UNIVERSITY CARLOS III
Master in Industrial Economics and Markets
Game Theory **January 22, 2016**

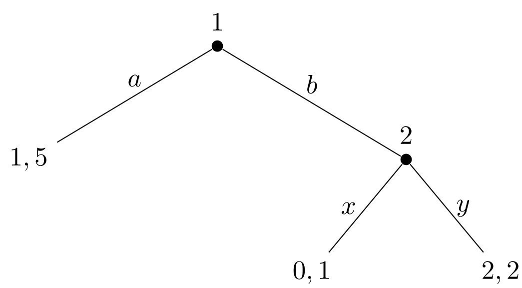
FINAL EXAM

INSTRUCTIONS (Please read)

- TIME: 2 hours
- There are 4 exercises. Each exercise is worth 2.5 points.
- Write your answers in this booklet.

NAME:

Problem 1: *Consider the following extensive form game.*



(1) *Write it as a normal form game.*

Solution:

	x	y
a	$1,5$	$1,5$
b	$0,1$	$2,2$

(2) *Find all the Nash equilibria in pure strategies of the above game.*

Solution: The NE are $\{(a,x),(b,y)\}$.

(3) *Find all the Nash equilibria in mixed strategies of the above game.*

Solution: Note first that $\mathrm{BR}_2(b) = y$. So, player 2 will not use a mixed strategy if he anticipates that player 1 will chose b. Thus, if there is a mixed strategy it must be of the form $(a, px + (1-p)y)$, with $0 \le p \le 1$. Suppose player 2 uses the mixed strategy $s_2 = px + (1-p)y$. The expected payoffs for Player 1 are,

$$\begin{aligned} u(a, s_2) &= 1 \\ u(b, s_2) &= p \times 0 + 2(1-p) = 2 - 2p \end{aligned}$$

Thus, player 1 will chose a if $2-2p \geq 1$, that is if $p \leq \frac{1}{2}$. Thus, the set of mixed strategies is

$$\{(a, px + (1-p)y) : 0 \leq p \leq \frac{1}{2}\}$$

(4) *Describe all the subgames of the above game.*

Solution: There are two subgames. The whole game and the game that starts at the node at which player 2 enters the game.

(5) *Find all the sub-game perfect Nash equilibria of the above game.*

Solution: Since $\mathrm{BR}_2(b) = y$ and $\mathrm{BR}_2(y) = b$, the only SPNE is (b, y).

Problem 2: *Let $P = 2000 - 10Q$ be the inverse demand function. Assume that in the market there are two firms: 1 and 2, whose costs are, respectively, $c(q_1) = 100q_1$ and $c(q_2) = 300q_2$.*

(1) *If the market is such that the firms have to decide how much to produce and the total production is sold at the price at which consumers are willing to clear the market (quantity or Cournot competition) how much are the two firms going to produce in equilibrium? What are firm 1's and firm 2's equilibrium profits, Π_1^* and Π_2^*?*

Solution: Both firm engage in a Cournot competition in quantities. Firm 1 solves the problem

$$\max_{q_1} \qquad (1900 - 10q_1 - 10q_2)q_1$$

Its best response is to choose

$$q_1 = \mathrm{BR}_1(q_2) = \frac{190 - q_2}{2}$$

Firm 2 solves the problem

$$\max_{q_2} \qquad (1700 - 10q_1 - 10q_2)q_2$$

Its best response is to choose

$$q_2 = \mathrm{BR}_2(q_1) = \frac{170 - q_1}{2}$$

The NE is the solution of the following system of equations

$$\begin{aligned} q_1 &= \frac{190 - q_2}{2} \\ q_2 &= \frac{170 - q_1}{2} \end{aligned}$$

That is

$$q_1 = 70, \quad q_2 = 50$$

The profits are

$$\Pi_1 = 49000, \quad \Pi_2 = 25000$$

(2) *The manager of firm 2 is considering the possibility of investing in a new technology that will lower the cost of firm to $c(q_2) = 100q_2$. What is the maximum amount that the manager would be willing to invest to acquire the new technology?*

Solution:
In the new situation, both firms have the same marginal cost. Their Best response is

$$q_i = \mathrm{BR}_i(q_j) = \frac{190 - q_j}{2}, \quad i \neq j \quad i, j = 1, 2$$

By Symmetry, the NE is the solution of

$$q = \frac{190 - q}{2}$$

We obtain

$$\bar{q}_1 = \bar{q}_2 = \frac{190}{3} \approx 63.33$$

The profits are

$$\bar{\Pi}_1 = \bar{\Pi}_2 = \frac{361000}{9} \approx 40111.11$$

The maximum amount that the manager would be willing to invest to acquire the new technology is

$$\bar{\Pi}_2 - \Pi_2 = \frac{361000}{9} - 25000 = \frac{136000}{9} \approx 15111.11$$

Problem 3: *Two telecom companies, T and R, are offering the same services in the fixed-phone sector. Each firm announces his tariff and the clients choose which one to contract.*

(1) *Assume both firms have marginal costs equal to c. What are equilibrium prices? Why?*

Solution: Both firms enter into a Bertrand competition. They set $p_T = p_R = c$.

(2) *What are equilibrium profits?*

Solution: Equilibrium profits are $\Pi_T = \Pi_R = 0$.

