Evolutionary Games - Winter 2005

Chapter 1

From Evolutionary Biology to Game Theory

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Summary

- Evolutionarily Stable Strategies
- Dynamics - The replicator dynamics
- Stochastic evolutionary dynamics
- References

Evolutionarily Stable Strategies *(1/13)*

Intuition:

- Strategies are genetically encoded and inherited (as opposed to chosen).
- Reproduction is asexual, and individuals "breed true."
- Payoffs are the number of offspring (absolute or relative to some base level) that a given strategy produces, when matched with some othe strategy.
- A mixed strategy is often viewed as a list of "fractions of population" playing each strategy, specially in fully dynamic analysis.

Evolutionarily Stable Strategies *(2/13)*

- Players are matched to play the game with some other player taken at "random" from the population (alternatively "viscous" populations and group selection).

- An **Evolutionarily Stable Strategy** is a (mixed) strategy such that all small-frequency entrants have strictly lower payoff (so they cannot prosper and change the population).

- Why not do a explicitly dynamic analysis? We will show there is a deep connection.

- What does this have to do with economics? Wait until next chapter!!

Static theory: ESS

Let G be a two-person symmetric game, with common strategy set $S = \{s_1, s_2, ..., s_n\}$, and A the payoff matrix, with generic element a_{ij} for $i, j = 1, ..., n$. A mixed strategy/probability distribution over S is denoted by σ and the set of all such distributions is the $n-1$ dimensional simplex Δ^{n-1}. The payoff for strategy σ against strategy σ' is:

$$\sigma A \sigma' = \sum_{i=1}^{n} \sigma_i \left[\sum_{j=1}^{n} \sigma'_j a_{ij} \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma'_j a_{ij}$$

Notice that payoffs are bilinear in own and other strategy.

Evolutionarily Stable Strategies *(4/13)*

Definition 1 (Taylor and Jonker 1978) *A strategy $\sigma \in \Delta^{n-1}$ is an evolutionarily stable strategy (ESS) if for all $\sigma' \neq \sigma$ there is $\bar{\varepsilon}$ such that if* $0 < \varepsilon < \bar{\varepsilon}$

$$\sigma A \left[(1-\varepsilon)\sigma + \varepsilon\sigma'\right] > \sigma' A \left[(1-\varepsilon)\sigma + \varepsilon\sigma'\right]$$

- In principle, this is about *monomorphic* populations against *single* mutants, but given connection to dynamics, not much "harm" in thinking about mixed strategies as population polymorphisms.

Proposition 2 (Maynard Smith 1982) *A strategy $\sigma \in \Delta^{n-1}$ is an ESS in two-player game G if and only if:*

i *For all $\sigma' \in \Delta^{n-1}, \sigma A \sigma \geq \sigma' A \sigma$*

ii *For all $\sigma' \in \Delta^{n-1}, \sigma' \neq \sigma$, if $\sigma A \sigma = \sigma' A \sigma$, then $\sigma A \sigma' > \sigma' A \sigma'$.*

Proof. *Let σ be an ESS. Suppose, first (i) is not true. Then there is $\sigma' \neq \sigma$ such that*

$$\sigma A \sigma < \sigma' A \sigma$$

Then there must be $\bar{\varepsilon} > 0$ such that if $0 < \varepsilon < \bar{\varepsilon}$

$$(1-\varepsilon)\sigma A \sigma + \varepsilon \sigma A \sigma' < (1-\varepsilon)\sigma' A \sigma + \varepsilon \sigma' A \sigma'$$

Thus

$$\sigma A \left[(1-\varepsilon)\sigma + \varepsilon \sigma'\right] < \sigma' A \left[(1-\varepsilon)\sigma + \varepsilon \sigma'\right]$$

contradicting the definition of ESS. Suppose now that (ii) is not true.

