A Appendix B: Further proofs (not for publication)

1. Proof of Theorem 2

We determine first the consistent beliefs of buyers associated with the sellers' reporting strategy in equation (2). Then we study traders' strategies in each stage of the game, and establish their optimality given these beliefs.

Beliefs and behavior in the auction With the message strategy (2) there is one out-of-equilibrium message, the empty message 0. Thus, we can find the beliefs for an uninformed buyer, say buyer B_j, who receives a report from an informed buyer, say buyer B_i, using Bayes' rule in all cases, except when they receive message 0:

- When B_j receives from B_i a message $m_i = \theta_j$ he knows for sure that he likes the object (the message is truthful). That is, $\Pr(v = \theta_j | m_i = \theta_j) = 1$.

- When B_j receives a message announcing a variety different from his type, B_j knows again for sure that the message is truthful, and hence that he does not like the object, so that $\Pr(v = \theta_j | m_i \neq \theta_j, m_i \neq 0) = 0$.

- When B_j receives the empty message 0, beliefs are not pinned down by Bayes' rule, in this case we assume that $\Pr(v = \theta_1 | m_i = 0) = 1$.[34]

Finally, the buyers who neither acquired information directly, nor indirectly by purchasing it in the market, have beliefs equal to their prior beliefs. That is, $\Pr(v = \theta_j) = 1/K$ for all j. The beliefs of a buyer who is purchasing two (or more) distinct reports from two (or more) informed buyers are similarly obtained.

Given these beliefs, the optimality of a 'truthful bidding strategy' for each trader can be easily established similarly to the proof of Theorem 1 in Appendix A.

Stage 4: Behavior in the message stage

We show next that the reporting strategy we postulated for a seller of information is indeed optimal for such trader. A key element in the argument is that, by changing the message strategy, he cannot affect the outcome of the auction in his favor.

[34] The specific variety on which buyers' beliefs put all probability mass when $m_i = 0$ is immaterial, what is important is that the object is believed to be of one given variety with probability 1.

Seller: The novel deviation which needs to be considered for the seller of information is, when he likes the object, to announce the empty message 0. If he does that, the distribution of bids he will face in the auction is such that he will have to pay 1 if any buyer of information likes the object of variety 1, (i.e. if there is some j with $\theta_j = 1$, an event with probability $1 - [(K-1)/K]^{N-J-1}$, with $N - J - 1$ being the number of buyers of information) and $1/K$ otherwise. This is exactly the same distribution of bids he faces in equilibrium, in which case the price paid is 1 if there is some buyer j with $\theta_j = v$, and $1/K$ otherwise.

Stages 3 and 2: Purchase and sale of information.

Here there is no difference with the proof of Theorem 1, since the gains from purchasing information, given the reporting strategy in (2), are unchanged.

Stage 1: Information acquisition

Having determined the benefits for a buyer of acquiring information, we immediately find when this is profitable:

Claim 4 *When $c \geq c^{true}$, the cost of acquiring information exceeds the maximal gains that a monopolist seller of information can get from the sale of information ($[(N-2)/K][(K-1)/K]^{N-1}$) plus the gains from obtaining the object in the auction ($1/K[(K-1)/K]^{N-1}$), hence no buyer chooses to acquire information. On the other hand, when $c \leq c^{true}$ one buyer always acquires information.*

Proof of Claim 4. The distribution of prices in the auction facing the seller of information is given by a price of 1 with probability $1 - [(K-1)/K]^{N-J-1}$, with $N - J - 1$ being the number of buyers of information and $1/K$ otherwise. Since by the same argument as in the proof of Theorem 1 it is optimal to set the price of information so that $J = 1$ and the price at which information is sold is also given by the same expression, we have

$$\pi_{B_1} = \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-2}\left(1 - \frac{1}{K}\right) + (N-2)\min\left\{c, \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1}\right\} - c.$$

A necessary condition for $\pi_{B_1} \geq 0$ is that $c > \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1}$, in which $\pi_{B_1} \geq 0$ reduces to:

$$c \geq \frac{1}{K}\left(\frac{K-1}{K}\right) + (N-2)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1} = c^{true}. \tag{A.1}$$

This completes the proof of Theorem 2. ■

2. Equilibrium with intense competition among buyers: $K < N - 2$

The properties of the equilibrium when the seller of information is an informed buyer are as in Theorem 1 with two main exceptions: the threshold for information to be acquired is different, given by $c^{I'} \equiv \frac{1}{K}\left(\frac{K-1}{K}\right) + \left(\frac{K-1}{K}\right)^{K+1}$, and the monopolist seller of information typically chooses to set a price that is sufficiently high that more than one buyer decides to remain uninformed. Hence, when neither the seller nor any buyer of information like the object, this is gained by one, randomly chosen, uninformed buyer, at a price $1/K$. The equilibrium allocation is then now not always ex post efficient. Additionally, the threshold $c^{I'}$ still induces inefficient information acquisition. We present below the main changes in the proof with respect to that of Theorem 1.