(3) *If the firm colluded they could announce the same tariff that firm T set before firm R entry in the market. Aggregate profits would be $1,000$ million. Firm T proposes in a secret meeting with R to share such profits in the following way 90% to T and the remaining 10% to R in each and every period, and in case of deviation a reversion to the equilibrium found in (a). Let r to be the discount rate: compute the expected value of the profits for each one of the two firms.* **Note:** *The discount factor is $\delta = \frac{1}{1+r}$.*

Solution: We use millions as our unit of account. In the agreement, Firm's T payoff is 900 and Firm's R payoff is 100 in each period. Thus, the payoffs are

$$\begin{aligned} u_T &= 900 + 900\delta + 900\delta^2 + \cdots + +900\delta^t + \cdots = \frac{900}{1-\delta} = 900\left(1 + \frac{1}{r}\right) \\ u_R &= 100 + 100\delta + 100\delta^2 + \cdots + +100\delta^t + \cdots = \frac{100}{1-\delta} = 100\left(1 + \frac{1}{r}\right) \end{aligned}$$

(4) *If a firm deviates and it sets a tariff which is only a bit lower than the agreed one it earns* 950 *million and the other firm earns* 0*. What is the present value of the profits of each firm if it is the only one to deviate?*

Solution: If firm T deviates in period t, its profits are

$$\begin{array}{rl} 900 & \text{in periods } 1, \dots, t-1, \\ 950 & \text{in period } t, \\ 0 & \text{in periods } t+1, t+2, \dots \end{array}$$

So, its payoff is

$$\bar{u}_T = 900 + 900\delta + 900\delta^2 + \cdots + +900\delta^{t-1} + 950\delta^t = 900\frac{1-\delta^t}{1-\delta} + 950\delta^t$$

Similarly, if firm R deviates in period t, its payoff is

$$\bar{u}_R = 100\frac{1-\delta^t}{1-\delta} + 950\delta^t$$

(5) *If the discount rate is $r = 20\%$ is the strategy suggested by firm T an equilibrium? For which values of the discount rate r is the strategy suggested by firm T an equilibrium?*

Solution: The strategy suggested by firm T is an equilibrium if $u_T \geq \bar{u}_T$ and $u_R \geq \bar{u}_R$. In particular for firm R this implies that

$$\frac{100}{1-\delta} \geq 100\frac{1-\delta^t}{1-\delta} + 950\delta^t$$

This is the same as

$$\frac{100\delta^t}{1-\delta} \geq 950\delta^t$$

or

$$1 + \frac{1}{r} = \frac{1}{1-\delta} \geq \frac{95}{10}$$

That is

$$r \leq \frac{10}{85} \approx 11.7\%$$

So, the strategy suggested by firm T is not an equilibrium.

Problem 4: *Consider the situation in which player 1 knows what game is played (A or B below). However, player 2 only knows that A is played with probability $\frac{1}{2}$ and B is played with probability $\frac{1}{2}$.*

Game A:

		Player 2	
		S	B
Player 1	S	6,4	0,0
	B	0, 0	4,6

A

Game B:

		Player 2	
		S	B
Player 1	S	6,0	0,6
	B	0, 4	4,0

B

(1) *Describe the situation as a Bayesian game.*

Solution: The set of players is $N = \{1, 2\}$. The set of strategies is $S_1 = \{SS, SB, BS, BB\}$, $S_2 = \{S, B\}$. The types are $T_1 = \{A, B\}$, $T_2 = \{a\}$. The beliefs are

$$\begin{aligned} p_1(a|A) &= p_1(a|B) = 1 \\ p_2(A|a) &= p_2(B|a) = \frac{1}{2} \end{aligned}$$

(2) *Find the Bayesian equilibria in pure strategies.*

Solution: The table of expected payoffs is

	S	B
SS	$6, 2$	$0, 3$
SB	$3, 4$	$2, 0$
BS	$3, 0$	$2, 6$
BB	$0, 2$	$4, 3$

Note that $\text{BR}_1(S|A) = SS$, $\text{BR}_1(S|B) = BB$ and $\text{BR}_{SS} = B$, $\text{BR}_{BB} = B$. Hence, the unique BNE is (BB, B) The payoffs are $(4, 3)$.

(3) *Find the Bayesian equilibria in mixed strategies.*

Solution: Let us look for a mixed strategy of the form

$$((xS + (1-x)B, zS + (1-z)B), yS + (1-y)B)$$

Note that

$$\begin{aligned} u_1(S, yS + (1-y)B|A) &= 6y \\ u_1(B, yS + (1-y)B|A) &= 4 - 4y \end{aligned}$$

If player 1 type A is indifferent between S and B we must have that $6y = 4 - 4y$, that is $y = \frac{2}{5}$. Note now that

$$\begin{aligned} u_1(S, \frac{2}{5}S + \frac{3}{5}B|B) &= \frac{12}{5} \\ u_1(B, \frac{2}{5}S + \frac{3}{5}B|B) &= \frac{12}{5} \end{aligned}$$

so, player 1 type B is also indifferent between S and B. Now player 2 must be indifferent between his two strategies. Since,

$$\begin{aligned} u_2((xS + (1-x)B, zS + (1-z)B), S) &= \frac{1}{2}4x + \frac{1}{2}4(1-z) = 2 + 2x - 2z \\ u_2((xS + (1-x)B, zS + (1-z)B), T) &= \frac{1}{2}6(1-x) + \frac{1}{2}6z = 3 - 3x + 3z \end{aligned}$$

we must have that $2 + 2x - 2z = 3 - 3x + 3z$, that is $z = x - \frac{1}{5}$. The BNE in mixed strategies are

$$\left\{ \left(\left(xS + (1-x)B, \left(x - \frac{1}{5} \right) S + \left(\frac{6}{5} - x \right) B \right), \frac{2}{5}S + \frac{3}{5}B \right) : \frac{1}{5} \leq x \leq 1 \right\}$$