Then there must be $\sigma' \neq \sigma$ *such that* $\sigma A \sigma = \sigma' A \sigma$, *and* $\sigma A \sigma' \leq \sigma' A \sigma'$. *Multiplying the equality by* $(1-\varepsilon)$ *and the inequality by* ε *we have for all* ε

$$\sigma A \left[(1-\varepsilon)\sigma + \varepsilon\sigma' \right] \leq \sigma' A \left[(1-\varepsilon)\sigma + \varepsilon\sigma' \right]$$

contradicting the definition of ESS. To prove the converse, assume (i) and *(ii) hold. If (i) holds with strict inequality, the inequality in the definition of ESS holds for ε small. If (i) holds with equality, then (ii) holds, and then the definition of ESS holds for all ε.* ■

Remark 3 *From (i) we have that if a strategy* $\sigma \in \Delta^{n-1}$ *is an ESS, it is also a Nash equilibrium.*

Remark 4 *The proposition uses heavily the bilinearity, thus the two-player framework.*

Definition 5 $\sigma \in \Delta^{n-1}$ *is* **weakly dominated** *if* $\exists \sigma' \in \Delta^{n-1}$ *such that*

$$\begin{aligned} \sigma' A \sigma'' &\geq \sigma A \sigma'' \; \forall \sigma'' \in \Delta^{n-1} \\ \sigma' A \sigma'' &> \sigma A \sigma'' \text{ for some } \sigma'' \in \Delta^{n-1} \end{aligned}$$

Proposition 6 *Let* σ *be and ESS. Then* σ *is not weakly dominated.*

Proof. *Suppose not, then if* $\exists \sigma' \in \Delta^{n-1}$ *such that*

$$\sigma' A \sigma'' \geq \sigma A \sigma'' \; \forall \sigma'' \in \Delta^{n-1}$$

In particular, it must be true that

$$\sigma' A \sigma \geq \sigma A \sigma, \text{ and } \sigma' A \sigma' \geq \sigma A \sigma' \tag{1}$$

By corollary 3 σ *is a Nash equilibrium. Combining this with first of the inequalities in equation 1 we have*

$$\sigma' A \sigma = \sigma A \sigma$$

Thus we must have by (ii) in Proposition 2 that

$$\sigma' A \sigma' < \sigma A \sigma'$$

which contradicts the second inequality in 1. ■

Remark 7 *Remark 3 plus theorem 3.2.2. of Van Damme (1987) show that if a strategy* $\sigma \in \Delta^{n-1}$ *is an ESS, it is also a Nash equilibrium.*

Example 8 *Hawk-Dove Game (Maynard-Smith and Price 1973)*

sp, bp	H	D
H	$(V-C)/2$, $(V-C)/2$	V, 0
D	0, V	$V/2$, $V/2$

For $V > C$, clearly H is dominant, an a unique ESS exists. Suppose $V < C$. Then the unique Nash equilibrium is $\sigma^* = (\sigma^*_H, \sigma^*_D) = (V/C, 1 - V/C)$. This is the only candidate for an ESS, according to Remark 3. Let us check it is indeed one.

Since it is a Nash equilibrium in mixed strategies with full support, for all $\sigma \in \Delta^{n-1}$

$$\sigma A \sigma^* = \sigma^* A \sigma^*$$

Thus, from part (ii) in Proposition 2 we must show that for all $\sigma \in \Delta^{n-1}, \sigma \neq \sigma^*$,

$$\sigma^* A \sigma > \sigma A \sigma.$$

Then

$$\sigma^* A\sigma - \sigma A\sigma = (\sigma_H^* - \sigma_H)\left(\frac{V-C}{2}\sigma_H + V\sigma_D\right) + (\sigma_D^* - \sigma_D)\frac{V}{2}\sigma_D$$
$$= (\sigma_H^* - \sigma_H)\left(\frac{V-C}{2}\sigma_H + V\sigma_D - \frac{V}{2}\sigma_D\right) = (\sigma_H^* - \sigma_H)\left(\frac{V-C\sigma_H}{2}\right)$$

And since $\sigma_H^* = V/C$, this implies

$$\sigma^* A\sigma - \sigma A\sigma = \frac{1}{2C}(V - C\sigma_H)^2 > 0$$

Exercise 9 *Check that all symmetric 2X2 games have an ESS.*

Example 10 *Rock-Scissors-Paper*

1/2	*R*	*S*	*P*
R	*0,0*	*1,-1*	*-1,1*
S	*-1,1*	*0,0*	*1,-1*
P	*1,-1*	*-1,1*	*0,0*

This game has a unique Nash equilibrium $\sigma^* = (\sigma^*_R, \sigma^*_S, \sigma^*_P) = (1/3, 1/3, 1/3)$. This is the only candidate for an ESS, according to Remark 3. Let us check it is not one.

Choose $\sigma = (1, 0, 0)$. Then

$$0 = \sigma A \sigma^* = \sigma^* A \sigma^*$$

but also

$$0 = \sigma A \sigma = \sigma^* A \sigma$$

So there are games without an ESS.

