UNIVERSITY CARLOS III

Master in Economics **Master in Industrial Economics and Markets**

Game Theory

FINAL EXAM–January 15th, 2020

NAME:

Problem 1:
Consider a market with one good and two firms. The firms decide prices p_1 and p_2 independently. Given those prices, the amount sold by each company is

$$\begin{aligned} x_1(p_1, p_2) &= a - 2p_1 + p_2 \\ x_2(p_1, p_2) &= a - 2p_2 + p_1 \end{aligned}$$

where $a \in \{48, 96\}$ is a parameter which is only known precisely by firm 1. Firm 2 thinks that, with probability 1/2, $a = 48$ and, with probability 1/2, $a = 96$. This situation is common knowledge for both firms. Assume that the cost of production is 0 for both firms.

(a) **(5 points.)** Describe the situation as a Bayesian game

Solution: *There are two players $N = \{1, 2\}$. There are two types of player 1: $T_1 = \{c_l, c_h\}$, where $c_l = 4$ and $c_h = 8$. There is one type of player 2: $T_2 = \{c\}$. The sets of strategies are $S_1 = S_2 = S_{c_l} = S_{c_h} = [0, \infty)$. The beliefs of the players are*

$$\begin{aligned} p_1(t_2 = c|t_1 = c_l) &= p_1(t_2 = c|t_1 = c_l) = 1 \\ p_2(t_1 = c_l|t_2 = t) &= p_2\big(t_1 = c_|t_2 = t\big) = 1/2 \end{aligned}$$

The payoffs are

$$\begin{aligned} u_h(p_h, p_2) &= (54 - p_h + \frac{p_2}{2})(p_h - c_h) \\ u_l(p_l, p_2) &= (54 - 2p_l + \frac{p_2}{2})(p_l - c_l) \\ u_2 &= (p_2 - 6)\left(\frac{1}{2}\left(54 - p_2 + \frac{p_h}{2}\right) + \frac{1}{2}\left(54 - p_2 + \frac{p_l}{2}\right)\right) \end{aligned}$$

(b) **(10 points.)** Compute the best reply of each firm. You must compute the best reply of each type of the firms.

Solution: *Agent 1, type a_h, maximizes $\max_{p_h} u_h = (96 - 2p_h + p_2)p_h$. The first order condition is*

$$96 - 4p_h + p_2 = 0.$$

Note that the second derivative with respect to p_h is

$$\frac{\partial^2 u_h}{\partial p_h^2} = -4 < 0$$

Hence, the first order condition corresponds to a maximum of u_h. The best reply of agent 1, type a_h, is

$$\text{BR}_h(p_2) = \frac{96 + p_2}{4}$$

Likewise, agent 1, type a_l, maximizes $\max_{p_l} u_l = (96 - 2p_lh + p_2)p_l$. *The first order condition is*

$$48 - 4p_l + p_2 = 0.$$

Note that the second derivative with respect to p_l is

$$\frac{\partial^2 u_l}{\partial p_l^2} = -4 < 0$$

Hence, the first order condition corresponds to a maximum of u_l. The best reply of agent 1, type a_l, is

$$\mathrm{BR}_l(p_2) = \frac{48 + p_2}{4}$$

Finally, agent 2 maximizes

$$\max_{p_2} \frac{1}{2}(96 - 2p_2 + p_h)\, p_2 + \frac{1}{2}(48 - 2p_2 + p_l)\, p_2$$

The first order condition is

$$\frac{1}{2}(p_h - 2p_2 + 48) + \frac{1}{2}(p_l - 2p_2 + 96) - 2p_2 = 0$$

The best reply of agent 2 is

$$\mathrm{BR}_2(p_h, p_l) = \frac{144 + p_h + p_l}{8}$$

(c) **(10 points.)** Compute the Bayes–Nash equillibrium, the quantities sold in this equilibrium and the profits of each firm.

Solution: *The NE is the solution to*

$$p_h = \frac{96 + p_2}{4} \qquad p_l = \frac{48 + p_2}{4} \qquad p_2 = \frac{144 + p_h + p_l}{8}$$

We obtain

$$p_h^* = 30 \quad p_l^* = 18 \quad p_2^* = 24$$

the utilities of the agents are

$$u_h^* = 1800 \quad u_l^* = 648 \quad u_2^* = 1152$$

and the quantities sold are

$$x_h^* = 60 \quad x_l^* = 36 \quad x_2^* = 48$$

Problem 2: Two firms operate in a market: the incumbent (player 1) who has to decide wether to build (B) a plant or not (N) and a potential entrant (player 2) who has to decide wether to enter (E) the market or not (N). Player 1 knows the cost of building the plant, but player 2 does not know. Player 2 only knows that the cost is high with probability $0 < p < 1$ and low with probability $1 - p$. Both firms have take their decision simultaneously. The profits of the firms are given in the following tables.

