Name (Print): ______________________________

University Carlos III de Madrid

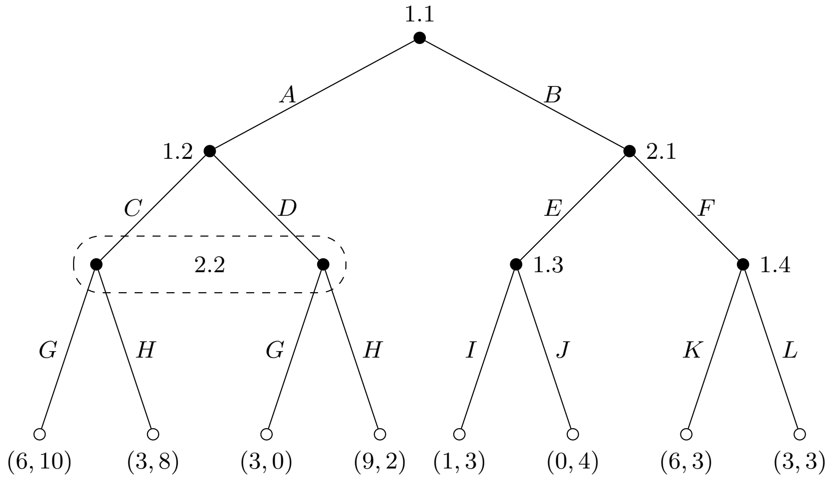
Master in Economics
Master in Industrial Economics and Markets

Final Exam. Game Theory. 01/13/2023.

Time Limit: 120 Minutes.

Exercise	Points	Score
1	35	
2	45	
3	35	
4	40	
Total:	155	

1. Consider the following game in extensive form



   (a) (5 points) What are the sub-games of the above game? It is enough to write the node at which each sub-game starts.
   Solution: *There are five sub-games that start at the nodes 1.1, 1.2 2.1, 1.3 and 1.4.*

   (b) (10 points) Write the normal form of the sub-game that starts at at node 1.2. Find the Nash equilibria (in pure and mixed strategies) of this sub-game.
   Solution: *The normal form of the sub-game that starts at at node* 1.2 *is,*

	G	H
C	$6, 10$	$3, 8$
D	$3, 0$	$9, 2$

There are two NE in pure strategies: (C, G) *with payoffs* $(6, 10)$ *and* (D, H) *with payoffs* $(9, 2)$. *In addition, there is mixed strategy NE*

$$\left(\left(\frac{1}{2}C+\frac{1}{2}D\right),\left(\frac{2}{3}G+\frac{1}{3}H\right)\right)$$

with payoffs

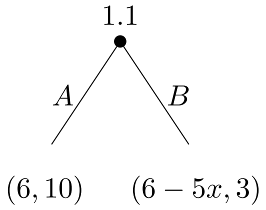
$$u_1 = u_2 = 5$$

(c) (20 points) Write the subgame perfect Nash equilibria of the whole game

Solution: *The SPNE of the sub-game that starts at at node* 2.1 *are of the form* $\sigma_2 = xE + (1-x)F$, $0 \le x \le 1$ *with payoffs* $u_1 = x + 6(1-x) = 6 - 5x$ *and* $u_2 = 3$. *We use the notation* $(1.2, 2.1, 1.3, 1.4)$. *All the SPNE are of the form*

$$(*, xE + (1-x)F, I, K)$$

a. Let us look for SPN in which in the subgame that starts at 1.2 *the NE* (C, G) *is played.*



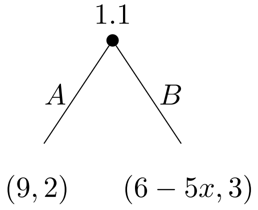
We obtain the SPNE

$$(A, xE + (1-x)F, I, K)\,, 0 < x \le 1, \quad u_1 = 6, u_2 = 10$$

and

$$(yA + (1-y)B, F, I, K)\,, 0 \le y \le 1, \quad u_1 = 6, u_2 = 10 + 3(1-y)$$

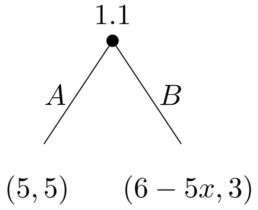
b. Let us look for SPN in which in the subgame that starts at 1.2 *the NE* (D, H) *is played.*



We obtain the SPNE

$$(A, xE + (1-x)F, I, K)\,, 0 \le x \le 1 \quad u_1 = 9, u_2 = 2$$

c. Let us look for SPN in which in the subgame that starts at 1.2 *the NE* $\left(\left(\frac{1}{2}C + \frac{1}{2}D\right), \left(\frac{2}{3}G + \frac{1}{3}H\right)\right)$ *is played.*