Claim 5 *When $K < N-2$ the* ***revenue from the sale of information*** *for a monopolist seller of information is maximized by setting the price p at a level such that the number J of agents who do not buy information is larger than one ($J = N - K - 1$ or, when c is low, J equals what is essentially the highest value of J that is sustainable).*

Proof of Claim 5 Suppose first that

$$c \geq \frac{1}{K}\left(\frac{K-1}{K}\right)^2,$$

so that all configurations are sustainable, since (15) holds for all $J > 1$, and $p(J) = \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}$ for all $J = 1, .., N-2$. We now find under what conditions the revenue from the sale of information is higher in configuration J than in $J+1$:

$$\begin{aligned} (N-(J+1))p(J) &\geq (N-(J+2))p(J+1) \qquad \text{(A.2)} \\ &\iff (N-(J+1))\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J} \geq (N-(J+2))\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)} \\ &\iff \frac{(N-(J+1))}{(N-(J+2))} \geq \frac{K}{K-1} \\ &\iff K-1 \geq N-(J+2). \end{aligned}$$

When $K < N-2$ this condition is only satisfied for $J \geq N-K-1$, while for $J < N-K-1$ the revenue is always higher at $J+1$ than at J. Hence if $K < N-2$ the maximum obtains at $J = N-K-1$, larger than 1.

Consider next the case where

$$\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-\bar{J}+1} \leq c < \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-\bar{J}} \qquad \text{(A.3)}$$

for some $\bar{J} \in \{2, .., N-2\}$, so that only configurations $J = 1, .., \bar{J}$ are sustainable, $p(J) = \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}$ for $J = 1, .., \bar{J}-1$ and $p(\bar{J}) = c$. When $K < N-2$, again by the same argument as in equation (A.2), the revenue is higher at $J^* = \min\{N-K-1, \bar{J}-1\}$ than at any other $J = 2, .., \bar{J}-1$. Thus it still suffices to compare the revenue in configuration $J = J^*$ with the one at $J = \bar{J}$ (where $p(\bar{J}) = c$), for all $\bar{J} \in \{2, .., N-2\}$. If $J^* = N-K-1$, the maximal revenue obtains at $J = J^*$ since:

$$\begin{aligned}(N - J^* - 1)p(J^*) &= \frac{1}{K}\left(\frac{K-1}{K}\right)^{K+1} \geq \big(N - (\bar{J}+1)\big)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-\bar{J}} \\ &\geq \big(N - (\bar{J}+1)\big)p(\bar{J}) = \big(N - (\bar{J}+1)\big)c,\end{aligned}$$

where the first inequality holds by (16) and the second one by (17). On the other hand, if $J^* = \bar{J}-1$ the maximum revenue is attained either at $J = \bar{J}-1$ or at $\bar{J}$.[35] ■

Claim 6 *When $K < N-2$ a buyer acquires information in equilibrium when $c \leq c^{I'}$.*

Proof of Claim 6 Information is not gathered in equilibrium when $0 \geq \pi^1_{B_1}$. When (A.3) holds, we have

$$\pi^1_{B_1} = \frac{1}{K}\left(\frac{K-1}{K}\right) + \max\left\{\frac{1}{K}\left(\frac{K-1}{K}\right)^{K+1}, \big(N - (\bar{J}+1)\big)c\right\} - c$$

which is clearly positive. Hence, for information not to be acquired it must be that

$$c \geq \frac{1}{K}\left(\frac{K-1}{K}\right)^2.$$

In this case by Claim 5 the optimal value of J obtains at $N-K-1$ and thus

$$\begin{aligned}\pi^1_{B_1} &= \frac{1}{K}\left(\frac{K-1}{K}\right) + (N - (N-K-1+1))\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(N-K-1)} - c \\ &= \frac{1}{K}\left(\frac{K-1}{K}\right) + \left(\frac{K-1}{K}\right)^{K+1} - c\end{aligned}$$

and so $0 \geq \pi^1_{B_1}$ iff

$$c \geq \frac{1}{K}\left(\frac{K-1}{K}\right) + \left(\frac{K-1}{K}\right)^{K+1} = c^{I'}$$

■

[35] In particular, it will be at $\bar{J}-1$ when c is sufficiently closer to $\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-\bar{J}+1}$ than to $\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-\bar{J}}$ and at $\bar{J}$ otherwise.

Claim **7** *When $K < N-2$ the equilibrium is inefficient (there is underinvestment in information) when*

$$\left(\frac{K-1}{K}\right)\left(1-\left(\frac{K-1}{K}\right)^{N-1}\right) \geq c > c^{I\prime}.$$

Proof of Claim 7 Given our previous findings, it suffices to show the interval above is nonempty, that is

$$\left(\frac{K-1}{K}\right)\left(1-\left(\frac{K-1}{K}\right)^{N-1}\right) > c^{I\prime}$$

This is equivalent to:

$$\begin{aligned}
\left(\frac{K-1}{K}\right)-\frac{1}{K}\left(\frac{K-1}{K}\right) &= \left(\frac{K-1}{K}\right)^{2} > \left(\frac{K-1}{K}\right)^{N}+\left(\frac{K-1}{K}\right)^{K+1} \\
&\Leftrightarrow\ 1 > \left(\frac{K-1}{K}\right)^{N-2}+\left(\frac{K-1}{K}\right)^{K-1} > 2\left(\frac{K-1}{K}\right)^{K-1} \\
&\Leftrightarrow\ \frac{1}{2} > \left(\frac{K-1}{K}\right)^{K-1}
\end{aligned}$$

