UNIVERSITY CARLOS III

Master in Economics **Master in Industrial Economics and Markets**

Game Theory

FINAL EXAM–January 19th, 2022

NAME:

Problem 1: Consider a Cournot duopoly with two firms which operate in a Market with one good and inverse demand function $P = 108 - Q$ where Q is the total output in the market. Firm 1 has a constant marginal cost $c_1 = 27$. Firm 2 has a constant marginal cost which can be either $c_l = 18$ (with probability 1/3) or $c_h = 36$ (with probability 2/3). Firm 2 knows its cost and the cost of firm 1. Firm 1 knows its cost but the only information it has about the cost of firm 2 is the above probability.

(a) **(5 points.)** Describe the situation as a Bayesian game

Solution: *There are two players $N = \{1, 2\}$, There is one type of player for player 1, $T_1 = \{c\}$ and two types for player 2 $T_2 = \{l, h\}$. The beliefs of the players are*

$$\begin{aligned} \mu_1(t_2 = l|t_1 = c) &= 1/3 \quad \mu_1(t_2 = h|t_1 = c) = 2/3 \\ \mu_1(t_1 = c|t_2 = l) &= \mu_1(t_1 = c|t_2 = h) = 1 \end{aligned}$$

The strategy sets are $S_1 = S_l = S_h = [0, +\infty)$. The utilities of the players are

$$\begin{aligned} u_1(q_1, q_l, q_h; c) &= \frac{1}{3}(108 - q_1 - q_l)\, q_1 + \frac{2}{3}(108 - q_1 - q_h)\, q_1 - 27q_1 = \\ u_l(q_1, q_l) &= u_2(q_1, q_l, q_l; l) = (90 - q_1 - q_l)\, q_l = 90q_l - q_1 q_l - q_l^2 \\ u_h(q_1, q_h) &= u_2(q_1, q_l, q_h; h) = (72 - q_1 - q_h)\, q_h = 72q_h - q_1 q_h - q_h^2 \end{aligned}$$

(b) **(10 points.)** Compute the best reply of each firm. You must compute the best reply of each type of the firms.

Solution: *Agent 2, type h, maximizes* $\max_{q_h} u_h = (72 - q_h - q_1)q_h$. *The first order condition is*

$$72 - 2q_h - q_1 = 0.$$

Note that the second derivative with respect to q_h is

$$\frac{\partial^2 u_h}{\partial q_h^2} = -2 < 0$$

Hence, the first order condition corresponds to a maximum of u_h. The best reply of agent 2, type h, is

$$\text{BR}_h(q_1) = \frac{72 - q_1}{2}$$

Likewise, agent 2, type l, maximizes $\max_{q_l} u_l = (90 - 2q_l h - q_1)q_l$. *The first order condition is*

$$90 - 2q_l - q_1 = 0.$$

Note that the second derivative with respect to q_l is

$$\frac{\partial^2 u_l}{\partial q_l^2} = -2 < 0$$

Hence, the first order condition corresponds to a maximum of u_l. The best reply of agent 2, type l, is

$$\text{BR}_l(q_1) = \frac{90 - q_1}{2}$$

Finally, agent 1 maximizes

$$\max_{q_1} \frac{1}{3}(108 - q_1 - q_l)\, q_1 + \frac{2}{3}(108 - q_1 - q_h)\, q_1 - 27q_1$$

The first order condition is

$$81 - 2q_1 - \frac{1}{3}q_1 - \frac{2}{3}q_h = 0$$

The best reply of agent 1 is

$$\text{BR}_1(q_h, q_l) = \frac{1}{6}(243 - 2q_h - q_l)$$

(c) **(10 points.)** Compute the Bayes–Nash equillibrium, the quantities sold in this equilibrium and the profits of each firm.

Solution: *The NE is the solution to*

$$q_h = \frac{72 - q_1}{2} \qquad q_l = \frac{90 - q_1}{2} \qquad q_1 = \frac{1}{6}(243 - 2q_h - q_l)$$

We obtain

$$q_h^* = 22 \quad q_l^* = 31 \quad q_1^* = 28$$

the utilities of the agents are

$$u_h^* = 484 \quad u_l^* = 961 \quad u_1^* = 784$$

(d) **(10 points.)** Suppose now that it is known that firm 2 has cost $c_l = 18$. What is the Cournot equilibrium and the profits of the firms? Which firm is better off and which one is worse off with respect to case of asymmetric information?