We obtain the SPNE

$$(A, xE + (1-x)F, I, K)\,, \frac{6}{25} < x < 1 \quad u_1 = 5, u_2 = 5$$

$$(B, xE + (1-x)F, I, K), 0 \le x < \frac{6}{25} \quad u_1 = 6 - 5x, u_2 = 3$$

and

$$\left(yA + (1-y)B, \frac{6}{25}E + \frac{19}{25}F, I, K\right), 0 \le y \le 1, \quad u_1 = 5, u_2 = 5y + 3(1-y)$$

2. Two firms produce an externality on each other, that is the production of firm $i = 1, 2$ affects negatively the profits of firm $j \neq i$. Assume that, if each day, firm 1 produces q_1 and firm 2 produces q_2, the profits of the firms are

$$\begin{aligned} u_1 &= (24 - q_2)q_1 - q_1^2 \\ u_2 &= (24 - q_1)q_2 - q_2^2 \end{aligned}$$

Both firms decide simultaneous and independently the quantities q_1 and q_2.

(a) (10 points) Compute the best reply of each firm if they interact only one day. Compute the NE (q_1^*, q_2^*) and the profits of the firms in the NE.

Solution: *The best reply functions of the firms are*

$$\mathrm{BR}_1(q_2) = \max\{0, \frac{24 - q_2}{2}\}, \quad \mathrm{BR}_2(q_1) = \max\{0, \frac{24 - q_1}{2}\}$$

The NE is the solution to $q_1 = \mathrm{BR}_1(q_2)$, $q_2 = \mathrm{BR}_2(q_1)$. *That is,*

$$q_1 = \max\{0, \frac{24 - q_2}{2}\}, \quad q_2 = \max\{0, \frac{24 - q_1}{2}\}$$

We obtain $q_1 = q_2 = 8$. *The profits are* $u_1 = u_2 = 64$.

(b) (10 points) Suppose firms interact only one day and can credibly agree on a production plan $(\bar{q}_1, \bar{q}_2)$ that maximizes the joint profit, with $\bar{q}_1 = \bar{q}_2$. What would that agreement be. What would be the profits achieved by the firms?

Solution: *The companies maximize*

$$\max_{q_1, q_2} u_1 + u_2$$

The first order conditions are

$$-2q_1 - 2q_2 + 24 = 0, \quad -2q_1 - 2q_2 + 24 = 0$$

and we obtain $q_2 = 12 - q_1$. *Assuming* $q_1 = q_2$, *we obtain* $\bar{q}_1 = \bar{q}_2 = 6$. *The profits would be* $\bar{u}_1 = \bar{u}_2 = 72$.

(c) (5 points) Suppose firms cannot commit to a production plan. Why the result of the previous part is not reasonable?

Solution: *Because* $\mathrm{BR}_i(6) = 9 \neq 6$. *Note also that* $u_1(9, 6) = u_2(6, 9) = 81 > 72$.

(d) (20 points) Suppose now that firms interact an infinite number of days and that each day each firm decides simultaneous and independently a new production plan for the day. Is there a SPNE in which, in the equilibrium path, the production plan $(\bar{q}_1, \bar{q}_2)$ is carried out every day.

Solution: *Consider the trigger strategy:*

- *At* $t = 1$ *play* $(6, 6)$.
- *If* $t > 1$ *play* $(6, 6)$ *if* $(6, 6)$ *was played at previous periods. Play* $(8, 8)$ *otherwise.*

We show that the trigger strategy is a NE of the whole game. With the trigger strategy the payoffs of the players are

$$u_t == 72 + 72\delta + 72\delta^2 + \cdots + 72\delta^k = \frac{72}{1-\delta}$$

If one player deviates at period 1, her payoff would be at most

$$u_t = 81 + 64\delta + 64\delta^2 + \cdots + 64\delta^k = 81 + \frac{64\delta}{1-\delta}$$

So, the trigger strategy is a NE of the whole game iff

$$\frac{72}{1-\delta} \geq 81 + \frac{64\delta}{1-\delta}$$

that is if $\delta \geq \frac{9}{17}$. *Now the standard argument shows that for* $\delta \geq \frac{9}{17}$, *the trigger strategy is also a NE of every sub-game.*

3. Consider the situation in which player 2 knows which game is played (a or b below). However, player 1 only knows that table a is played with probability $\frac{1}{2}$ and table b is played with probability $\frac{1}{2}$.

		Player 2	
		C	D
Player 1	A	3 ,7	1, 1
	B	2, 1	4,3

a

		Player 2	
		C	D
Player 1	A	5 ,1	1, 3
	B	4, 5	2, 1

b

(a) (5 points) Describe the situation as a Bayesian game.

