UNIVERSITY CARLOS III

Master in Economics | **Master in Industrial Economics and Markets**

Game Theory

FINAL EXAM–January 23rd, 2019

NAME:

Problem 1: Two graduate students share an apartment. They have to spend time cleaning the apartment. If each of them spends x_1, x_2 hours cleaning, their utilities are

$$\begin{aligned} u_1(x_1, x_2) &= (20 + x_2)x_1 - 3x_1^2 \\ u_2(x_1, x_2) &= (20 + x_1)x_2 - 3x_2^2 \end{aligned}$$

Note that the time spent in cleaning increases the utility of both tenants. That is, cleaning has a positive externality. On the other hand, the time spent cleaning imposes a personal cost.

(a) **(5 points.)** Suppose first that they consider only the possibility of cleaning the apartment on a single given day. If both students decide independently the time spent cleaning, how much time will they devote to cleaning? What are their utilities?

Solution:

Agent 1 maximizes $\max_{x_1} u_1 = (20 + x_2)x_1 - 3x_1^2$. *The first order condition is*

$$\frac{\partial u_1}{\partial x_1} = 20 + x_2 - 6x_1 = 0$$

Note that the second derivative with respect to x_1 *is*

$$\frac{\partial^2 u_1}{\partial x_1^2} = -6 < 0$$

Hence, the first order condition corresponds to a maximum of u_1. *The best reply of agent 1 is*

$$\mathrm{BR}_1(x_2) = \frac{20 + x_2}{6}$$

Likewise, agent 2 maximizes $\max_{x_2}(20 + x_1)x_2 - 3x_2^2$. *The best reply of agent 2 is*

$$\mathrm{BR}_2(x_1) = \frac{20 + x_1}{6}$$

The NE is the solution to

$$q_1 = \frac{20 + x_2}{6}, \quad q_2 = \frac{20 + x_1}{6}$$

The NE is $x_1^* = x_2^* = 4$. *The utilities of the agents are* $u_1^* = u_2^* = 48$.

(b) **(5 points.)** Suppose now they could make a joint agreement on how much time each should spend on cleaning. Which is the amount of time each should spend on cleaning that maximizes their joint welfare?

Solution:

Now, the agents maximize

$$\max_{x_1, x_2}(20 + x_2)x_1 - 3x_1^2 + (20 + x_1)x_2 - 3x_2^2$$

The solution is $\bar{x}_1 = \bar{x}_2 = 5$. *The utilities of the agents are* $\bar{u}_1^* = \bar{u}_2^* = 50$.

(c) **(5 points.)** Would they have incentives to deviate from the agreement reached in part (2)?

Solution:

The agreement in part (b) is not a NE. For example,

$$\mathrm{BR}_1(5) = \frac{25}{6} = 4.16667 \neq \bar{x}_1$$

with utility

$$u_1\left(\frac{25}{6}, 5\right) = \frac{625}{12} = 52.0833$$

Thus, the agents have incentives to deviate.

(d) **(10 points.)** Suppose now that they have to clean the apartment every day. The discount factor is $\delta = 2/3$. Can you find a SPNE of the infinitely repeated game with the above game as the stage game?

Solution:

Let us consider the following (trigger strategies) strategy profile T. Firm $i = 1, 2$ choose the following x_i,

- *At $t = 1$, choose $\bar{x}_i = 5$;*
- *At $t > 1$, if*

  $$\bar{x}_1 = \bar{x}_2 = 5$$

  was chosen at $t = 1, \ldots, t-1$ then chose $\bar{x}_i = 5$. Otherwise, choose the Nash equilibrium

  $$x_i^* = 4$$

The utility obtained by any player under the strategy profile T is

$$u_T = 50 + 50\delta + 50\delta^2 + \cdots = \frac{50}{1-\delta} = 150$$

If an agent $i = 1, 2$ deviates in stage $t = 1$, then its best option is to deviate to

$$\mathrm{BR}_i(5) = \frac{25}{6}$$

with a utility of

$$\frac{625}{12}$$

Thus, if agent $i = 1, 2$ deviates in stage $t = 1$, the maximum payoff he can obtain is

$$u_d = \frac{625}{12} + 48\delta + 48\delta^2 \cdots = \frac{625}{12} + \frac{48\delta}{1-\delta} = \frac{1777}{12} = 148.083$$

Thus, the strategy profile T is a NE of the repeated game. We show next that it is also a NE in every subgame. There are two types of subgames starting at a stage t.

