UNIVERSITY CARLOS III

Master in Economics **Master in Industrial Economics and Markets**

Game Theory

FINAL EXAM–January 20th, 2021

NAME:

Exercise 1:
Consider A market with one good and two firms. The firms decide prices p_1 and p_2 simultaneous and independently. Given those prices, the amount sold By each company is

$$\begin{aligned} x_1(p_1, p_2) &= 54 - p_1 + \frac{p_2}{2} \\ x_2(p_1, p_2) &= 54 - p_2 + \frac{p_1}{2} \end{aligned}$$

Firm 2 has constant marginal cost $c_2 = 6$. Firm 2 does not know the cost of firm 1. Firm 2 thinks that with probability $\frac{1}{2}$ firm 1 has constant marginal cost $c_l = 4$ and with probability $\frac{1}{2}$ firm 1 has constant marginal cost $c_h = 8$. Firm 1 knows its costs and the costs of firm 2. This situation is common knowledge for Both firms.

(a) **(5 points.)** Describe the situation as A Bayesian game

Solution: *There are two players $N = \{1, 2\}$. There are two types of player 1: $T_1 = \{c_l, c_h\}$, where $c_l = 4$ and $c_h = 8$. There is one type of player 2: $T_2 = \{c\}$. The sets of strategies are $S_1 = S_2 = S_{c_l} = S_{c_h} = [0, \infty)$. The beliefs of the players are*

$$\begin{aligned} p_1\left(t_2 = c | t_1 = c_l\right) &= p_1\left(t_2 = c | t_1 = c_l\right) = 1 \\ p_2\left(t_1 = c_l | t_2 = t\right) &= p_2\left(t_1 = c_| t_2 = t\right) = 1/2 \end{aligned}$$

The payoffs are

$$\begin{aligned} u_h\left(p_h, p_2\right) &= (54 - p_h + \frac{p_2}{2})(p_h - c_h) \\ u_l\left(p_l, p_2\right) &= (54 - 2p_l + \frac{p_2}{2})(p_l - c_l) \\ u_2 &= (p_2 - 6)\left(\frac{1}{2}\left(54 - p_2 + \frac{p_h}{2}\right) + \frac{1}{2}\left(54 - p_2 + \frac{p_l}{2}\right)\right) \end{aligned}$$

(b) **(10 points.)** Compute the Best reply of each firm. You must compute the Best reply of each type of the firms.

Solution: *Agent 1, type c_h, maximizes $\max_{p_h} u_h = (54 - p_h + \frac{p_2}{2})(p_h - c_h)$. The first order condition is*

$$\frac{p_2}{2} - 2p_h + 62 = 0.$$

Note that the second derivative with respect to p_h is

$$\frac{\partial^2 u_h}{\partial p_h^2} = -2 < 0$$

Hence, the first order condition corresponds to a maximum of u_h. The best reply of agent 1, type c_h, is

$$\mathrm{BR}_h(p_2) = \frac{p_2 + 124}{4}$$

Likewise, agent 1, type c_l, maximizes $\max_{p_l} u_l = (p_l - 4)\left(\frac{p_2}{2} - p_l + 54\right)$. *The first order condition is*

$$\frac{p_2}{2} - 2p_l + 58 = 0.$$

Note that the second derivative with respect to p_l is

$$\frac{\partial^2 u_l}{\partial p_l^2} = -2 < 0$$

Hence, the first order condition corresponds to a maximum of u_l. The best reply of agent 1, type c_l, is

$$\mathrm{BR}_l(p_2) = \frac{p_2 + 116}{4}$$

Finally, agent 2 maximizes

$$\max_{p_2}(p_2 - 6)\left(\frac{1}{2}\left(54 - p_2 + \frac{p_h}{2}\right) + \frac{1}{2}\left(54 - p_2 + \frac{p_l}{2}\right)\right)$$

The first order condition is

$$\frac{1}{2}\left(54 - p_2 + \frac{p_h}{2}\right) + \frac{1}{2}\left(54 - p_2 + \frac{p_l}{2}\right) - p_2 + 6 = 0$$

The best reply of agent 2 is

$$\mathrm{BR}_2(p_h, p_l) = \frac{1}{8}(p_h + p_l + 240)$$

(c) **(10 points.)** Compute the Bayes–Nash equillibrium, the quantities sold in this equilibrium and the profits of each firm.