- Static analysis has problems: nonexistence, inflexibility (two player symmetric games, monomorphism, point valued) .
- Some problems can be solved: extension to more than two player games or asymmetric not difficult, set valued extension available, monomorphism an interpretation.
- But the best solution is to go fully dynamic.
- At the cost of introducing dynamical systems.

Replicator dynamics:

Let now a game G with N players (each player is a continuum of individuals - a population).

The set of pure strategies for the ith player is S^i which has n_i strategies.

Player i's payoff function is $u^i : \prod_{k=1}^{N} S^k \rightarrow \Re$.

Let Δ^i denote the $n_i - 1$ dimensional simplex, x^i a generic member of Δ^i, and x^{-i} a generic element of $\Delta^{-i} = \prod_{j \neq i} \Delta^j$.

u^i is extended in the usual way, so $u^i(x^i, x^{-i})$ is payoff to i of using x^i againts x^{-i}.

We denote $\alpha \in \Delta^i$ the mixture giving probability one to $\alpha \in S^i$.

Dynamics - The replicator dynamics *(3/20)*

Let r^i_α be the measure of i players using α, and let $R^i = \sum_{s=1}^{n_i} r^i_s$.

Let x^i_α be the proportion of i players using α, that is, $x^i_\alpha = r^i_\alpha / R^i$.

Divide time into discrete periods of length τ.

At any instant an i player is randomly a player from each of the other $N-1$ populations.

Total payoffs for an i player playing pure strategy α in period t is $u^i(\alpha, x^{-i}(t))\tau$.

Every period all players reproduce after playing the game.

Reproduction is asexual and strategies breed true.

The number of successors is the sum of the background fitness $B^i(t, r(t))\tau$, plus the payoffs from playing the game.

After reproduction, a fraction $D^i(t, r(t))\tau$ of the users of all strategies (except the newborn) dies in every period. We have then

$$r^i_\alpha(t+\tau) = r^i_\alpha(t)\left[1 - D^i(t, r(t))\tau + B^i(t, r(t))\tau + u^i(\alpha, x^{-i}(t))\tau\right]. \qquad (2)$$

By letting the period length τ go to zero, we can obtain the continuous time version of equation (2),

$$\frac{\partial r^i_\alpha(t)}{\partial t} = r^i_\alpha(t)\left[B^i(t, r(t)) - D^i(t, r(t)) + u^i(\alpha, x^{-i}(t))\right].$$

Using the chain rule, since $x^i_\alpha(t) = r^i_\alpha(t)/R^i(t)$,

$$\begin{aligned}\frac{\partial x^i_\alpha(t)}{\partial t} &= \frac{\partial r^i_\alpha(t)}{\partial t}\frac{1}{R^i(t)} - \sum_{\beta=1}^{n_i}\frac{r^j_\beta(t)}{\left(R^i(t)\right)^2}\frac{\partial r^i_\beta(t)}{\partial t} \\ &= x^i_\alpha(t)\left[B^i(t,r(t)) - D^i(t,r(t)) + u^i(\alpha, x^{-i}(t))\right] \\ &\quad - \sum_{\beta=1}^{n_i} x^i_\beta(t)\left[B^i(t,r(t)) - D^i(t,r(t)) + u^i(\beta, x^{-i}(t))\right] \\ &= x^i_\alpha(t)\left[u^i(\alpha, x^{-i}(t)) - \sum_{\beta=1}^{n_i} x^i_\beta(t) u^i(\beta, x^{-i}(t))\right]\end{aligned}$$

so that

$$\frac{\partial x^i_\alpha(t)}{\partial t} = x^i_\alpha(t)\left[u^i(\alpha, x^{-i}(t)) - u^i(x^i(t), x^{-i}(t))\right]$$

IS Invariance of simplex faces: $x^i_\alpha(0) > 0 \Leftrightarrow \forall t > 0, x^i_\alpha(t) > 0$.

IA Invariance to additive shifts in payoffs: the game with $u^i(\alpha, x^{-i}(t))$, and $u^i(\alpha, x^{-i}(t)) + H^i(t, x^{-i}(t))$ for all α is equivalent.