We show next that the term on the right hand side,

$$\left(\frac{K-1}{K}\right)^{K-1}$$

is decreasing. Let

$$f\left(K\right) \equiv \left(K-1\right)\ln\frac{K-1}{K}$$

$$\begin{aligned}
\frac{\partial f\left(K\right)}{\partial K} &= \ln\frac{K-1}{K}+\left(\frac{1}{K-1}-\frac{1}{K}\right)\left(K-1\right) \\
&= \ln\frac{K-1}{K}+\frac{1}{K} < -\frac{1}{K}-\frac{1}{K^{2}}+\frac{1}{K} = -\frac{1}{K^{2}} < 0
\end{aligned}$$

Since $K \geq 3$, to establish the result it suffices then to show that

$$\frac{1}{2} > \left(\frac{3-1}{3}\right)^{3-1} = \left(\frac{2}{3}\right)^{2} = \frac{4}{9},$$

always satisfied.■

3. Remaining elements of the Proof of Proposition 3.

With the message structure in (24), (25), we can find the beliefs of the uninformed buyers using Bayes' rule in all cases except when the empty message, 0, is received. Let n_l be the number of buyers in layer l and N_l the total number of buyers in layers 1 through l plus the seller of information B_1 : $N_l = \sum_{j=0}^{l} n_j$, where we adopt the convention $n_0 = 1$.

The beliefs of buyer B_j purchasing a report of type l are:

1. When B_j receives a message $m_l = \theta_j$ he knows for sure he likes the object. That is[36]

$$\Pr(v = \theta_j | m_l = \theta_j) = 1$$

2. On the other hand, when B_j receives a message $m_l \neq \theta_j$, this may happen either because $v = m_l$, or because the sender or somebody in an earlier layer $t < l$ likes the object, in which case the sender does not tell the truth and randomizes over the types which are absent from the population of buyers of information, including B_1. Given this:

$$\Pr(m_l = \theta_j) = \left(\frac{K-1}{K}\right)^{N_{l-1}} \frac{1}{K}, \quad \Pr(m_l \neq \theta_j) = 1 - \left(\frac{K-1}{K}\right)^{N_{l-1}} \frac{1}{K}$$

and

$$\begin{aligned}\Pr(v &= \theta_j | m_l \neq \theta_j) = \frac{\Pr(v = \theta_j \cap m_l \neq \theta_j)}{\Pr(m_l \neq \theta_j)} = \frac{\Pr(v = \theta_j) - \Pr(v = \theta_j \cap m_l = \theta_j)}{\Pr(m_l \neq \theta_i)} \\ &= \frac{\frac{1}{K} - \left(\frac{K-1}{K}\right)^{N_{l-1}} \frac{1}{K}}{1 - \left(\frac{K-1}{K}\right)^{N_{l-1}} \frac{1}{K}} = \frac{1 - \left(\frac{K-1}{K}\right)^{N_{l-1}} \frac{1}{K}}{K - \left(\frac{K-1}{K}\right)^{N_{l-1}} \frac{1}{K}}\end{aligned}$$

3. If B_j receives the empty message $m_l = 0$, we assume that beliefs are as in point 2. above (message 0 is only received off the equilibrium path, hence beliefs can be set arbitrarily in this case).

The beliefs of buyers who do not purchase information nor acquire it directly are then unchanged:

$$\Pr(v = \theta_j) = \frac{1}{K}, \quad \Pr(v \neq \theta_j) = \frac{K-1}{K}$$

Behavior in the auction We begin again by analyzing the last stage of the game, where we characterize the agents' behavior in the auction.

1. When an agent B_j receives a message $m_l = \theta_j$, by the same argument as in Theorem 1 the only weakly undominated strategy consists in making a bid equal to the agent's posterior beliefs about his valuation of the object. That is his bid is equal to 1.

[36]Even though a buyer receiving message m_l also receives messages $m_j, j > l$, given the nested structure of the information these reports have no additional informational value and can be ignored in the derivation of the conditional expectations.

2. In contrast, when an agent B_j receives a message $m_l \neq \theta_j$ (or the empty message $m_l = 0$, something that only happens off the equilibrium path) the optimal bid is not equal to his posterior belief $\Pr(v = \theta_j | m_i \neq \theta_j)$. An agent may receive in fact this signal for two reasons. Either $v = \theta_i$ for some B_i in an earlier layer, in which case he cannot win the object in the auction as agent B_i, who is better informed, will make a higher bid, or $v \neq \theta_i$ for all B_i in earlier layers and $v \neq \theta_j$.In the latter case he may indeed win the auction by bidding a positive price, but that will entail a negative surplus. This implies that the optimal bid is zero when $m_l \neq \theta_j$ since the information conveyed by winning the auction should also be taken into account (because of the correlation in the information of bidders): zero is the expected value of the object conditional on winning the auction.

3. Finally, the buyers who do not listen to the reports of the directly informed seller do not suffer from any problem of correlation in information. Thus, the only weakly undominated strategy optimal bid in that case is again equal to $\Pr(v = \theta_j) = \frac{1}{K}$.