Solution: *The best reply of agent 2, type l, is still*

$$\text{BR}_l(q_1) = \frac{90 - q_1}{2}$$

But now agent 1 maximizes

$$\max_{q_1} (108 - q_1 - q_l)\, q_1 - 27q_1$$

The first order condition is

$$81 - 2q_1 - q_h = 0$$

The best reply of agent 1 is

$$\text{BR}_1(q_h, q_l) = \frac{81 - q_2}{2}$$

The NE is the solution to

$$q_l = \frac{90 - q_1}{2} \qquad q_1 = \frac{81 - q_2}{2}$$

We obtain

$$q_l^* = 33 \quad q_1^* = 24$$

the utilities of the agents are

$$u_l^* = 1089 \quad u_1^* = 576$$

Firm 2 type l is better off in comparison with the asymmetric situation. Firm 1 is worse off.

(e) **(10 points.)** Suppose now that it is known that firm 2 has cost $c_h = 36$. What is the Cournot equilibrium and the profits of the firms? Which firm is better off and which one is worse off with respect to case of asymmetric information?

Solution: *The best reply of agent 2, type h, is*

$$\mathrm{BR}_h(q_1) = \frac{72 - q_1}{2}$$

And now agent 1 maximizes

$$\max_{q_1} \left(108 - q_1 - q_h\right) q_1 - 27 q_1$$

The first order condition is

$$81 - 2q_1 - q_h = 0$$

The best reply of agent 1 is

$$\mathrm{BR}_1(q_h, q_h) = \frac{81 - q_2}{2}$$

The NE is the solution to

$$q_h = \frac{72 - q_1}{2} \quad q_1 = \frac{81 - q_2}{2}$$

We obtain

$$q_h^* = 21 \quad q_1^* = 30$$

the utilities of the agents are

$$u_h^* = 441 \quad u_1^* = 900$$

Firm 2 type l is worse off in comparison with the asymmetric situation. Firm 1 is better off.

(f) **(10 points.)** Suppose now that firm 1 does not know the cost of firm 2. But, firm 1 can hire a research institute that will publish the cost of firm 2. How much would firm 1 be willing to pay to make that information public?

Solution: *Without the information the profit of firm 1 is 784. If it is common knowledge that firm 2 is of type l, the profit of firm 1 would be 576. If it is common knowledge that firm 2 is of type h, the profit of firm 1 would be 900. Thus, the expected value of revealing the information is*

$$\frac{1}{3} \times 576 + \frac{2}{3} \times 900 = 792$$

Therefore, the maximum amount firm 1 would be willing to pay is $792 - 784 = 8$.

Problem 2: Consider the situation in which player 1 knows what is the game played (a or b below). But, player 2 only knows that a is played with probability 1/2 and b is played with probability 1/2.

Player 1 \ Player 2	X	Y
C	18, 8	0, 0
D	0, 0	12 , 12

(a, 1/2)

Player 1 \ Player 2	X	Y
C	18, 0	0, 12
D	0, 8	12 , 0

(b, 1/2)

(10 points.) Find the Bayesian equilibria in pure strategies and the payoffs of the players in those equilibria.
Solution: *Note that* $\mathrm{BR}_1(X) = CC$ *and*

$$\begin{aligned} u_2(CC, X) &= 4 \\ u_2(CC, Y) &= 6 \end{aligned}$$

Hence, $\mathrm{BR}_2(CC) = Y$*. We conclude that there is no pure strategy NE in which player 2 plays* X*. On the other hand,* $\mathrm{BR}_1(Y) = DD$ *and*

$$\begin{aligned} u_2(DD, X) &= 4 \\ u_2(DD, Y) &= 6 \end{aligned}$$

Hence, $\mathrm{BR}_2(DD) = Y$*. We conclude that* (DD, Y) *is a BNE in pure strategies. The payoffs are* $u_a(DD, Y) = 12$, $u_b(DD, Y) = 12$. $u_2(DD, Y) = 6$.

