be ill then he would travel the the town $T$. In there, one can find two bars where it is allowed to play cards: $A$ and $B$. If both agents go to bar $A$ their utilities are $u_1 = 1$ and $u_2 = 4$. Whereas, if they both go to $B$ their utilities are $u_1 = 4$ and $u_2 = 1$. If they do not go to the same bar, their utilities would be 0 for both of them.

(1) Assume first that agent 001 cannot communicate with agent 002 to coordinate, because the message would be intercepted by some hackers that would leak the fraud.

(a) **(5 points)** Describe the situation as a dynamic game. How many subgames does this game have? **Solution:**




*There are two subgames. They start at the nodes* 1.1 *and* 2.1.

(b) **(5 points)** Compute the Nash equilibria of all the proper subgames of the above game and the payoffs of the players.

**Solution:** *There is only one proper subgame. It starts at node* 2.1. *We write this game in normal form*

| | $A$ | $B$ |
|---|---|---|
| $A$ | *1,4* | *0,0* |
| $C$ | *0,0* | *4,1* |

*There are two NE in pure strategies,*

- $(A, A)$ *with payoffs* $u_1(A, A) = 1$, $u_2(A, A) = 4$; *and*
- $(B, B)$ *with payoffs* $u_1(B, B) = 4$, $u_2(B, B) = 1$.

*Let us now search for a mixed strategy Ne of the form*

$$(xA + (1-x)B, yA + (1-y)B)$$

*Note that*

$$\begin{aligned} u_1(A, yA + (1-y)B) &= y \\ u_1(B, yA + (1-y)B) &= 4 - 4y \\ u_2(xA + (1-x)B, A) &= 4x \\ u_2(xA + (1-x)B, B) &= 1 - x \end{aligned}$$

*from where we obtain that* $x = \frac{1}{5}$, $y = \frac{4}{5}$. *So,*

$$\left(\frac{1}{5}A + \frac{4}{5}B, \frac{4}{5}A + \frac{1}{5}B\right)$$

*is a NE with payoffs*

$$u_1\left(\frac{1}{5}A + \frac{4}{5}B, \frac{4}{5}A + \frac{1}{5}B\right) = u_2\left(\frac{1}{5}A + \frac{4}{5}B, \frac{4}{5}A + \frac{1}{5}B\right) = \frac{4}{5}$$

(c) **(5 points)** Compute the subgame perfect Nash equilibria of the above game and the payoffs of the players.

**Solution:** *There are three SPNE*

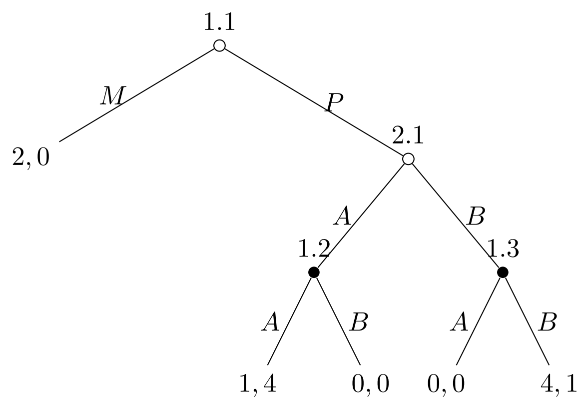
- $((M, A), A)$ *with payoffs* $u_1 = 2$, $u_2 = 0$;

- $((P, B), B)$ *with payoffs* $u_1 = 4$, $u_2(B, B) = 1$; *and*
- $\left((M, \frac{1}{5}A + \frac{4}{5}B), \frac{4}{5}A + \frac{1}{5}B\right)$ *with payoffs* $u_1 = 2$, $u_2 = 0$.

(2) Assume now that agent 002 is able to leave a message for agent 001 at the train station of town $T$ with the bar that agent 2 will go to.

(a) **(5 points)** Describe the new situation as a dynamic game. How many subgames does this game have?