- *Subgames in which at every stage $1, 2, \ldots, t-1$ the strategy profile $(5, 5)$ was played . Then, the situation in the subgame that starts at this node is exactly like in the original game, except that the payoffs are multiplied by δ^{t-1}. The above argument shows that the strategy profile T is also a NE of that subgame.*
- *Subgames in which at some stage $1, 2, \ldots, t-1$ the strategy profile $(5, 5)$ was not played. In these subgames the strategy profile T prescribes that $(4, 4)$, a NE of the stage game, is played in every stage. Therefore, it is a SPNE of this subgame.*

	Player 2	
Player 1	C	D
A	6,4	0,8
B	0, 8	6,4

(i)

	Player 2	
Player 1	C	D
A	12,8	0,4
B	24, 16	12,12

(ii)

Therefore, the trigger strategy constitutes a SPNE.

Problem 2: Consider the situation in which player 1 knows what game is played ((*i*) or (*ii*) below). But player 2 only knows that A is played with probability 1/2 and B is played with probability 1/2.
(25 points.) Find the Bayesian equilibria in pure and mixed strategies.
Solution:

		C	D	
		q	$1-q$	
AA	xz	$9,6$	$0,6$	$9q$
AB	$x(1-z)$	$15,10$	$6,10$	$6+9q$
BA	$(1-x)z$	$6,8$	$3,4$	$3(1+q)$
BB	$(1-x)(1-z)$	$12,12$	$9,8$	$3(3+q)$
		$2(6-x-2z)$	$2(4+x-2z)$	

Since $9q < 6+9q$ *and* $3(1+q) < 3(3+q)$ *for every* $q \in [0,1]$ *we must have* $xz = (1-x)z = 0$. *Hence,* $z = 0$ *and we obtain*

		C	D	
		q	$1-q$	
AB	x	$15,10$	$6,10$	$6+9q$
BB	$1-x$	$2,12$	$9,8$	$3(3+q)$
		$2(6-x)$	$2(4+x)$	

We obtain

$$\mathrm{BR}_1(q) = \begin{cases} BB \quad (x=0) & \text{if } q < \frac{1}{2} \\ \{AB, BB\} \quad (0 \le x \le 1) & \text{if } q = \frac{1}{2} \\ AB \quad (x=1) & \text{if } q > \frac{1}{2} \end{cases}$$

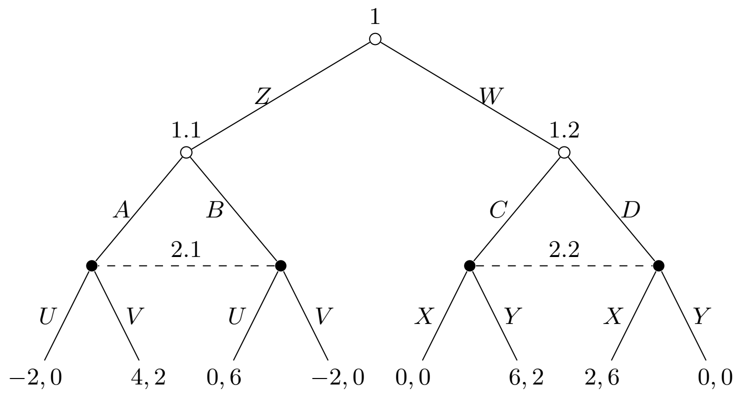
and

$$\mathrm{BR}_2(x,z) = \begin{cases} C \quad (q=1) & \text{if } 0 \le x < 1 \\ \{C, D\} \quad (0 \le q \le 1) & \text{if } x = 1 \end{cases}$$

We get the following BNE

- (AB, C), *with payofs* $u_1 = 15$, $u_2 = 10$.
- $(AB, qC + (1-q)D)$, $\frac{1}{2} \le q \le 1$ *with payofs* $u_1 = \frac{21}{2}$, $u_2 = 10$.