Solution: *There are two players* $N = \{1, 2\}$. *There are two types of player 2:* $T_2 = \{a, b\}$. *There is one type of player 1:* $T_1 = \{t\}$. *The sets of strategies are* $S_2 = \{CC, CD, DC, DD\}$, $S_1 = \{A, B\}$. *The beliefs of the players are*

$$\begin{aligned} p_2\left(t_1 = t | t_2 = a\right) &= p_2\left(t_1 = t | t_2 = b\right) = 1 \\ p_1\left(t_2 = a | t_1 = t\right) &= p_1\left(t_2 = c_{|} t_1 = t\right) = 1/2 \end{aligned}$$

The payoffs are given by the above tables.

(b) (10 points) Find the Bayesian–Nash equilibria in pure strategies and the payoffs of the players.

Solution: *The associated normal form game is*

	CC	CD	DC	DD
A	$(4,4)$	$(2,5)$	$(3,1)$	$(1,2)$
B	$(3,3)$	$(2,1)$	$(4,4)$	$(3,2)$

and we see that there are two BNE in pure strategies (A, CD) *with payoffs* $(2,5)$ *and* (B, DC) *with payoffs* $(4,4)$.

(c) (20 points) Find the Bayesian–Nash equilibria in mixed strategies and the payoffs of the players.

Solution: *Let us look for a BNE of the form*

$$(xA + (1-x)B, (yC + (1-y)D, zC + (1-z)D))$$

Let

$$\begin{aligned} s_1 &= xA + (1-x)B \\ s_a &= yC + (1-y)D \\ s_b &= zC + (1-z)D \end{aligned}$$

We have that

$$\begin{aligned} u_1(A; s_a, s_b) &= \frac{1}{2}(3y + 1 - y) + \frac{1}{2}(5z + 1 - z) = 1 + y + 2z \\ u_1(B; s_a, s_b) &= \frac{1}{2}(2y + 4(1-y)) + \frac{1}{2}(4z + 2(1-z)) = 3 - y + z \\ u_a(s_1, C) &= 7x + 1 - x = 1 + 6x \\ u_a(s_1, D) &= x + 3(1-x) = 3 - 2x \\ u_b(s_1, C) &= x + 5(1-x) = 5 - 4x \\ u_b(s_1, D) &= 3x + 1 - x = 1 + 2x \end{aligned}$$

Suppose first that player 2a is using a completely mixed strategy. Then $u_a(s_1, C) = u_a(s_1, D)$. *Hence,* $1 + 6x = 3 - 2x$ *and we conclude that* $x = \frac{1}{4}$. *For this value of* x *we have that* $u_b(s_1, C) = (5 - 4x)|_{x=\frac{1}{4}} = 4$ *and* $u_b(s_1, D) = (1 + 2x)|_{x=\frac{1}{4}} = \frac{3}{2}$, *so* $z = 1$. *We check if there is a BNE of the form*

$$\left(\frac{1}{4}A + \frac{3}{4}B; (yC + (1-y)D, C)\right)$$

Player 1 must be indifferent between A *and* B. *Hence,* $1 + y + 2z = 3 - y + z$. *Since* $z = 1$, *we obtain that* $y = \frac{1}{2}$. **And we have checked that**

$$\left(\frac{1}{4}A + \frac{3}{4}B; \left(\frac{1}{2}C + \frac{1}{2}D, C\right)\right)$$

is BNE in mixed strategies with payoffs $u_1 = \frac{7}{2}$, $u_a = \frac{5}{2}$, $u_b = 4$.

Suppose now that player 2b is using a completely mixed strategy. Then $u_b(s_1, C) = u_b(s_1, D)$. *Hence,* $5 - 4x = 1 + 2x$ *and we conclude that* $x = \frac{2}{3}$. *For this value of* x *we have that* $u_a(s_1, C) = (1 + 6x)|_{x=\frac{2}{3}} = 5$ *and* $u_a(s_1, D) = (3 - 2x)|_{x=\frac{2}{3}} = \frac{5}{3}$, *so* $y = 1$.