RD Relative dynamics: Let α, β be such that for some $t, x^i_\alpha(t), x^i_\beta(t) > 0$. Then

$$\frac{\partial x^i_\alpha(t)}{\partial t}\frac{1}{x^i_\alpha(t)} - \frac{\partial x^i_\beta(t)}{\partial t}\frac{1}{x^i_\beta(t)} = u^i(\alpha, x^{-i}(t)) - u^i(\beta, x^{-i}(t)) \qquad (3)$$

This last observation immediately leads to the following result:

Proposition 11 *Assume α, β are such that, $x^i_\alpha(0), x^i_\beta(0) > 0$ and α strictly dominates β. Then*

$$\lim_{t \to \infty} x^i_\beta(t) = 0$$

Proof. *By IS we have that since $x^i_\alpha(0), x^i_\beta(0) > 0$, for all t, $x^i_\alpha(t), x^i_\beta(t) > 0$. Denote by $U_{\alpha\beta} = \max_{x^{-i} \in \Delta^{-i}} \left\{ u^i(\beta, x^{-i}) - u^i(\alpha, x^{-i}) \right\}$. $U_{\alpha\beta} < 0$, since α strictly dominates β.*

Then, by integrating 3:

$$\begin{aligned}
\ln \frac{x^i_\beta(t)}{x^i_\alpha(t)} - \ln \frac{x^i_\beta(0)}{x^i_\alpha(0)} &= \int_0^t \left(u^i(\beta, x^{-i}(t)) - u^i(\alpha, x^{-i}(t))\right) ds \\
\frac{x^i_\beta(t)}{x^i_\alpha(t)} &= \exp\left[\int_0^t \left(u^i(\beta, x^{-i}(t)) - u^i(\alpha, x^{-i}(t))\right) ds\right] \frac{x^i_\beta(0)}{x^i_\alpha(0)} \\
\frac{x^i_\beta(t)}{x^i_\alpha(t)} &\leq \exp\left[\int_0^t U_{\alpha\beta} ds\right] \frac{x^i_\beta(0)}{x^i_\alpha(0)} = \exp\left[U_{\alpha\beta} t\right] \frac{x^i_\beta(0)}{x^i_\alpha(0)}
\end{aligned}$$

Since $\lim_{t\to\infty} \exp\left[U_{\alpha\beta} t\right] = 0$ *the result follows.* ■

- This can be easily generalized to iterated deletion ($x^i_\beta(t)$ is small for t large),

- Also to mixed strategies σ^i (function $V^i_{\sigma^i}(t) = \prod_{\alpha=1}^{n_i} x_i(t)^{\sigma^i_\alpha}$ goes to zero).

- It DOES NOT generalize to weakly dominated strategies. See example.

Proposition 12 *Let σ^* be a Nash equilibrium of game G. Then, the state $x^* = \sigma^*$ is a stationary state of the replicator dynamics for that game, that is, for all $i \in N$ and $\alpha \in S^i$ we have $\frac{\partial x^i_\alpha(t)}{\partial t} = 0$*

Proof. *Remember in the replicator dynamics*

$$\frac{\partial x^i_\alpha(t)}{\partial t} = x^i_\alpha(t)\left[u^i(\alpha, x^{-i}(t)) - u^i(x^i(t), x^{-i}(t))\right]$$

Then, if a strategy $x^{i*}_\alpha = 0$, $\frac{\partial x^i_\alpha(t)}{\partial t} = 0$. For strategies with $x^{i*}_\alpha > 0$, since this is a Nash equilibrium the payoff α have to be identical to the payoff for x^{i*}, thus $u^i(\alpha, x^{-i*}) = u^i(x^{i*}, x^{-i*})$ and the result follows. ■

Remark 13 *Let a pure strategy $x^i = (0, ..., 1, ..., 0)$. It is immediate that $\frac{\partial x^i_\alpha(t)}{\partial t} = 0$ for all $\alpha \in S^i$. So any pure strategy profile is stationary for the replicator dynamics.*

Definition 14 *Let* $\frac{\partial x(t)}{\partial t} = F(x(t))$ *be a dynamical system in a set* $W \in \Re^m$. *A stationary point* x^* *(i.e. a point with* $F(x^*) = 0$*) is* asymptotically stable *if:*

I *For all neighborhood* U_1 *of* x^*, *there is some other neighborhood of it* U_2 *such that for any solution of the differential equation,* $x(.)$, *if* $x(0) \in U_2$ *then* $x(t) \in U_1$ *for all* $t > 0$.