Behavior in the message game Next we consider the agents' behavior in the message game. Given the reporting strategy described in equations (24) and (25) of the seller of information, we now show that the optimal reporting strategy of every buyer who is purchasing the information is to truthfully report his type. Next we show that the reporting strategy of the seller of information is also optimal for this agent.

Optimality of truthful reporting for the buyers of information First of all, notice that a change in the message strategy of a buyer of information can only change the outcome of the auction, not the price paid for information. A deviation by the buyer of information consists in reporting anything other than his type. We divide this discussion in two cases.

1. Let the buyer of information be agent B_j, and suppose he purchased report l. If $v = \theta_1$ or $v = \theta_i$, for some buyer of information B_i in some layer $t < l$, either B_1 or B_i will bid 1 in the auction, no matter what is the report of B_j. Hence there is no possibility for B_j to obtain any extra surplus by misreporting his type or reporting the empty message.

2. If $v \neq \theta_1$ and $v \neq \theta_i$ for all buyers of information B_i in any layer $t < l$, then (25) prescribes that $m_t = v$ for all $t = 1, .., l$, no matter what is the report sent by agents

who purchase report l. The reports of agents in layer l only affect the reports sent by B_1 to agents in layers $t' > l$. Under truthful reporting by B_j all buyers in layers $t > l$ receive a message inducing them to bid zero when $v = \theta_j$. The effect of B_j's misreporting his type is that buyers in layers $t > l$ will receive instead a message which will induce them either to bid zero, or a positive amount, thus lowering, at least weakly $B_j's$ expected gains from the auction. So misreporting or reporting the empty message is not optimal for B_j.

Optimality of the message strategy for the seller of information As for the buyers of information, a change in the seller's reporting strategy has no effect on the revenue from the sale of information, only on the outcome of the auction.

1. When $v = \theta_1$, i.e. the seller of information likes the object, he can deviate and send, for some $l \in \{1, .., L\}$ a message $m_l = \theta_k$ for some buyer B_k purchasing a report of some type. In this case, the bid of B_k will equal 1, and the seller of information has to pay more for the object than if he had followed the reporting strategy (24) and (25), so that deviation is clearly not optimal. He can also deviate by reporting the empty message, but since that does not change the receivers' beliefs, this does not change his payoffs.

2. When $v \neq \theta_1$, the seller is not interested in the object. Since any deviation from the messages prescribed by his reporting (24) and (25) only changes the outcome in the auction, in which he is not interested, the seller can never gain from such a deviation.

Sale of information

Payoffs for the monopolist selling information to all uninformed traders The single informed trader acts as a monopolist in the market for information. The maximal rent he can extract from the $N-1$ uninformed buyers purchasing information from him, for any given layer structure, is determined by comparing the payoff a buyer can get by acquiring the information of the quality associated to the layer he is in with the alternative payoff he could get by not purchasing the information.

To evaluate the payoff of a buyer in layer l we need to distinguish the case (i) where in layer l there is a single buyer from the case (ii) where in that layer there is more than a single buyer. In case (i) the buyer in layer l will always get the commodity when he likes it and no other buyer in the layers above likes it (an event with probability $\left(\frac{K-1}{K}\right)^{N_{l-1}} \frac{1}{K}$), and the price he will pay will be the second highest bid after his, equal to 0[37]. On the other hand in case (ii) the same is true only when the buyer likes the object and no other buyer in the layers above *as well as in his own layer* l likes it (an event with probability $\left(\frac{K-1}{K}\right)^{N_l-1}\frac{1}{K}$). When some other buyer in layer l likes the object, the buyer will get the object with some probability but will pay an amount equal to his valuation so that his payoff will be zero. Note that in case (i) we have $N_l = N_{l-1}+1$; hence in both cases the payoff in the auction for a buyer in layer l is given by the following expression:

$$\left(\frac{K-1}{K}\right)^{N_l-1}\frac{1}{K}$$

If we subtract from this the price paid to acquire the information, we obtain the following expression for the total payoff to a buyer B_i of acquiring information of quality l at a price p_l, when all the $N-1$ uninformed buyers purchase information of some quality from the single seller of information:

$$\pi_{B_i} = \left(\frac{K-1}{K}\right)^{N_l-1}\frac{1}{K} - p_l.$$

The buyer's alternative payoff if he chooses not to buy any information and hence remain uninformed, is given by $\pi_u = \left(\frac{K-1}{K}\right)^{N-1}\frac{1}{K}$. Alternatively, the buyer could also choose to acquire directly the information, after the market for information closes, as a cost c, in which case his payoff would be $\pi_c = \left(\frac{K-1}{K}\right)^{N-1}\frac{1}{K} - c$. Note that both π_u and π_c are independent of the number L of layers. Also, $\max\{\pi_u, \pi_c\} = \pi_u = \left(\frac{K-1}{K}\right)^{N-1}\frac{1}{K}$.