We check if there is a BNE of the form

$$\left(\frac{2}{3}A + \frac{1}{3}B; (C, zC + (1-z)D)\right)$$

Player 1 must be indifferent between A and B. Hence, $1 + y + 2z = 3 - y + z$. *Since* $y = 1$, *we obtain that* $z = 0$. **And we have checked that**

$$\left(\frac{2}{3}A + \frac{1}{3}B; (C, D)\right)$$

is the other BNE in mixed strategies with payoffs $u_1 = 2$**,** $u_a = \frac{5}{3}$**,** $u_b = \frac{7}{3}$**.**

4. Consider a market with one good and two firms. The firms decide prices p_1 and p_2 simultaneous and independently. Given those prices, the amount sold by each company is

$$\begin{aligned} x_1(p_1, p_2) &= 54 - p_1 + \frac{p_2}{2} \\ x_2(p_1, p_2) &= 54 - p_2 + \frac{p_1}{2} \end{aligned}$$

Firm 2 has constant marginal cost $c_2 = 6$. Firm 2 does not know the cost of firm 1. Firm 2 thinks that with probability $\frac{1}{2}$ firm 1 has constant marginal cost $c_l = 4$ and with probability $\frac{1}{2}$ firm 1 has constant marginal cost $c_h = 8$. Firm 1 knows its costs and the costs of firm 2. This situation is common knowledge for both firms.

(a) (10 points) Write the payoffs of the firms.

Solution: *There are two players* $N = \{1, 2\}$. *There are two types of player 2:* $T_2 = \{c_l, c_h\}$, *where* $c_l = 4$ *and* $c_h = 8$. *There is one type of player 1:* $T_1 = \{c\}$. *The sets of strategies are* $S_2 = \{CC, CD, DC, DD\}$, $S_1 = \{A, B\}$. *The beliefs of the players are*

$$\begin{aligned} p_2\left(t_1 = c | t_2 = c_l\right) &= p_2\left(t_1 = c | t_2 = c_l\right) = 1 \\ p_1\left(t_2 = c_l | t_1 = t\right) &= p_1\big(t_2 = c_{|}t_1 = t\big) = 1/2 \end{aligned}$$

The payoffs are

$$\begin{aligned} u_h\left(p_h, p_1\right) &= (54 - p_h + \frac{p_1}{1})(p_h - c_h) \\ u_l\left(p_l, p_1\right) &= (54 - 1p_l + \frac{p_1}{1})(p_l - c_l) \\ u_1 &= (p_1 - 6)\left(\frac{1}{2}\left(54 - p_1 + \frac{p_h}{1}\right) + \frac{1}{2}\left(54 - p_1 + \frac{p_l}{1}\right)\right) \end{aligned}$$

(b) (20 points) Compute the best reply of each firm. You must compute the best reply of each type of the firms.

Solution: *Agent 1, type* c_h*, maximizes* $\max_{p_h} u_h = (54 - p_h + \frac{p_2}{2})(p_h - c_h)$*. The first order condition is*

$$\frac{p_2}{2} - 2p_h + 62 = 0.$$

Note that the second derivative with respect to p_h *is*

$$\frac{\partial^2 u_h}{\partial p_h^2} = -2 < 0$$

Hence, the first order condition corresponds to a maximum of u_h*. The best reply of agent 1, type* c_h*, is*

$$\mathrm{BR}_h(p_2) = \frac{p_2 + 124}{4}$$

Likewise, agent 1, type c_l*, maximizes* $\max_{p_l} u_l = (p_l - 4)\left(\frac{p_2}{2} - p_l + 54\right)$*. The first order condition is*

$$\frac{p_2}{2} - 2p_l + 58 = 0.$$

Note that the second derivative with respect to p_l *is*

$$\frac{\partial^2 u_l}{\partial p_l^2} = -2 < 0$$

Hence, the first order condition corresponds to a maximum of u_l*. The best reply of agent 1, type* c_l*, is*

$$\mathrm{BR}_l(p_2) = \frac{p_2 + 116}{4}$$

Finally, agent 2 maximizes

$$\max_{p_2}(p_2 - 6)\left(\frac{1}{2}\left(54 - p_2 + \frac{p_h}{2}\right) + \frac{1}{2}\left(54 - p_2 + \frac{p_l}{2}\right)\right)$$

The first order condition is

$$\frac{1}{2}\left(54 - p_2 + \frac{p_h}{2}\right) + \frac{1}{2}\left(54 - p_2 + \frac{p_l}{2}\right) - p_2 + 6 = 0$$

The best reply of agent 2 is

$$\mathrm{BR}_2(p_h, p_l) = \frac{1}{8}(p_h + p_l + 240)$$

(c) (10 points) Compute the Bayes–Nash equillibrium, the quantities sold in this equilibrium and the profits of each firm.

Solution: *The NE is the solution to*

$$p_h = \frac{p_2 + 124}{4} \quad p_l = \frac{p_2 + 116}{4} \quad p_2 = \frac{1}{8}(p_h + p_l + 240)$$

We obtain

$$p_h^* = 41 \quad p_l^* = 39 \quad p_2^* = 40$$

the utilities of the agents are

$$u_h^* = 1089 \quad u_l^* = 1225 \quad u_2^* = 1156$$

and the quantities sold are

$$x_h^* = 33 \quad x_l^* = 35 \quad x_2^* = 34$$