II *There is some neighborhood* V *of* x^* *such that for any solution* $x(.)$, *if* $x(0) \in V$ *then* $\lim_{t \to \infty} x(t) = x^*$

Paths starting close to x^* remain close to it, and all paths converge to x^*.

Proposition 15 *If x^* is a an asymptotically stable point of the replicator dynamics, it is a Nash equilibrium.*

Proof. *Suppose not. Then there exists x^* which is asymptotically stable, but not a NE.*

Thus there is a player i and strategy α such that for some β with $x_\beta^{i*} > 0$

$$u^i(\alpha, x^{-i*}) > u^i(\beta, x^{-i*})$$

By condition II in the definition, there is a V and $x(0)$ such that for $\lim_{t\to\infty} x(t) = x^*$.
Thus, continuity of $u^i(.)$, guarantees there is a t^* such that for $t > t^*$ there is $\delta > 0$ with $u^i(\beta, x^{-i}(t)) - u^i(\alpha, x^{-i}(t)) < -\delta$. Therefore, by integrating 3

on can show (as in proof of Proposition 12):

$$\frac{x^i_\beta(t)}{x^i_\alpha(t)} \leq \exp\left[\int_0^t -\delta ds\right] \frac{x^i_\beta(0)}{x^i_\alpha(0)} = \exp\left[-\delta t\right] \frac{x^i_\beta(0)}{x^i_\alpha(0)}$$

Since $\lim_{t\to\infty} \exp[-\delta t] = 0$, $\lim_{t\to\infty} x^i_\beta(t) = 0$, *which is a contradiction.* ■

So asymptotically stable points have to be Nash equilibria. Now we focus on the relationship between ESS and the replicator dynamics.

Proposition 16 (Hofbauer, Schuster and Sigmund 1980) *Let $\sigma^* \in \Delta^{n-1}$ be an ESS. Then, the state $x^* = \sigma^*$ is an asymptotically stable state of the replicator dynamics.*

Proof. One typical way of showing a state is asymptotically stable is finding a Liapunov function $\psi : \Delta^{n-1} \longrightarrow \Re$ for that state. Such functions have two key properties:

(a) ψ has a unique maximum at x^* in a neighborhood V of x^*.

(b) $\psi(x(t))$ increases for any path $x(t)$ starting at $x(0) \in V$.

(a) and (b) guarantee I and II hold (the increasing part guarantees not to go far, the maximum guarantees convergence).

Let $\psi(x) = \sum_{q=1}^{n} x_q^* \log x_q$. This function obviously satisfies (a).

Then

$$\begin{aligned}
\frac{\partial \psi(x(t))}{\partial t} &= \sum_{q=1}^{n} x_q^* \frac{\partial x_q(t)}{\partial t} \frac{1}{x_q(t)} \\
&= \sum_{q=1}^{n} x_q^* \left(\sum_{r=1}^{n} a_{qr} x_r(t) - x(t) A x(t) \right) \\
&= x^* A x(t) - x(t) A x(t)
\end{aligned}$$

But notice that it is almost immediate from the definition of ESS (plus continuity) that for x close enough to x^* we must have $x^* A x > x A x$, thus $x^* A x(t) - x(t) A x(t) > 0$ and thus (b) is satisfied. ■

To finish, notice that the feature that has been used most is:

$$\frac{\partial x^i_\alpha(t)}{\partial t}\frac{1}{x^i_\alpha(t)} - \frac{\partial x^i_\beta(t)}{\partial t}\frac{1}{x^i_\beta(t)} = u^i(\alpha, x^{-i}(t)) - u^i(\beta, x^{-i}(t))$$

but the only thing that mattered here is that we wanted

$$\frac{\partial x^i_\alpha(t)}{\partial t}\frac{1}{x^i_\alpha(t)} - \frac{\partial x^i_\beta(t)}{\partial t}\frac{1}{x^i_\beta(t)} > 0$$

whenever

$$u^i(\alpha, x^{-i}(t)) - u^i(\beta, x^{-i}(t)) > 0$$

Thus we can formulate an alternative requirement on the dynamics. Let

$$\frac{\partial x^i_\alpha(t)}{\partial t} = F^i_\alpha(x(t)) \text{ for all } i \in N \text{ and } \alpha \in S^i$$

Provided that

$$\sum_{\alpha=1}^{n_i} F^i_\alpha(x) = 1 \text{ for all } i \in N \text{ and } x$$

and the initial conditions were such that $x^i(0) \in \Delta^i$ for all $i \in N$, the dynamics will not leave the simplex.