Problem 3:
Consider the following game in extensive form



1. **(5 points.)** What are the sub-games of the above game? It is enough to write the node at which each sub-game starts.

   Solution: *There are three sub-games: The whole game and the sub-games that start at the nodes 1.1 and 1.2.*

2. **(5 points.)** Write the normal form of the sub-game that starts at at node 1.1. Find the Nash equilibria in pure strategies of this sub-game.

   Solution: *The normal form of the sub-game that starts at at node* 1.1 *is,*

	U	V
A	$-2, 0$	$4, 2$
B	$0, 6$	$-2, 0$

There are two NE in pure strategies: (A, V) *with payoffs* $(4, 2)$ *and* (B, U) *with payoffs* $(0, 6)$.

3. **(5 points.)** Write the normal form of the sub-game that starts at at node 1.2. Find the Nash equilibria in pure strategies of this sub-game.

   Solution: *The normal form of the sub-game that starts at at node* 1.2 *is,*

	X	Y
C	$0, 0$	$6, 2$
D	$2, 6$	$0, 0$

There are two NE: (C, Y) *with payoffs* $(6, 2)$ *and* (D, X) *with payoffs* $(2, 6)$.

4. **(10 points.)** Find the pure strategy sub-game perfect Nash equilibria of the complete game.

   Solution: *Let us write an strategy as* $(1, (1.1, 2.1), (1.2, 2.2))$. *At node* 1*, player 1 chooses* Z *and* W *anticipating the subsequent NE which gives him the highest payoff. Thus, the sub-game perfect Nash equilibria are:*

   (a) $(W, (A, V), (C, Y))$, *with payoffs* $(6, 2)$.
   (b) $(Z, (A, V), (D, X))$, *with payoffs* $(4, 2)$.
   (c) $(W, (B, U), (C, Y))$, *with payoffs* $(6, 2)$.
   (d) $(W, (B, U), (D, X))$, *with payoffs* $(2, 6)$.

Problem 4: Consider the following stage game

	C	D
C	$4,4$	$0,6$
D	$6,0$	$2,2$

And consider the repeated game which consists in playing the above stage game infinitely many times with discount factor δ.

(a) **(5 points.)** Can you find a subgame perfect Nash Equilibrium for every $0 < \delta < 1$?

Solution: *The strategy (D, D) is a NE of the stage game. Thus, playing (D, D) in every period is a SPNE.*

(b) **(10 points.)** Describe the trigger strategy. For what values of δ does the trigger strategy constitute a subgame perfect Nash equilibrium?

Solution:
If players follow the trigger strategy their stream of payoffs is

$$4 + 4\delta + 4\delta^2 + \cdots = \frac{4}{1-\delta}$$

If one player deviates at the first period his stream of payoffs will be

$$6 + 2\delta + 2\delta^2 + \cdots = 6 + \frac{2\delta}{1-\delta}$$

The trigger strategy is a NE of the repeated game iff

$$\frac{4}{1-\delta} \geq 6 + \frac{2\delta}{1-\delta}$$

that is iff $\delta \geq \frac{1}{2}$. The standard argument shows it is a SPNE.

(c) **(5 points.)** Let now $\delta = \frac{3}{4}$. And consider the **tit-for-tat:** strategy profile: Player $i = 1, 2$ at

- $t = 1$ plays C.
- $t > 1$ plays whatever the other played at $t - 1$.