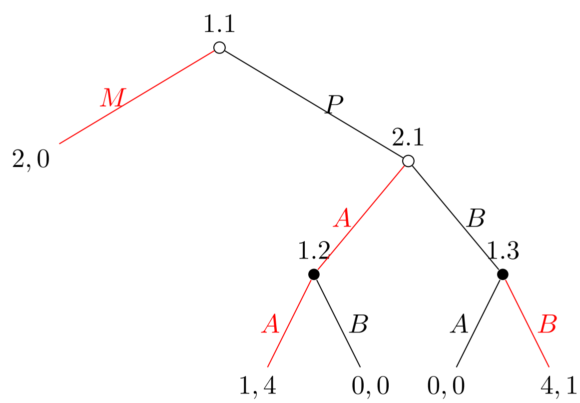
**Solution:**




*There are four subgames. They start at the nodes* 1.1 *and* 2.1, 1.2 *and* 1.3.

(b) **(5 points)** Compute the subgame perfect Nash equilibria of this new game and the payoffs of the players.

**Solution:**




*The SPNE is* $((M, A, B), A)$. *The payoffs are* $u_1 = 2$, $u_2 = 0$.

**Problem 3:** Consider the following game,with two players, in normal form.

| Player 1 \ Player 2 | $C$ | $D$ |
|---|---|---|
| $A$ | 5,4 | 1,5 |
| $B$ | 6, 2 | 3,3 |

and consider the repeated game with the above game as a stage game.

(1) **(5 points)** Suppose that the above game is played two times. How many information sets does each player have?, How many strategies does each player have? Give an example of a possible strategy for player 1. How many subgames are in the game repeated two times?

**Solution:**

- *At the first stage each player has one information set. There are four possible outcomes in the game. Hence, at the second stage each player has four information sets. Therefore, each player has a total of five information sets.*
- *There are also 5 subgames; and*
- *Each player can choose among 2 strategies in every subgame. Hence, there is a total of $2^5 = 32$ strategies per player.*
- *An example of a strategy for player 1 could be*
  - (a) *play $A$ in the first period.*
  - (b) *In the second period play*
    - – *$A$ if in the first period $(A, C)$ was played.*
    - – *$B$ if in the first period $(A, D)$ was played.*
    - – *$A$ if in the first period $(B, C)$ was played.*
    - – *$B$ if in the first period $(B, D)$ was played.*

(2) **(5 points)** Suppose that the above game is played 2 times. Compute all the subgame perfect Nash equilibria and the payoffs of the players.

**Solution:** *The game has a unique NE: $(B, D)$, with payoffs $u_1(B, D) = u_2(B, D) = 3$. Hence, the game repeated finitely many times has a unique SPNE which consists in playing the unique NE of the stage game at every period. Since it is repeated two times, the payoffs of the players are $u_1 = u_2 = 6$.*

(3) **(5 points)** Suppose that the above game is played 198743926508 times. Compute all the subgame perfect Nash equilibria and the payoffs of the players.

**Solution:** *By the argument above the payoffs are $u_1 = u_2 = 3 \times 198743926508$.*

(4) **(10 points)** Suppose that the above game is played infinitely many times and both players have the same discount factor $\delta$. Compute the smallest $\delta$ such that there is a subgame perfect Nash equilibria, using trigger strategies, in which at every stage players play the strategy $(A, C)$.

**Solution:** *The trigger strategy for player 1 is the following,*

(a) *At period $t = 1$ play $A$.*

(b) *If in every period $1 \leq s < t$ it was played $(A, C)$, then in period $t$ play $A$. Otherwise, play $B$.*

*And the trigger strategy player 2 is*

(a) *At period $t = 1$ play $C$.*

(b) *If in every period $1 \leq s < t$ it was played $(A, C)$, then in period $t$ play $C$. Otherwise, play $D$.*

*If both players follow the trigger strategy, the payoffs are*

$$
\begin{aligned}
u_1 &= 5 + 5\delta + 5\delta^2 + 5\delta^3 + \cdots \\
u_2 &= 4 + 4\delta + 4\delta^2 + 4\delta^3 + \cdots
\end{aligned}
$$