Definition 17 *The dynamics* $\frac{\partial x(t)}{\partial t} = F(x(t))$ *are payoff monotonic if:*

$$\frac{F^i_\alpha(x)}{x^i_\alpha} - \frac{F^i_\beta(x)}{x^i_\beta} > 0 \Leftrightarrow u^i(\alpha, x^{-i}) - u^i(\beta, x^{-i}) > 0$$

Remark 18 *Payoff monotonic dynamics can be easily seen to satisfy suitably modified versions of the theorems above.*

Example 19 *Let the game G of Sjöstrom (1994):*

m_3^0:

	m_2^0	m_2^1
m_1^0	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{1}{3}, 0, \frac{1}{3}$
m_1^1	$0, \frac{1}{3}, \frac{1}{3}$	$0, 0, \frac{1}{3}$

m_3^1:

	m_2^0	m_2^1
m_1^0	$0, 0, \frac{1}{2}$	$0, \frac{1}{3}, \frac{1}{2}$
m_1^1	$\frac{1}{3}, 0, \frac{1}{2}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

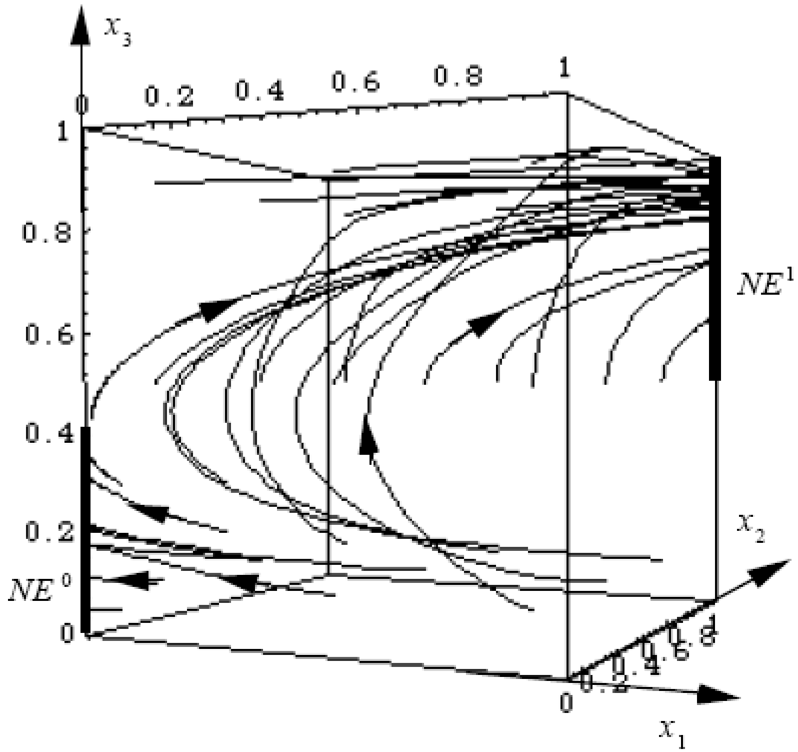
This is *weakly dominance solvable*, as m_3^1 is weakly dominated for player 3. After m_3^1 is eliminated, m_1^0 and m_2^0 are strictly dominant.

In the figure one can see that weakly dominated strategies need not go to zero.

The intuition is that the proof requires the difference in payoffs for two strategies $U_{\alpha\beta} > 0$ for all t.

Dynamics - The replicator dynamics *(19/20)*

That is not guaranteed with weak domination, as the x^{-i} against which a certain strategy does badly could go to zero.



x_3
0
0.2
0.4
0.6
0.8
1
1
0.8
0.6
0.4
0.2
0
NE^1
NE^0
x_2
x_1
0

(Kandori, Mailath, and Rob 1993, Young 1993)

- Let 2×2 coordination game (i.e. (A, A) and (B, B) are equilibria):

	A	B
A	$X_1,\ X_2$	$U_1,\ U_2$
B	$V_1,\ V_2$	$Y_1,\ Y_2$

- In 2X2 games *risk dominant equilibrium* - largest product of deviation payoffs (losses from mistakes).