(20 points.) Find the Bayesian equilibria in mixed strategies and the payoffs of the players in those equilibria.
Solution: *Let us look for a BNE in mixed strategies of the form*

$$\begin{aligned} \sigma_a &= xC + (1-x)D, \quad 0 \le x \le 1 \\ \sigma_b &= yC + (1-y)D \quad 0 \le y \le 1 \\ \sigma_2 &= zX + (1-z)Y \quad 0 \le z \le 1 \end{aligned}$$

Suppose first that $0 < x < 1$*. Then* $u_a(C, \sigma_2) = u_a(D, \sigma_2)$*, Since,*

$$\begin{aligned} u_a(C, \sigma_2) &= 18z \\ u_a(D, \sigma_2) &= 12 - 12z \end{aligned}$$

we must have $z = 2/5$*. Given this value of* z*, we have*

$$\begin{aligned} u_b(C, \sigma_2) &= 18z = \frac{36}{5} \\ u_b(D, \sigma_2) &= 12 - 12z = \frac{36}{5} \end{aligned}$$

and player b *is also indifferent between* C *and* D *for* $z = 2/5$*. Now, for* $z = 2/5$*, player 2 must be indifferent between* X *and* Y*.*

$$\begin{aligned} u_2(\sigma_a, \sigma_b; X) &= \frac{1}{2}18x + \frac{1}{2}8(1-y) = 4 + 4x - 4y \\ u_2(\sigma_a, \sigma_b; Y) &= \frac{1}{2}18(1-x) + \frac{1}{2}12y = 6 - 6x + 6y \end{aligned}$$

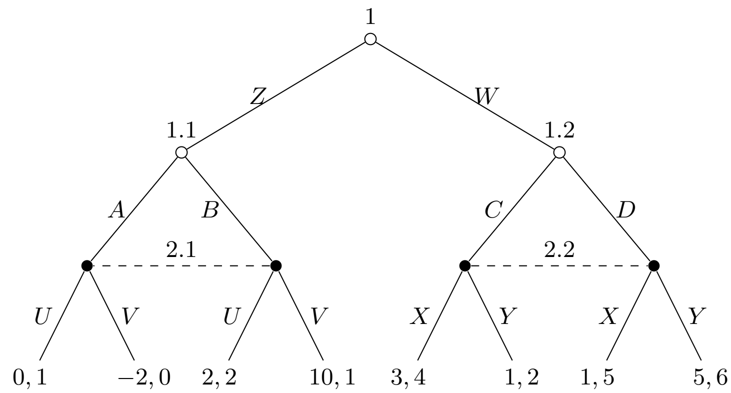
equating $4 + 4x - 4y = 6 - 6x + 6y$ *we obtain* $5x - 5y = 1$*. Since,* $x, y \ge 0$ *we obtain a family of BNE*

$$\left\{ \left(\left(xC + (1-x)D, \left(x - \frac{1}{5}\right) C + \left(\frac{6}{5} - x\right) D \right), \frac{2}{5}X + \frac{3}{5}Y \right) : \frac{1}{5} \le x \le 1 \right\}$$

with payoffs

$$u_a = u_b = \frac{36}{5}, \quad u_2 = \frac{24}{5},$$

Problem 3:
Consider the following game in extensive form



1. **(5 points.)** What are the sub-games of the above game? It is enough to write the node at which each sub-game starts.

   Solution: *There are three sub-games: The whole game and the sub-games that start at the nodes 1.1 and 1.2.*

2. **(5 points.)** Write the normal form of the sub-game that starts at at node 1.1. Find the Nash equilibria in pure strategies of this sub-game.

   Solution: *The normal form of the sub-game that starts at at node* 1.1 *is,*

	U	V
A	$0,1$	$-2,0$
B	$2,2$	$10,1$

   There is a unique NE in pure strategies: (B,U) *with payoffs* $(2,2)$.

3. **(5 points.)** Write the normal form of the sub-game that starts at at node 1.2. Find the Nash equilibria in pure strategies of this sub-game.

   Solution: *The normal form of the sub-game that starts at at node* 1.2 *is,*

	X	Y
C	$3,4$	$1,2$
D	$1,5$	$5,6$

   There are two NE in pure strategies: (C,X) *with payoffs* $(3,4)$ *and* (D,Y) *with payoffs* $(5,6)$.

4. **(10 points.)** Find the pure strategy sub-game perfect Nash equilibria of the complete game.

   Solution: *Let us write an strategy as* $(1,(1.1,2.1),(1.2,2.2))$. *At node* 1, *player 1 chooses* Z *and* W *anticipating the subsequent NE which gives him the highest payoff. Thus, the sub-game perfect Nash equilibria are:*

   (a) $(W,(B,U),(C,X))$, *with payoffs* $(3,4)$.