Solution: *The NE is the solution to*

$$p_h = \frac{p_2 + 124}{4} \quad p_l = \frac{p_2 + 116}{4} \quad p_2 = \frac{1}{8}(p_h + p_l + 240)$$

We obtain

$$p_h^* = 41 \quad p_l^* = 39 \quad p_2^* = 40$$

the utilities of the agents are

$$u_h^* = 1089 \quad u_l^* = 1225 \quad u_2^* = 1156$$

and the quantities sold are

$$x_h^* = 33 \quad x_l^* = 35 \quad x_2^* = 34$$

Exercise 2:
Two players face the following Bayesian game. Player 1 knows that game S is played with probability $\frac{3}{4}$ and game T is played with probability $\frac{1}{4}$. Player 2 is informed of the game that is played.

Game $(S, \frac{3}{4})$:

Player 1 \ Player 2	X	Y
A	2, 4	2, 3
B	4, 2	3 , 6

Game $(T, \frac{1}{4})$:

Player 1 \ Player 2	X	Y
A	4, 3	2, 4
B	3, 2	-2, 4

(15 points.) Find the Bayesian equilibria in pure strategies and the payoffs of the players in those equilibria.
Solution: *Note that* $\mathrm{BR}_2(A) = XY$ *and*

$$\begin{aligned} u_1(A, XY) &= \frac{3}{4} \times 2 + \frac{1}{4} \times 2 = 2 \\ u_1(B, XY) &= \frac{3}{4} \times 4 + \frac{1}{4} \times (-2) = \frac{10}{4} > 2 \end{aligned}$$

Hence $\mathrm{BR}_1(XY) = B$. *And there is no BNE in which player 2 plays* XY *On the other hand,* $\mathrm{BR}_2(B) = YY$ *and*

$$\begin{aligned} u_1(A, YY) &= \frac{3}{4} \times 2 + \frac{1}{4} \times 2 = 2 \\ u_1(B, YY) &= \frac{3}{4} \times 3 + \frac{1}{4} \times (-2) = \frac{7}{4} < 2 \end{aligned}$$

Hence $\mathrm{BR}_1(YY) = A$. *We conclude that here is no BNE in which player 2 plays* YY. *Hence, there are no BNE in pure strategies.*

(20 points.) Find the Bayesian equilibria in mixed strategies and the payoffs of the players in those equilibria.
Solution: *Note that, for player 2, type* T, X *is dominated by strategy* Y. *So, in no NE will this player use strategy* X. *Let us look for a BNE in mixed strategies of the form*

$$\begin{aligned} \sigma_T &= Y \\ \sigma_S &= xX + (1-x)Y, \quad 0 < x < 1 \\ \sigma_1 &= pA + (1-p)B \quad 0 < p < 1 \end{aligned}$$

The expected utilities of player S *are*

$$\begin{aligned} u_S(\sigma_1, X) &= 4p + 2(1-p) = 2 + 2p \\ u_S(\sigma_1, Y) &= 3p + 6(1-p) = 6 - 3p \end{aligned}$$

We must have that $2 + 2p = 6 - 3p$. *That is,* $p = \frac{4}{5}$. *We also have that,*

$$u_1(A, \sigma_2, Y) = \frac{3}{4}(2x + 2(1-x)) + \frac{1}{4}2 = 2$$

$$u_1(B, \sigma_2, Y) = \frac{3}{4}(4x + 3(1-x)) + \frac{1}{4} \times (-2) = \frac{7+3x}{4}$$