- (A, A) is *risk dominant* if:

$$(X_1 - V_1)(X_2 - U_2) > (Y_1 - U_1)(Y_2 - V_2)$$

- In a symmetric game this is just $(X_1 - V_1) > (Y_1 - U_1)$

- Example (B, B) *risk dominant* :

	A	B
A	$100, 100$	$0,\ 99$
B	$99,\ 0$	$10, 10$

$$(100 - 99)(100 - 99) < (10 - 0)(10 - 0)$$

- Interpretation: If I think mistakenly equilibrium is (A, A) but in reality it is (B, B), I lose 10.
  If I think mistakenly equilibrium is (B, B) but in reality it is (A, A), I lose 1. Thus, (B, B) looks better in a world of strategic uncertainty.

- In a symmetric game the mixed equilibrium is the mixed strategy: $(q, 1-q) = \left(\frac{Y_1 - U_1}{Y_1 - U_1 + X_1 - V_1}, \frac{X_1 - V_1}{Y_1 - U_1 + X_1 - V_1}\right)$.

- Thus the (A, A) equilibrium is *risk dominant* if the mixed equilibrium is "*further*" from it (i.e. if $q < \frac{1}{2}$).

- 2×2 symmetric coordination game with (A, A) risk dominant.

- N players.

- Mixed equilibrium $(q^*, 1 - q^*)$, $(q^* < \frac{1}{2}$, since (A, A) risk dominant).

Step 0 *State space*: State θ is the number of A players.

Step 1 *Deterministic dynamics*: $\theta_{t+1} = BR(\theta_t) = \begin{cases} N & \\ \theta_t & \text{for} \\ 0 & \end{cases} \begin{matrix} u_A(\theta_t) > u_B(\theta_t) \\ u_A(\theta_t) = u_B(\theta_t) \\ u_A(\theta_t) < u_B(\theta_t) \end{matrix}$

- Two stable and ESS steady states: $\theta = N$ and $\theta = 0$.

Step 2 *Add noise verify ergodicity*: Every player can "mutate" with probability $\frac{1}{2\varepsilon}$ at every point in time.

- Ergodic, because irreducible (all states reachable from one another) and recurrent (coming back a.s.), aperiodic (not in a $k-$period cycle for any k) states.

- Existence of a unique invariant distribution follows from *ergodicity*.

Step 3 *Compute invariant distribution*: Let N^* least integer bigger than Nq^*. If $\theta_t \geq N^*$, best response is A.

Proposition 20 *If N large enough so $N^* \leq \frac{1}{2}N$, then limit ϕ^* of invariant distribution is point mass on $\theta = N$ (all use A).*

- $D_A = \{\theta_0 \geq N^*\}$, $D_B = \{\theta_0 < N^*\}$ basin of attraction of A and B (i.e. states θ_t generating the same distribution in $t+1$).

- Sufficient to know transitions between states in D_A and D_B.

- Let $q_{BA} = \Pr(\theta_{t+1} \in D_B | \theta_t \in D_A)$, $q_{AB} = \Pr(\theta_{t+1} \in D_A | \theta_t \in D_B)$. Then:

$$\begin{bmatrix} \phi_A \\ \phi_B \end{bmatrix} = \begin{bmatrix} 1 - q_{BA} & q_{AB} \\ q_{BA} & 1 - q_{AB} \end{bmatrix} \begin{bmatrix} \phi_A \\ \phi_B \end{bmatrix}$$

- Thus:

$$\frac{\phi_B}{\phi_A} = \frac{q_{BA}}{q_{AB}}$$

Stochastic evolutionary dynamics *(6/15)*

- q_{BA} we want to end with all B. Most likely we need $N-N^*$ "mutations"., probability $\binom{N}{N^*}\varepsilon^{N-N^*}(1-\varepsilon)^{N^*}$.

- q_{AB} we want to end with all A. Most likely we need N^* "mutations", probability $\binom{N}{N^*}\varepsilon^{N^*}(1-\varepsilon)^{N-N^*}$.

- Thus

$$\frac{\phi_B}{\phi_A} \simeq \frac{\binom{N}{N^*}\varepsilon^{N-N^*}(1-\varepsilon)^{N^*}}{\binom{N}{N^*}\varepsilon^{N^*}(1-\varepsilon)^{N-N^*}}$$

- Since $N^* \leq \frac{1}{2}N$

$$\lim_{\varepsilon \to 0} \frac{\phi_B}{\phi_A} = 0$$

- ε_A could be different from ε_B but $\varepsilon_A/\varepsilon_B$ cannot tend to zero (Bergin and Lippman).