Player 1 \ Player 2	E	N
B	0, -2	4, 0
N	4, 2	6,0

(high cost, p)

Player 1 \ Player 2	E	N
B	3, -2	7, 0
N	4, 2	6, 0

(low cost, $1 - p$)

(15 points.) Find the Bayesian equilibria in pure strategies and the payoffs of the players in those equilibria.
Solution: *Note that* $\mathrm{BR}_1(E) = NN$ *and*

$$\begin{aligned} u_2(NN, E) &= 2 \\ u_2(NN, N) &= 0 \end{aligned}$$

Hence $\mathrm{BR}_2(NN) = E$. *We conclude that for any* $0 < p < 1$,

$$(NN, E) \quad \text{is BNE with payoffs} \quad u_1(NN, E|h) = 4, u_1(NN, E|l) = 4, u_2(NN, E) = 2$$

On the other hand, $\mathrm{BR}_1(N) = NB$ *and*

$$\begin{aligned} u_2(NB, E) &= 2p - 2(1-p) = 4p - 2 \\ u_2(NB, N) &= 0 \end{aligned}$$

We consider three cases:

1. *If* $0 \leq p < 1/2$, *then,* $\mathrm{BR}_1(NB) = N$ *and*

   $$(NB, N) \quad \text{is also a BNE with payoffs} \quad u_1(NB, N|h) = 6, u_1(NB, N|l) = 7, u_2(NB, N) = 0$$

2. *If* $p = 1/2$, *then,* $\mathrm{BR}_1(NB) = \{E, N\}$ *and again*

   $$(NB, N) \quad \text{is also a BNE with payoffs} \quad u_1(NB, N|h) = 6, u_1(NB, N|l) = 7, u_2(NB, N) = 0$$

3. *If* $p > 1/2$, *then,* $\mathrm{BR}_1(NB) = E$. *We conclude that, for these values of* p, *there is no BNE in pure strategies in which player 1 plays* NB.

(20 points.) Find the Bayesian equilibria in mixed strategies and the payoffs of the players in those equilibria.
Solution: *Note that, for player 1 with high cost, strategy* B *is dominated by strategy* N. *So, in no NE will this player use strategy* B. *Let us look for a BNE in mixed strategies of the form*

$$\begin{aligned} \sigma_h &= N \\ \sigma_l &= xB + (1-x)N, \quad 0 < x < 1 \\ \sigma_2 &= yE + (1-y)N \quad 0 < x < 1 \end{aligned}$$

The expected utilities of player l *are*

$$\begin{aligned} u_l(B, \sigma_2) &= 3y + 7(1-y) = 7 - 4y \\ u_l(N, \sigma_2) &= 4y + 6(1-y) = 6 - 2y \end{aligned}$$

We must have that $7 - 4y = 6 - 2y$. *That is,* $y = 1/2$. *The expected utilities of player 2 are*

$$\begin{aligned} u_2(\sigma_h, \sigma_l; E) &= 2p + (1-p)(-2x + 2(1-x)) = 2(1 - 2x(1-p)) \\ u_2(\sigma_h, \sigma_l; N) &= 0 \end{aligned}$$

We must have that $1 - 2x(1-p) = 0$. *Hence,*

$$x = \frac{1}{2(1-p)}$$

Since, $0 < x < 1$ *we must have that* $0 < \frac{1}{2(1-p)} < 1$. *That is, for*

$$0 \leq p \leq \frac{1}{2}$$

we have the BNE

$$\left(\left(N, \frac{1}{2(1-p)}B + \left(1 - \frac{1}{2(1-p)}\right)N\right), \frac{1}{2}E + \frac{1}{2}N\right)$$

with payoffs

$$u_h = \frac{1}{2} \times 4 + \frac{1}{2} \times 6 = 5, \quad u_l = 5 \quad u_2 = 0$$

Finally, recall from the previous part, that for $p = 1/2$ *we have*

$$u_2(NB, E) = u_2(NB, N) = 0$$

and we also have that

$$u_l\left(B, \sigma_2\right) = 7 - 4y \geq u_l\left(N, \sigma_2\right) = 6 - 2y$$

