3402 CONTRACTS AND GAME THEORY
Final Examination
Universitat Pompeu Fabra – Winter 1995
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1. (20) Armies 1 and 2 are fighting over an island initially held by a battalion of army 2. Army 1 has K battalions and army 2 has L. Whenever the island is occupied by one army the opposing army can launch an attack. The outcome of the attack is that the occupying battalion and one of the attacking battalions are destroyed; the attaking army wins and, so long as it has battalions left, occupies the island with one battalion. The commander of each army is interested in maximizing the number of surviving battalions but also regards the occupation of the island as worth more than one battalion but less than two. (If after an attack, neither army has battalions left, then the payoff of each commander is 0.)
   (a) Draw the extensive form for this game, for $(K, L) = (2, 2)$ and also for $(2, 3)$ and find their subgame-perfect equilibria.
   (b) What will the subgame-perfect outcome be in general, as a function of K and L, and why is this subgame-perfect?

2. (15) Consider the following two-player game. First player 1 can choose either *Stop* or *Continue*. If she chooses *Stop* then the game ends with the pair of payoffs (1, 1). If she chooses *Continue* then the players simultaneously announce nonnegative integers and each player's payoff is the product of the numbers. What are the subgame-perfect equilibria of this game?

3. (25) Consider an infinitely repeated game where each player maximizes her expected δ-discounted sum of payoffs, for some δ close to but less than 1. Payoffs at each round depend on the players' moves as shown below.

	x_2	y_2
x_1	2, 2	0, 6
y_1	6, 0	1, 1

Consider the following strategies. At the first stage, and at each stage where the outcome of the preceding round was (x_1, x_2) or (y_1, y_2), the players independently randomize, each player i choosing y_i with probability q or x_i with probability $1 - q$. At each round where the outcome of the preceding round was (x_1, y_2) the players choose y_1 and x_2. At each round where the outcome of the preceding round was (y_1, x_2) the players choose x_1 and y_2.

(a) Write the continuation payoff (the value function) after (y_1, x_2) and the one after (x_1, y_2), given the strategies described above.

(b) Write the continuation payoffs (the value functions) after (x_1, x_2) or (y_1, y_2), given the strategies described above.

(c) Write the equations (actually, an equation and two inequalities) that make these strategies a subgame-perfect equilibrium, once δ is specified, as long as δ is not too small.

4. (20) Let the following Bayesian game:

1. Nature determines whether the payoffs are as in Game 1 or as in Game 2, each game being equally likely.

2. Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not.

3. Player 1 chooses either T or B; player 2 simultaneously chooses either L or R.

4. Payoffs are given by the game drawn by nature.

Game 1

	L	R
T	1,1	0,0
B	0,0	0,0

Game 2

	L	R
T	0,0	0,0
B	0,0	2,2

(a) Find all the pure strategy and mixed strategy Bayesian Nash equilibria of this game.

5. (15)

A buyer and a seller have valuations v_b and v_s. It is common knowledge that there are gains from trade (i.e., that $v_b > v_s$), but the size of the gains is private information, as follows: the seller's valuation is uniformly distributed on [0, 1]; the buyer's valuation $v_b = kv_s$ where $k > 1$ is common knowledge; the seller knows v_s (and hence v_b) but the buyer does not know v_b (or v_s). Suppose the buyer makes a single offer, p, which the seller either accepts or rejects. What is the perfect Bayesian equilibrium when $k < 2$? When $k > 2$?

6. (20) The following is the game of the "crazy crab" we analyzed in class:

Assume $v > c$ and write the payoff matrix of the normal form.

(a) What strategies are strictly dominated?

(b) What are the Nash equilibria when $0.4v - 0.6k < 0$?

(c) What are the Nash equilibria when $0.4v - 0.6k > 0$?

7. (20)

Find all the pure-strategy perfect Bayesian Nash equilibria for the following signaling game.

8. (25) Viva Bus is only one busline allowed to drive between the cities of Pinto and Valdemoro. There are two types of customers, soldiers and peasants. Peasants are willing to pay more than soldier travelers. The busline people cannot directly tell whether a ticket purchaser is a soldier or a peasant (in Spain soldiers do not travel in uniform). The two types do differ in how much they are willing to pay to avoid having to purchase their tickets in advance.

   More concretely, the utility levels of each of the two types net of the price of the tickets, P, for ay given amount of time W prior to the trip that the ticket is purchased are given by:

$$\text{Peasants} : U_p(P, W) = 1 - \theta_P P - W^2 \tag{1}$$

$$\text{Soldiers} : U_s(P, W) = 1 - \theta_S P - W \tag{2}$$

   The proportion of travelers who are soldiers is λ. Assume that the cost of transporting a passenger is zero. The variable W does not enter directly in the preferences of the busline.

   (a) Draw the indifference curves of the two types in the (P, W) space. Draw the busline's iso-profit curves. What would be the contracts offered by the busline if they could tell a soldier from a peasant?

   (b) Do the contracts you just draw respect the Incentive Compatibility and Individual Rationality constraints once the types cannot be distinguished? If yes, explain why, if not draw a couple of contracts that do respect them.

   (c) Formulate the profit maximizing price discrimination problem mathematically that Viva Bus would want to solve.

   (d) Solve the problem you just formulated. (Hint: start assuming that the Individual Rationality constraint for one type is binding, and that the Incentive Compatibility constraint for the other type is binding, solve the problem that way and then check that the other constraints are satisfied).