*It is enough to consider deviations in the first period. If player 1 deviates in period 1, and player 2 follows the trigger strategy the payoff of player 1 is*

$$u_1^d = 6 + 3\delta + 3\delta^2 + 3\delta^3 + \cdots$$

*Hence, player 1 does not have incentives to deviate iff*

$$5 + 5\delta + 5\delta^2 + 5\delta^3 + \cdots \geq 6 + 3\delta + 3\delta^2 + 3\delta^3 + \cdots$$

*That is if*

$$2\delta + 2\delta^2 + 2\delta^3 + \cdots = \frac{2\delta}{1-\delta} \geq 1$$

*which is equivalent to* $\delta \geq \frac{1}{3}$.

*If player 2 deviates in period 1, and player 1 follows the trigger strategy the payoff of player 2 is*

$$u_2^d = 5 + 3\delta + 3\delta^2 + 3\delta^3 + \cdots$$

*Hence, player 2 does not have incentives to deviate iff*

$$4 + 4\delta + 4\delta^2 + 4\delta^3 + \cdots \geq 5 + 3\delta + 3\delta^2 + 3\delta^3 + \cdots$$

*That is if*

$$\delta + \delta^2 + \delta^3 + \cdots = \frac{\delta}{1-\delta} \geq 1$$

*which is equivalent to* $\delta \geq \frac{1}{2}$. *So, we need* $\delta \geq \max\{\frac{1}{2}, \frac{1}{3}\} = \frac{1}{2}$.

**Problem 4:** Consider the situation in which player 2 knows which game is played ($a$ or $b$ below). However, player 1 only knows that table $a$ is played with probability $\frac{1}{2}$ and table $b$ is played with probability $\frac{1}{2}$.

| | | Player 2 | |
|---|---|---|---|
| | | $C$ | $D$ |
| Player 1 | $A$ | 7,3 | 1,2 |
| | $B$ | 2, 1 | 4,6 |

a

| | | Player 2 | |
|---|---|---|---|
| | | $C$ | $D$ |
| Player 1 | $A$ | 7,1 | 1,6 |
| | $B$ | 2, 3 | 4,2 |

b

(1) **(5 points)** Describe the situation as a Bayesian game.

**Solution:** *The set of players is* $N = \{1, 2\}$. *The set of strategies is* $S_1 = \{A, B\}$, $S_2 = \{CC, CD, DC, DD\}$. *The types are* $T_1 = \{c\}$, $T_2 = \{a, b\}$. *The beliefs are*

$$\begin{aligned} p_1(a|c) &= p_1(b|c) = \frac{1}{2} \\ p_2(c|a) &= p_2(c|b) = 1 \end{aligned}$$

*The payoffs are described in the above tables.*

(2) **(10 points)** Find the Bayesian–Nash equilibria in pure strategies and the payoffs of the players.

**Solution:** *We remark that* $\mathrm{BR}_2(A) = CD$ *and* $\mathrm{BR}_2(B) = DC$. *Also*

$$\begin{aligned} u_1(A, CD) &= \frac{1}{2} \times 7 + \frac{1}{2} \times 1 = 4 \\ u_1(B, CD) &= \frac{1}{2} \times 2 + \frac{1}{2} \times 4 = 3 \end{aligned}$$

*Hence,* $\mathrm{BR}_1(CD) = A$ *and we see that* $(A, CD)$ **is a BNE with payoffs** $u_1(A, CD) = 4$**,** $u_2(A, CD|a) = 3$**,**$u_2(A, CD|b) = 6$**.** *On the other hand,*

$$\begin{aligned} u_1(A, DC) &= \frac{1}{2} \times 1 + \frac{1}{2} \times 7 = 4 \\ u_1(B, DC) &= \frac{1}{2} \times 4 + \frac{1}{2} \times 2 = 3 \end{aligned}$$

*Hence,* $\mathrm{BR}_1(DC) = A$ *and we see that there are no other BNE in pure strategies.*

(3) **(10 points)** Find the Bayesian–Nash equilibria in mixed strategies and the payoffs of the players.