GENERALIZATION

- Game with N strategies.
- Dynamics spend most time in $\omega-$limit sets (roughly limits of unperturbed dynamics).
- Ω set of $\omega-$limit sets, with K elements.
- Let a sequence of ergodic Markov chains $P^{\varepsilon} \rightarrow P$
- To know ϕ^{ε} we need to know the transitions $P^{\varepsilon}_{\theta_i \theta_j}$ for every two states θ_i, θ_j .

- *Cost of transition*: its order in ε. That is, for every two states θ_i, θ_j,

$$c(\theta_i|\theta_j) \equiv \lim_{\varepsilon \to 0} \left(\frac{\log P^{\varepsilon}_{\theta_i \theta_j}}{\log \varepsilon} \right)$$

that is the number of mutations from θ_j to θ_i

- An $\theta_i\theta_j$*-path* is a sequence of states $\overrightarrow{\theta}_{ij} = (\theta_j, \theta_2, ..., \theta_i)$ that begins in θ_j and ends in θ_i.

- *Minimal cost of a transition* between θ_i and θ_j is denoted $\overrightarrow{c}(\theta_i|\theta_j) = \sum_{\overrightarrow{\theta}_{ij}} c(\theta_i|\theta_j)$

- For each $\omega' \in \Omega$, an ω'*-tree*, denoted by h, is a complete directed graph with K vertices (one for each $\omega \in \Omega$), ending in ω'.

- Let $h(\omega)$ be the succesor of ω in an h tree.
- Let H_ω be the set of all ω-trees.
- The *resistance* of an $\omega-$tree h is $r(h) = \sum_{\omega' \in \Omega \backslash \omega} \overrightarrow{c}(h(\omega')|\omega)$

Proposition 21 *(Young 1993) The limit distribution ϕ^* is concentrated in ω-limit sets which solve*

$$\min_{\omega \in \Omega} \min_{h \in H_\omega} r(h)$$

Example 22 *A game with two non-trivial $\omega-$limit sets*

	L	*M*	*R*
U	*4,4*	*4,4*	*0,0*
M	*4,4*	*4,4*	*0,0*
D	*0,0*	*0,0*	*3,3*

Example 23 *A game with three $\omega-$limit sets and non-trivial minimal resistance trees. The 2×2 symmetric coordination game with (A, A) risk dominant played in an N-player ring with 2 nearest neighbors.*

Step 0 *State space*: $\Theta = \{A, B\}^N$.

Step 1 *Deterministic dynamics:*

- All neighbors of one A, play A at $t+1$, so number of A never decreases.
- Two adjacent A, lead to all A.

- Three steady states:

  1. ω_1–All A. *Basin*: At least all states with two adjacent A, and any with a string $ABBA$.

  2. ω_2–All B. *Basin*: Just itself.

  3. ω_3–(When N even). Cycle between $ABAB$..., and $BABA$....*Basin*: At least those two states.

Step 2 *Add noise verify ergodicity*: Simple from ε-mutations..

Step 3 *Compute invariant distrbution*: Let N^* least integer bigger than Nq^*. If $\theta_t \geq N^*$, best response is A.

$$\min_{\omega \in \Omega} \min_{h \in H_\omega} r(h) = 2$$

and the arg min is the h tree - $\omega_2 \rightarrow \omega_3 \rightarrow \omega_1$

- Given ω_2, a mutation leads to ω_3 $(BBBB^{1\text{ mut}} \rightarrow ABBB \rightarrow BABA \rightarrow ABAB \rightarrow BABA \rightarrow ...)$.

- Given ω_3, a mutation leads to ω_1 $(ABAB^{1\text{ mut}} \rightarrow AAAB \rightarrow AAAA \rightarrow AAAA \rightarrow ...)$

- The other trees $\omega_3 \rightarrow \omega_2 \rightarrow \omega_1$ and $\omega_3 \rightarrow \omega_1 \leftarrow \omega_2$ are more costly (1 mutation from ω_2 does not go to ω_1).

- The ω_2−trees and ω_3−trees have cost higher than 2.

  1. Since number of A never decreases in deterministic, need at least N mutations from ω_2 to ω_1.

  2. For same reasons at least $N/2$ mutations from ω_3 to ω_1.

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Chapter 1

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