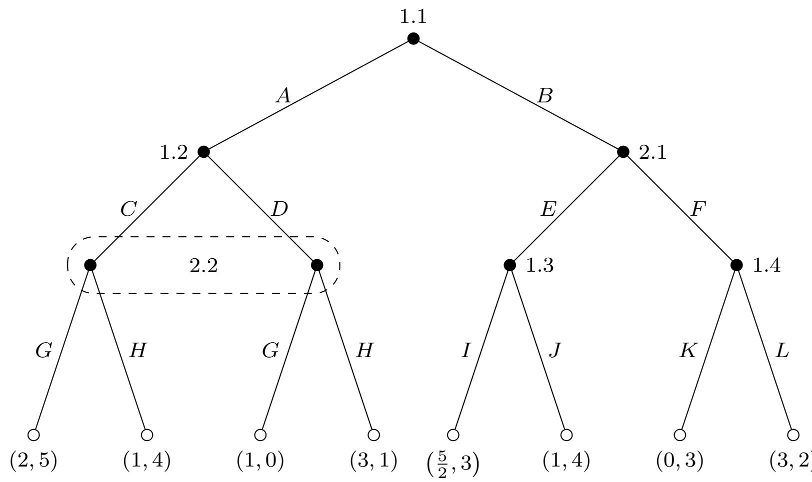
Hence, $7 + 3x = 8$, *that is* $x = \frac{1}{3}$. *We have the BNE*

$$\left(\frac{4}{5}A + \frac{1}{5}B; \left(\frac{1}{3}X + \frac{2}{3}Y\right), Y\right)$$

with payoffs

$$u_S = \frac{18}{5}, \quad u_T = 4 \quad u_1 = 2$$

Exercise 3:
Consider the following game in extensive form



1. **(5 points.)** What are the sub-games of the above game? It is enough to write the node at which each sub-game starts.

   Solution: *There are five sub-games that start at the nodes 1.1, 1.2 2.1, 1.3 and 1.4.*

2. **(5 points.)** Write the normal form of the sub-game that starts at at node 1.2. Find the Nash equilibria (in pure and mixed strategies) of this sub-game.

   Solution: *The normal form of the sub-game that starts at at node* 1.2 *is,*

	G	H
C	$2,5$	$1,4$
D	$1,0$	$3,1$

There are two NE in pure strategies: (C,G) *with payoffs* $(2,5)$ *and* (D,H) *with payoffs* $(3,1)$. *In addition, there is mixed strategy NE*

$$\left(\left(\frac{1}{2}C+\frac{1}{2}D\right),\left(\frac{2}{3}G+\frac{1}{3}H\right)\right)$$

with payoffs

$$u_1=\frac{5}{3},\quad u_2=\frac{5}{2}$$

3. **(5 points.)** Write the normal form of the sub-game that starts at at node 2.1. Find the Nash equilibria in pure strategies of this sub-game.

   Solution: *The normal form of the sub-game that starts at at node* $2-1$ *is,*

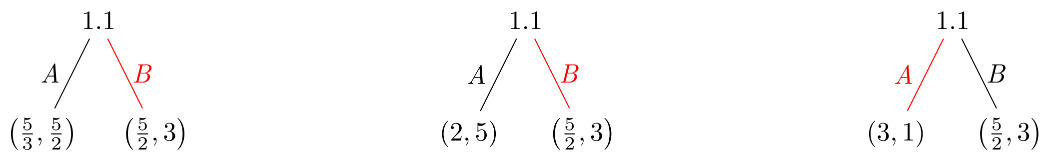
	E	F
IK	$5/2,3$	$0,3$
IL	$5/2,3$	$3,2$
JK	$1,4$	$0,3$
JL	$1,4$	$3,2$

There are two NE in pure strategies: (IK, E) *with payoffs* $\left(\frac{5}{2}, 3\right)$ *and* (IL, E) *with payoffs* $\left(\frac{5}{2}, 3\right)$.

4. **(10 points.)** Find the pure strategy sub-game perfect Nash equilibria of the complete game.