The maximal rent the monopolist selling the information can extract from buyer B_i, and hence the maximal value of the price he can charge, is thus given by

$$p_l = \frac{1}{K}\left(\frac{K-1}{K}\right)^{N_l-1} - \max\{\pi_u, \pi_c\} = \frac{1}{K}\left(\left(\frac{K-1}{K}\right)^{N_l-1} - \left(\frac{K-1}{K}\right)^{N-1}\right) \tag{A.4}$$

The total payoff of B_1, the informed buyer, from gaining the object when $v = \theta_1$ and from selling differentiated information to all the $N.-1$ other buyers is thus given by:

$$\pi_{B_1} = \frac{1}{K} + \sum_{l=1}^{L} n_l p_l - c \tag{A.5}$$

where p_l is as in expression (A.4). Note that the second term depends on the distribution of buyers across layers, i.e. on all the values n_l, $l = 0, 1, .., L$. To obtain the optimal differentiation of information, that is the optimal distribution of buyers across the various layers, we only need to consider then the effects of this distribution on the revenue from the sale of information.

[37] Note that in this case the buyer will be the only one receiving a message equal to his own type.

Lemma 2 *The optimal differentiation of information when information is sold to all $N-1$ uninformed buyers is to have $N-1$ layers of information.*

The proof is the same as for Lemma 1 in Appendix A. Let us denote the payoff of the seller of information in this possible equilibrium configuration, where there are no uninformed traders, by $\pi_{B_1}(0)$. From (A.5), setting $n_l = 1$ for all l and replacing p_l with the expression (A.4) we get:

$$\begin{aligned}\pi_{B_1}(0) &= \frac{1}{K}\left(1+\sum_{l=1}^{L} n_l \left(\left(\frac{K-1}{K}\right)^{N_l-1} - \left(\frac{K-1}{K}\right)^{N-1}\right)\right) - c \\ &= 1-\left(\frac{K-1}{K}\right)^{N-1} - (N-2)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1} - c\end{aligned}$$

Payoffs for the monopolist selling information to all but one uninformed trader The maximal rent the monopolist can extract in this case from the $N-2$ buyers purchasing information from him, for any given layer structure, is again determined by comparing the payoff a buyer can get by acquiring the information of the quality associated to the layer he is in with the alternative payoff he could get by not purchasing the information.

In determining the payoff of a buyer in layer l the only difference with respect to the previous case is that the price at which the buyer in layer l will get the object when he likes it and no other buyer in the layers above likes it is equal to the bid made by the uninformed buyer, given by $\frac{1}{K}$. Hence the payoff in the auction for a buyer in layer l is now given by:

$$\left(\frac{K-1}{K}\right)^{N_l-1}\frac{1}{K}\left(1-\frac{1}{K}\right) = \left(\frac{K-1}{K}\right)^{N_l}\frac{1}{K}$$

The total payoff for buyer B_i of acquiring information of quality l, at a price p_l, when $N-2$ of the uninformed traders purchase information of some quality from the seller of information and one trader remains uniformed, is

$$\pi_{B_i} = \left(\frac{K-1}{K}\right)^{N_l}\frac{1}{K} - p_l.$$

The buyer's alternative payoff if he chooses not to buy any information, and hence remains uninformed, is given by $\pi_u = 0$, while his payoff if he acquires directly the information, at a cost c, is $\pi_c = \left(\frac{K-1}{K}\right)^{N-2}\frac{1}{K}\left(1-\frac{1}{K}\right) - c = \left(\frac{K-1}{K}\right)^{N-1}\frac{1}{K} - c$.

The maximal rent the monopolist seller of the information can extract from buyer B_i, and hence the maximal value of the price he can charge, is thus given by

$$\begin{aligned} p_l &= \frac{1}{K}\left(\frac{K-1}{K}\right)^{N_l} - \max\{\pi_u, \pi_c\} \\ &= \min \frac{1}{K}\left(\left(\frac{K-1}{K}\right)^{N_l}, \left(\frac{K-1}{K}\right)^{N_l} - \left(\frac{K-1}{K}\right)^{N-1} + c\right) \end{aligned} \tag{A.6}$$

The total payoff of B_1, the informed buyer, from gaining the object when $v = \theta_1$ and from selling the information to the other buyers, is now:

$$\begin{aligned} \pi_{B_1}(1) &= \frac{1}{K}\left(1 - \frac{1}{K}\right) + \sum_{l=1}^{L} n_l p_l - c \\ &= \frac{1}{K}\left(1 - \frac{1}{K} + \sum_{l=1}^{L} n_l \min\left(\left(\frac{K-1}{K}\right)^{N_l}, \left(\frac{K-1}{K}\right)^{N_l} - \left(\frac{K-1}{K}\right)^{N-1} + c\right)\right) - c, \end{aligned}$$

where we substituted the expression of p_l in (A.6). We still find that the optimal differentiation of information is maximal:

Lemma 3 *The optimal differentiation of information when information is sold to $N-2$ uninformed buyers is to have $N-2$ layers of information..*

The proof is again the same as for Lemma 1 in Appendix A.

Regarding the value of the price at which information is sold, stated in (A.6), we should note that $\left(\frac{K-1}{K}\right)^{N_l-1} \geq \left(\frac{K-1}{K}\right)^{N_l-1} - \left(\frac{K-1}{K}\right)^{N-1} + c$ if and only if

$$\left(\frac{K-1}{K}\right)^{N-1} \geq c.$$

When does the monopolist prefer to sell to all, or all but one uninformed traders?