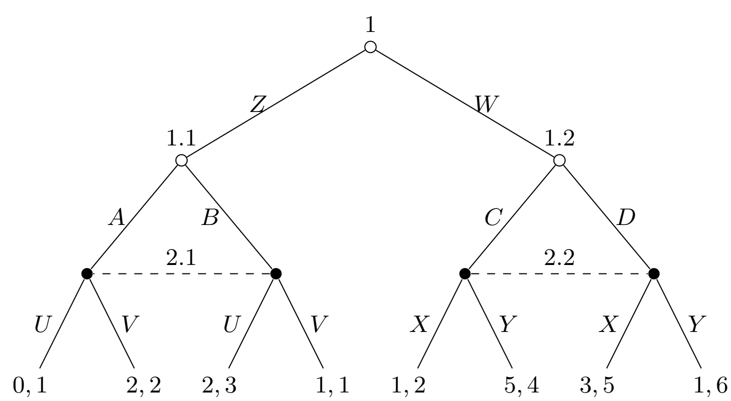
iff $0 \leq y \leq \frac{1}{2}$. *Thus, for* $p = 1/2$, *we we have the BNE*

$$\left(NB, yE + (1-y)N\right), \quad 0 \leq y \leq \frac{1}{2}$$

with payoffs

$$u_h = 6 - 2y, \quad u_l = 7 - 4y \quad u_2 = 0$$

Problem 3:
Consider the following game in extensive form



1. **(5 points.)** What are the sub-games of the above game? It is enough to write the node at which each sub-game starts.

   Solution: *There are three sub-games: The whole game and the sub-games that start at the nodes 1.1 and 1.2.*

2. **(5 points.)** Write the normal form of the sub-game that starts at at node 1.1. Find the Nash equilibria in pure strategies of this sub-game.

   Solution: *The normal form of the sub-game that starts at at node* 1.1 *is,*

	U	V
A	$0, 1$	$2, 2$
B	$2, 3$	$1, 1$

There are two NE in pure strategies: (A, V) *with payoffs* $(2, 2)$ *and* (B, U) *with payoffs* $(2, 3)$.

3. **(5 points.)** Write the normal form of the sub-game that starts at at node 1.2. Find the Nash equilibria in pure strategies of this sub-game.

   Solution: *The normal form of the sub-game that starts at at node* 1.2 *is,*

	X	Y
C	$1,2$	$5,4$
D	$3,5$	$1,6$

   There is one NE in pure strategies: (C, Y) *with payoffs* $(5, 4)$.

4. **(10 points.)** Find the pure strategy sub-game perfect Nash equilibria of the complete game.

   Solution: *Let us write an strategy as* $(1, (1.1, 2.1), (1.2, 2.2))$. *At node* 1, *player 1 chooses* Z *and* W *anticipating the subsequent NE which gives him the highest payoff. Thus, the sub-game perfect Nash equilibria are:*

   (a) $(W, A, C), (V, Y))$, *with payoffs* $(5, 4)$.
   (b) $(W, B, C), (U, Y)))$, *with payoffs* $(5, 4)$.

Problem 4: Consider the following stage game

	C	D
C	$6,6$	$3,3$
D	$8,3$	$4,4$

(a) **(5 points.)** Find all the NE of the above game.

Solution: *The strategy* C *is strictly dominated for player 1. Once we eliminate this stragegy,* D *is strictly dominated for player 2. Hence, the unique NE is* (D, D) *with payoff* $u_1 = u_2 = 4$.

(b) **(5 points.)** Consider the repeated game which consists in playing the above stage game 57 many times. What are the subgame perfect Nash Equilibria?

Solution: *The strategy* (D, D) *is the unique NE of the stage game. Thus, playing* (D, D) *in every period is the unique SPNE.*

Consider now the repeated game which consists in playing the above stage game infinitely many times with discount factor δ.

(c) **(5 points.)** Can you find a subgame perfect Nash Equilibrium for every $0 < \delta < 1$?

Solution: *The strategy* (D, D) *is the unique NE of the stage game. Thus, playing* (D, D) *in every period is the unique SPNE.*

(c) **(10 points.)** Describe the trigger strategy.

(c) **(15 points.)** For what values of δ does the trigger strategy constitute a subgame perfect Nash equilibrium? You must justify why the trigger strategy is a SPNE for those values of δ.

Solution:

If players follow the trigger strategy their stream of payoffs is

$$4 + 4\delta + 4\delta^2 + \cdots = \frac{4}{1-\delta}$$

If one player deviates at the first period his stream of payoffs will be

$$6 + 2\delta + 2\delta^2 + \cdots = 6 + \frac{2\delta}{1-\delta}$$

The trigger strategy is a NE of the repeated game iff

$$\frac{4}{1-\delta} \geq 6 + \frac{2\delta}{1-\delta}$$

that is iff $\delta \geq \frac{1}{2}$. *The standard argument shows it is a SPNE.*