**Solution:** *Let us look for a BNE of the form*

$$(xA + (1-x)B, (yC + (1-y)D, zC + (1-z)D))$$

*Let*

$$\begin{aligned} s_1 &= xA + (1-x)B \\ s_a &= yC + (1-y)D \\ s_b &= zC + (1-z)D \end{aligned}$$

*We have that*

$$\begin{aligned} u_1(A; s_a, s_b) &= \frac{1}{2}(7y + 1 - y) + \frac{1}{2}(7z + 1 - z) = 1 + 3y + 3z \\ u_1(B; s_a, s_b) &= \frac{1}{2}(2y + 4(1-y)) + \frac{1}{2}(2z + 4(1-z)) = 4 - y - z \\ u_2(s_1, C|a) &= 3x + 1 - x = 1 + 2x \\ u_2(s_1, D|a) &= 2x + 6(1-x) = 6 - 4x \\ u_2(s_1, C|b) &= x + 3(1-x) = 3 - 2x \\ u_2(s_1, D|b) &= 6x + 2(1-x) = 2 + 4x \end{aligned}$$

*Suppose first that player 2a is using a completely mixed strategy. Then* $u_2(s_1, C|a) = u_2(s_1, D|a)$*. Hence,* $1 + 2x = 6 - 4x$ *and we conclude that* $x = \frac{5}{6}$*. For this value of* $x$ *we have that* $u_2(s_1, C|b) = (3 - 2x)|_{x=\frac{5}{6}} = \frac{4}{3}$ *and* $u_2(s_1, D|b) = (2 + 4x)|_{x=\frac{5}{6}} = \frac{16}{3}$*, so* $z = 0$*. We check if there is a BNE of the form*

$$\left(\frac{5}{6}A + \frac{1}{6}B, (yC + (1-y)D, D)\right)$$

*Player 1 must be indifferent between* $A$ *and* $B$*. Hence,* $1 + 3y + 3z = 4 - y - z$*. Since* $z = 0$*, we obtain that* $y = \frac{3}{4}$*.* **And we have checked that**

$$\left(\frac{5}{6}A + \frac{1}{6}B, \left(\frac{3}{4}C + \frac{1}{4}D, D\right)\right)$$

**is BNE in mixed strategies with payoffs** $u_1 = \frac{13}{4}$**,** $u_2(\cdot|a) = \frac{8}{3}$**,** $u_2(\cdot|b) = \frac{16}{3}$**.**

*Suppose now that player 2b is using a completely mixed strategy. Then* $u_2(s_1, C|b) = u_2(s_1, D|b)$*. Hence,* $3 - 2x = 2 + 4x$ *and we conclude that* $x = \frac{1}{6}$*. For this value of* $x$ *we have that* $u_2(s_1, C|a) = (1 + 2x)|_{x=\frac{1}{6}} = \frac{4}{3}$ *and* $u_2(s_1, D|a) = (6 - 4x)|_{x=\frac{1}{6}} = \frac{16}{3}$*, so* $y = 0$*. We check if there is a BNE of the form*

$$\left(\frac{1}{6}A + \frac{5}{6}B, (D, zC + (1-z)D)\right)$$

*Player 1 must be indifferent between $A$ and $B$. Hence,* $1+3y+3z=4-y-z$. *Since* $y=0$, *we obtain that* $z=\frac{3}{4}$. **And we have checked that**

$$\left(\frac{1}{6}A+\frac{5}{6}B,\left(D,\frac{3}{4}C+\frac{1}{4}D\right)\right)$$

**is the other BNE in mixed strategies with payoffs** $u_1=\frac{13}{4}$**,** $u_2(\cdot|a)=\frac{16}{3}$**,** $u_2(\cdot|b)=\frac{8}{3}$**.**