1. Consider first the case where

$$\left(\frac{K-1}{K}\right)^{N-1} \geq c$$

   Then we have:

$$\pi_{B_1}(1) = \left(\frac{K-1}{K}\right) - \left(\frac{K-1}{K}\right)^{N} - (N-2)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1} + (N-2)\frac{c}{K} - c$$

   Therefore the monopolist prefers to sell information to all uninformed traders when

$$\pi_{B_1}(0) = 1 - \left(\frac{K-1}{K}\right)^{N-1} - (N-2)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1} - c \geq \pi_{B_1}(1)$$

that is, iff

$$1-\left(\frac{K-1}{K}\right)^{N-1} \geq \left(\frac{K-1}{K}\right)-\left(\frac{K-1}{K}\right)^{N}+(N-2)\frac{c}{K}$$

which is satisfied when c is low enough so that:

$$1-\left(\frac{K-1}{K}\right)^{N-1} \geq (N-2)c$$

2. When $\left(\frac{K-1}{K}\right)^{N-1} < c$, we have

$$\begin{aligned}
\pi_{B_1}(1) &= \frac{1}{K}\left(1-\frac{1}{K}+\sum_{l=1}^{N-2}\left(\frac{K-1}{K}\right)^{l+1}\right)-c \\
&= \left(\frac{K-1}{K}\right)\left(\frac{1}{K}+\left(\frac{K-1}{K}\right)-\left(\frac{K-1}{K}\right)^{N-1}\right)-c
\end{aligned}$$

So $\pi_{B_1}(1) \geq \pi_{B_1}(0)$ iff

$$\begin{aligned}
\left(\frac{K-1}{K}\right)\left(1-\left(\frac{K-1}{K}\right)^{N-1}\right) &\geq 1-\left(\frac{K-1}{K}\right)^{N-1}-(N-2)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1} \quad \text{(A.7)} \\
&\Leftrightarrow (N-1) \geq \left(\frac{K}{K-1}\right)^{N-1} = \left(1+\frac{1}{K-1}\right)^{N-1}.
\end{aligned}$$

Notice that

$$\begin{aligned}
\left(1+\frac{1}{K-1}\right)^{N-1} &= \sum_{t=0}^{N-1}\binom{N-1}{t}\frac{1}{(K-1)^t} \\
&= 1+\sum_{t=1}^{N-1}\frac{(N-1)(N-2)...(N-t)}{(1\cdot 2\cdot ...\cdot t)\,(K-1)^t} \\
&\leq 1+\frac{(N-1)}{(K-1)}+\sum_{t=2}^{N-1}\frac{1}{2}\frac{(N-1)^t}{(K-1)^t} \\
&\leq 2+\frac{(N-2)}{2}=\frac{(N+2)}{2}
\end{aligned}$$

Since by assumption $N \geq 4$, we have $(N-1) \geq (N+2)/2$ and so (A.7) is always satisfied. Hence in this case $\pi_{B_1}(1) \geq \pi_{B_1}(0)$, it is always optimal to sell information to all but one trader.

The previous analysis shows that the configuration with all traders purchasing information from the monopolist seller is optimal when:

$$1-\left(\frac{K-1}{K}\right)^{N-1} \geq (N-2)c, \tag{A.8}$$

otherwise the monopolist prefers to sell information to all but one trader.

This completes the proof of Proposition 3.■

4. Equilibrium with one buyer of known type as seller of information (Section 6.3)

Since most of the elements in the characterization are similar to the proof of Theorem 1, we concentrate on its novel elements. Also, for simplicity we will concentrate in the exposition on the case where $c \geq \frac{1}{K}\left(\frac{K-1}{K}\right)^2$ - the relevant one to determine whether or not information is acquired in equilibrium and, thus, for efficiency - and where a slightly stronger condition is imposed on K, N: $K \geq N+1$.

We show first that the equilibrium where the seller of information B_1 follows the message strategy (1) is no longer the most profitable equilibrium for B_1 and other reporting strategies may yield a higher payoff. Next, we show that in any case the payoff of the seller of information is lower when his type is known than when it is unknown.

Under (1), given that in this case B_1's type is known, upon hearing the empty message the other buyers will know that $v = B_1$, and they will bid 1 if $v = \theta_i$ and 0 otherwise. Hence when $v = B_1$ and there are $J \geq 1$ buyers not purchasing information the price is now $1/K$ with probability $\left(\frac{K-1}{K}\right)^{N-J}$ and 1 with complementary probability.[38] It is easy to verify that the buyers' willingness to pay for information is such that the optimal price, given J, is again $p_1 = \min\left\{c, \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)}\right\}$.[39] This implies that payoffs for B_1 as a function of J (where

[38] This is identical to the case where the seller of information always reports truthfully the value of the object.