   Solution: *In the subgame that starts at node* 2.1*, only the NE* (IL, K) *is SPNE. We substitute the payoffs of the SPNE of the subgame that starts at node* 2.1 *and the (three) payoffs of the NE of the game that starts at node* 1.2*. We obtain the three following games.*

   For each of the NE of the game



   Let us write an strategy as $((1.1, 1.2, 1.3, 1.4), (2.1, 2.2))$. *We obtain The following SPNE*

   (a) $\left(\left(B, \left(\frac{1}{2}C + \frac{1}{2}D\right), I, L\right); \left(\frac{2}{3}G + \frac{1}{3}H, E\right)\right)$, *with payoffs* $(6, 2)$, *with payoffs* $u_1 = \frac{5}{2}$, $u_2 = 3$.

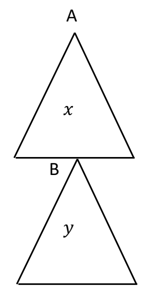
   (b) $((B, C, I, L); (G, E))$, *with payoffs* $u_1 = \frac{5}{2}$, $u_2 = 3$.

   (c) $((A, D, I, L); (H, E))$, *with payoffs* $u_1 = 3$, $u_2 = 1$.

Exercise 4: In A relationship, partner A cares about partner B. Partner A has a wealth of $w_1 = 10$ which it distributes between own consumption $w_1 - x$ and a transfer x to B. Partner B has a wealth of w_2, to which it adds the transfer x received from A. Partner B divides the total amount $w_2 + x$ Between gambling y and illegal drugs $z = w_2 + x - y$.
The utility function of B is $u_2(y, z) = \ln y + \ln z$, where y is the amount spent on gambling and z is the amount spent on illegal drugs. The utility function of A is $u_1(x; y, z) = (1 - a)\ln(w_1 - x) + au_2(y, z)$, where $w_1 - x$ is A's consumption. That is, A cares also about the welfare of B. Here, $0.1 \leq a < 1$ is the parameter of altruism of A towards B. First, A decides the transfer x to B, and then B decides its own consumption Bundle.

(a) **(5 points.)** Describe the situation as a sequential game and find the SPNE. Compute (only) the utility of player B. (The solution depends on a and w_2).

   Solution: *The game is*



$$u_A(x,y) = (1-a)\ln(10-x) + a\ln y + a\ln(w_2+x-y)$$
$$u_B(x,y) = \ln y + \ln(w_2+x-y)$$

Best reply of B:

$$y = \frac{w_2+x}{2}$$

Plug in into the utility of player 1:

$$\begin{aligned} u_A(x,(w_2+x)/2) &= (1-a)\ln(10-x) + a\ln\left(\frac{w_2+x}{2}\right) + a\ln\left(w_2+x-\frac{w_2+x}{2}\right) \\ &= (1-a)\ln(10-x) + 2a\ln\left(\frac{w_2+x}{2}\right) \end{aligned}$$

Best reply of A:

$$x^* = \frac{20a - w_2 + aw_2}{1+a}, \quad y^* = \frac{w_2+x^*}{2} = \frac{a(20+w_2)}{1+a}$$

Utility of player B:

$$u_B(x^*,y^*) = 2\ln\left(\frac{w_2+x^*}{2}\right) = 2\ln\left(\frac{a(10+w_2)}{1+a}\right)$$

(b) **(5 points.)** Suppose $w_2 = 1$, $a = 1/5$. What is the utility of B in the SPNE? Suppose if B buys flowers to A, then a increases from $1/5$ to $1/2$. How much would B be willing to pay for the flowers?

Solution: *Plugging $w_2 = 1$, $a = 1/5$, we obtain,*

$$u_B\left(x^*, y^*; w_2 = 1, a = \frac{1}{5}\right) = 2\ln\left(\frac{11}{6}\right)$$

Suppose B pays x to increase a from $1/5$ to $1/2$. Plugging $w_2 = 1-x$, $a = 1/2$ we obtain,

$$u_B\left(x^*, y^*; w_2 = 1-x, a = \frac{1}{2}\right) = 2\ln\left(\frac{11-x}{3}\right)$$

The maximum amount x that B would be willing to pay for the flowers satisfies

$$\frac{11}{6} = \frac{11-x}{3}$$

So,

$$x = \frac{11}{2}$$