   (b) $(W,(B,U),(D,Y)))$, *with payoffs* $(5,6)$.

Problem 4: Consider the following stage game

	C	D
C	$10, 10$	$0, 15$
D	$15, 0$	$5, 5$

(a) **(5 points.)** Find all the NE of the above game.

Solution: *The strategy C is strictly dominated for player 1. Once we eliminate this stragegy, C is strictly dominated for player 2. Hence, the unique NE is (D, D) with payoff $u_1 = u_2 = 5$.*

Consider now the repeated game which consists in playing the above stage game infinitely many times with discount factor δ.

(b) **(5 points.)** Can you find a subgame perfect Nash Equilibrium for every $0 < \delta < 1$?

Solution: *The strategy (D, D) is a NE of the stage game. Thus, playing (D, D) in every period is a SPNE.*

(c) **(10 points.)** Describe the trigger strategy.

(d) **(15 points.)** For what values of δ does the trigger strategy constitute a subgame perfect Nash equilibrium? You must justify why the trigger strategy is a SPNE for those values of δ.

Solution:

If players follow the trigger strategy their stream of payoffs is

$$10 + 10\delta + 10\delta^2 + \cdots = \frac{10}{1-\delta}$$

If one player deviates at the first period his stream of payoffs will be

$$15 + 5\delta + 5\delta^2 + \cdots = 15 + \frac{5\delta}{1-\delta}$$

The trigger strategy is a NE of the repeated game iff

$$\frac{10}{1-\delta} \geq 15 + \frac{5\delta}{1-\delta}$$

that is iff $\delta \geq \frac{1}{2}$. The standard argument shows it is a SPNE.

(d) Suppose now that $\delta = 3/4$. Let k pe a positive integer. Consider the following 'soft' trigger stragegy:

- At $t = 1$ play (C, C).
- If $2 \leq t \leq k$ play (C, C) if (C, C) was played at $t - 1, t - 2, \ldots, 1$. Play (D, D) otherwise.
- If $t > k$ play (C, C) if (C, C) was played at $t - 1, t - 2, \ldots, t - k$. Play (D, D) otherwise.

That is, if one player deviates from (C, C) he is punished for k periods with (D, D) and then we go back to (C, C). What is the smallest value of k for which the above strategy is a sub-game perfect Nash equilibrium?

(15 points.) **Solution:** *If players follow the 'soft' trigger strategy their stream of payoffs is*

$$u_t = 10 + 10\delta + 10\delta^2 + \cdots = \frac{10}{1-\delta}$$

If one player deviates at the first period his stream of payoffs will be

$$u_d = 15 + 5\delta + 5\delta^2 + \cdots + 5\delta^k + 10\delta^{k+1} + 10\delta^{k+2} + \cdots$$

The 'soft' trigger strategy is a NE of the repeated game iff

$$u_t - u_d = -5 + 5\delta + 5\delta^2 + \cdots + 5\delta^k \geq 0$$

Let

$$x = 5\delta + 5\delta^2 + \cdots + 5\delta^k$$

Then

$$\delta x = 5\delta^2 + 5\delta^3 + \cdots + 5\delta^{k+1}$$

So,

$$(1-\delta)x = 5\delta - 5\delta^{k+1}$$

Hence,

$$x = \frac{5\delta - 5\delta^{k+1}}{1-\delta}$$

That is,

$$5\delta + 5\delta^2 + \cdots + 5\delta^k = 5\frac{\delta - \delta^{k+1}}{1-\delta}$$

Therefore, the 'soft' trigger strategy is a NE of the repeated game iff

$$\frac{\delta - \delta^{k+1}}{1-\delta} \geq 1$$

That is, if

$$\delta - \delta^{k+1} \geq 1 - \delta$$

or

$$2\delta \geq 1 + \delta^{k+1}$$

with $\delta = \frac{3}{4}$ *the above inequality becomes*

$$1 + \left(\frac{3}{4}\right)^{k+1} \leq \frac{3}{2}$$

For $k = 1$ *we have*

$$1 + \left(\frac{3}{4}\right)^2 = \frac{25}{16} > \frac{3}{2}$$

For $k = 2$ *we have*

$$1 + \left(\frac{3}{4}\right)^3 = \frac{91}{64} < \frac{3}{2}$$

So, it is enough to take $k = 2$.