[39] It is also easy to see that a deviation from reporting strategy (1) is not profitable. When $v = \theta_1$ a deviation to any other message would still yield an auction price of $1/K$ with probability $((K-1)/K)^{N-J-1}$ and a price of 1 with complementary probability (although in different states).

we use the superscript m_1^0 to refer to reporting strategy (1)) is:

$$\begin{aligned}
\pi_{B_1}^{m_1^0}(N-J) &= \frac{1}{K}\left(1-\left(\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J-1}+\left(1-\left(\frac{K-1}{K}\right)^{N-J-1}\right)\right)\right)+ \\
&\quad (N-J-1)\min\left\{c,\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}\right\}-c \\
&= \frac{1}{K}\left(-\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J-1}+\left(\frac{K-1}{K}\right)^{N-J-1}\right)+ \\
&\quad (N-J-1)\min\left\{c,\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}\right\}-c \\
&= \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}+(N-J-1)\min\left\{c,\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}\right\}-c.
\end{aligned}$$

Since $c \geq \frac{1}{K}\left(\frac{K-1}{K}\right)^2$, $\min\left\{c,\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}\right\} = \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}$ and thus we have:

$$\begin{aligned}
\pi_{B_1}^{m_1^0}(N-J) &= (N-J)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}-c \geq \\
\pi_{B_1}^{m_1^0}(N-(J+1)) &= (N-(J+1))\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)}-c \\
&\Longleftrightarrow \frac{(N-J)}{(N-(J+1))} \geq \frac{K}{K-1} \Longleftrightarrow K \geq N-J,
\end{aligned}$$

implying that $\pi_{B_1}^{m_1^0}(N-J)$ is decreasing in J. Hence when $J=1$ the seller's payoff, given by

$$\pi_{B_1}^{m_1^0}(N-1) = (N-1)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1}-c \tag{A.9}$$

is higher than the payoff with any $J>1$. To show that $J=1$ is optimal note that with $J=0$ the willingness to pay for information is zero, but the auction price follows the same pattern as before, so that B_1's payoff is now:

$$\pi_{B_1}^{m_1^0}(N) = \frac{1}{K}\left(\frac{K-1}{K}\right)^{N}-c$$

which implies that $\pi_{B_1}^{m_1^0}(N) < \pi_{B_1}^{m_1^0}(N-1)$. Hence the seller's payoff at an equilibrium where he follows the reporting strategy (1) is given by $\pi_{B_1}^{m_1^0}(N-1)$.

Consider next the case where B_1 chooses the following message strategy, which we denote as m_1^L:

$$m_i = \begin{cases} \text{if } v \neq \theta_1 : m_i = v, \\ \text{if } v = \theta_1 : m_i = y, \text{ with probability } \frac{1}{K-1}, \text{ for all } y \neq \theta_1 \end{cases} \tag{A.10}$$

that is, he randomizes over all the types except his own. Thus in the event where the seller likes the object, rather than sending the empty message, he randomizes over all varieties different from θ_1. Under strategy (A.10) the expected valuation of a buyer B_i when he receives a report $m_1^L = \theta_i$ is only $(K-1)/K$ as he has to factor the probability that he hears $m_1^L = \theta_i$ because $v = \theta_1$. This implies that the payoffs for a buyer of information is;

$$\begin{aligned} \pi_{B_i}^J &= -p(J) + \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)}\left(1-\frac{1}{K}\right) \\ &\quad + \frac{1}{K}\left(\frac{K-1}{K}\right)\sum_{j=1}^{J-1}\frac{1}{j}\binom{J-1}{j}\left(\left(\frac{1}{K}\right)^j\left(\frac{K-1}{K}\right)^{N-J-j)}\right)\left(1-\frac{K-1}{K}\right), \end{aligned}$$

while the value of the outside options for these agents is

$$\begin{aligned} \pi_{IC}^J &= \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)}\left(1-\frac{1}{K}\right) \\ &\quad + \frac{1}{K}\left(\frac{K-1}{K}\right)\sum_{j=1}^{J-1}\frac{1}{j}\binom{J-1}{j}\left(\left(\frac{1}{K}\right)^j\left(\frac{K-1}{K}\right)^{N-J-j)}\right)\left(1-\frac{K-1}{K}\right) - c, \\ \pi_U^J &= 0 \end{aligned}$$

From the above expressions we obtain that the maximal willingness to pay for information of these traders is:

$$\begin{aligned} p(J) &= \min\left\{ \begin{array}{c} c, \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)}\left(1-\frac{1}{K}\right) + \\ \frac{1}{K}\left(\frac{K-1}{K}\right)\sum_{j=1}^{J-1}\frac{1}{j+1}\binom{J-1}{j}\left(\left(\frac{1}{K}\right)^j\left(\frac{K-1}{K}\right)^{N-J-j)}\right)\left(1-\frac{K-1}{K}\right) \end{array} \right\} \\ &> \min\left\{c, \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)}\left(1-\frac{1}{K}\right)\right\} \end{aligned}$$

This implies that

$$\begin{aligned}
\pi_{B_1}^{m_1^L}(N-J) &> \frac{1}{K}\left(1-\left(\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J-1}+\left(\frac{K-1}{K}\right)\left(1-\left(\frac{K-1}{K}\right)^{N-J-1}\right)\right)\right) \\
&\quad + (N-J-1)\min\left\{c, \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)}\left(1-\frac{1}{K}\right)\right\} - c \\
&= \frac{1}{K}\left(1-\left(\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J-1}+\left(1-\left(\frac{K-1}{K}\right)^{N-J-1}\right)\right)+\frac{1}{K}\right) \\
&\quad + (N-J-1)\min\left\{c, \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}\right\} - c \\
&= \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J} + \frac{1}{K^2} + (N-J-1)\min\left\{c, \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}\right\} - c
\end{aligned}$$

When $c \geq \frac{1}{K}\left(\frac{K-1}{K}\right)^2$, as noticed above we have $\min\left\{c, \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}\right\} = \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J}$ and so

$$\pi_{B_1}^{m_1^L}(N-J) > (N-J)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J} + \frac{1}{K^2} - c$$

Thus

$$\pi_{B_1}^{m_1^L}(N-1) > (N-1)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1} + \frac{1}{K^2} - c > \pi_{B_1}^{m_1^0}(N-1) = (N-1)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1} - c$$

This proves that the strategy (A.10) leads to higher payoffs than strategy (1).