Is the tit-for-tat strategy profile a **subgame perfect Nash** equilibrium of the repeated game? Why or Why not? (Hint: consider those subgames which start at a node t in which in the previous period (D, C) was played.)

Solution:
Consider a subgame which starts at a node t in which in the previous period (D, C) was played. If players follow the tit-for-tat strategy profile in that subgame the sequence of plays at every stage is

$$(C, D), (D, C), (C, D), (D, C), (C, D), (D, C), \ldots$$

with payoffs

$$(0, 6), (6, 0), (0, 6), (6, 0), (0, 6), (6, 0), \ldots$$

that is

$$u_1 = 0 + 6\delta + 0\delta^2 + 6\delta^3 + \cdots = \frac{6\delta}{1-\delta^2}$$

$$u_2 = 6 + 0\delta + 6\delta^2 + 0\delta^3 + \cdots = \frac{6}{1-\delta^2} = \frac{6}{1-\left(\frac{3}{4}\right)^2} = \frac{96}{7} = 13.7143$$

Suppose player 2 deviates and plays C in that period. The sequence of plays is

$$(C,C),(C,C),(C,C),(C,C),(C,C),(C,C),\ldots$$

The payoff of player 2 is

$$\bar{u}_2 = 4 + 4\delta + 4\delta^2 + 4\delta^3 + \cdots = \frac{4}{1-\delta} = \frac{4}{1-\frac{3}{4}} = 16$$

Hence, player 2 has incentives to deviate. We conclude that tit-for-tat is not a SPNE

(d) **(5 points.)** Let again $\delta = \frac{3}{4}$ Is the tit-for-tat strategy profile described above a **Nash equilibrium** of the repeated game? Why or Why not?

Solution: *Let r be the tit-for-tat strategy. Let s be a best reply of player 1 to r. Note that if, in the path determined by s and r, we reach a node in which player 2 plays C, then, in the subgame that starts at that node, s continues to be a best reply of player 1 against r.*

Let $x = u_1(s,r)$. Note that $u_1(r,r) = \frac{1}{1-\delta} = 16$. So $x \geq 16$. We will show $x = 16$. Suppose to the contrary that $x > 16$.

For simplicity, we assume that, in the strategy s, player 1 plays D for the first time in period 1.That is, we ignore all the initial periods in which r and s coincide.

Assume now that player 1 plays r and player 2 plays tit-for-tat. Thus, player 2 plays C in period 1 and D in period 2. There are two possible continuations for player 1 in period 2: C and D. That is,

$$\text{case 1:} \quad (D,C),(D,D),(*,D),\cdots$$

and

$$\text{case 2:} \quad (D,C),(C,D),(*,C),\cdots$$

Let us consider case 1. If player 1 plays D indefinitely the payoff for player 1 is

$$x = u_1(s,r) = 6 + 2\delta + 2\delta^2 + \cdots = 6 + \frac{2\delta}{1-\delta} = 12 < 16$$

This is not possible. Hence, player 1 switches to C after, say, k periods. That is,

$$\text{case 1:} \quad (D,C),\underbrace{(D,D),(D,D),\cdots,(D,D)}_{k \text{ times}},(C,D),(*,C)$$

Note that now case 2 is obtained from case 1 when $k = 0$.) The payoff for player 1 is

$$x = u_1(s,r) = 6 + 2\delta + 2\delta^2 + \cdots + 2\delta^k + 0 \times \delta^{k+1} + \delta^{k+2}x$$

Since $x > 2$, the optimal k is $k = 0$. And we have

$$x = 6 + 0 \times \delta + \delta^2 x = 6 + \delta^2 x$$

But this implies

$$x = \frac{6}{1-\delta^2} = \frac{6}{7} \times 16 < 16$$

a contradiction. Hence, case 1 is not possible. Therefore, case 2 is not possible either. We conclude that $x = 16$ and no profitable deviation exists.