We are now going to show that even with strategy (A.10) the equilibrium payoff of the seller of information is lower than when his type is not known to buyers, and hence the incentives to acquire information are also lower.

First we obtain an upper bound on the willingness to pay for information of the traders under (A.10):

$$\begin{aligned}
p(J) &= \min\left\{\begin{array}{c} c, \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)}\left(1-\frac{1}{K}\right) + \\ \frac{1}{K}\left(\frac{K-1}{K}\right)\sum_{j=1}^{J-1}\frac{1}{j+1}\binom{J-1}{j}\left(\left(\frac{1}{K}\right)^j\left(\frac{K-1}{K}\right)^{N-J-j)}\right)\left(1-\frac{K-1}{K}\right) \end{array}\right\} \\
&\leq \min\left\{\begin{array}{c} c, \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)}\left(1-\frac{1}{K}\right) + \\ \frac{1}{K^2}\left(\frac{K-1}{K}\right)\frac{1}{2}\left(1-\left(\frac{K-1}{K}\right)^{N-(J+1)}\right) \end{array}\right\}
\end{aligned}$$

This implies that

$$\begin{aligned}\pi_{B_1}^{m_1^L}(N-J) \;\; &\leq \;\; \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J} + \frac{1}{K^2} - c \\ &\quad + (N-J-1)\min\left\{c, \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J} + \frac{1}{K^2}\left(\frac{K-1}{K}\right)\frac{1}{2}\left(1-\left(\frac{K-1}{K}\right)^{N-(J+1)}\right)\right\}\end{aligned} \tag{A.11}$$

When $c \geq \frac{1}{K}\left(\frac{K-1}{K}\right)^2$ we also have $c \geq \frac{1}{K}\left(\frac{K-1}{K}\right)^2 + \frac{1}{K^2}\left(\frac{K-1}{K}\right)\frac{1}{2}\left(1-\left(\frac{K-1}{K}\right)\right)$ and hence

$$\begin{aligned}&(N-J)\tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-J} + \tfrac{1}{K^2} + \tfrac{N-J-1}{K^2}\left(\tfrac{K-1}{K}\right)\tfrac{1}{2}\left(1-\left(\tfrac{K-1}{K}\right)^{N-(J+1)}\right) - c \\ \geq\; &(N-(J+1))\tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-(J+1)} + \tfrac{1}{K^2} + \tfrac{N-J}{K^2}\left(\tfrac{K-1}{K}\right)\tfrac{1}{2}\left(1-\left(\tfrac{K-1}{K}\right)^{N-(J+2)}\right) - c\end{aligned}$$

if and only if

$$\begin{aligned}& (N-J)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J} + \frac{N-J-1}{2K}\frac{1}{K}\left(\left(\frac{K-1}{K}\right)-\left(\frac{K-1}{K}\right)^{N-J}\right) \\ = \;\; & (N-J-\frac{1}{2K})\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J} + \frac{N-J-1}{2K}\frac{1}{K}\left(\left(\frac{K-1}{K}\right)\right) \\ \geq \;\; & (N-(J+1))\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)} + \frac{N-J}{2K}\frac{1}{K}\left(\left(\frac{K-1}{K}\right)-\left(\frac{K-1}{K}\right)^{N-(J+1)}\right) \\ = \;\; & (N-(J+1)-\frac{1}{2K})\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)} + \frac{N-J}{2K}\frac{1}{K}\left(\left(\frac{K-1}{K}\right)\right).\end{aligned}$$

The above inequality holds if and only if

$$\left(N-J-\frac{1}{2K}\right)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-J} - \frac{(K-1)}{2K^3} \geq \left(N-(J+1)-\frac{1}{2K}\right)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)},$$

or

$$\begin{aligned}\frac{\left(N-J-\frac{1}{2K}\right)}{\left(N-(J+1)-\frac{1}{2K}\right)}\left(\frac{K-1}{K}\right) &\geq 1 + \frac{\frac{(K-1)}{2K^3}}{\left(N-(J+1)-\frac{1}{2K}\right)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-(J+1)}} \\ \Leftrightarrow \frac{\left(N-J-\frac{1}{2K}\right)}{\left(N-(J+1)-\frac{1}{2K}\right)} &\geq \left(1+\frac{1}{K-1}\right) + \frac{1}{\left(N-(J+1)-\frac{1}{2K}\right)2K\left(\frac{K-1}{K}\right)^{N-(J+1)}} \\ \Leftrightarrow N-J-\frac{1}{2K} &\geq \left(1+\frac{1}{K-1}\right)\left(N-(J+1)-\frac{1}{2K}\right) + \frac{1}{2K\left(\frac{K-1}{K}\right)^{N-(J+1)}} \\ \Leftrightarrow 1 &\geq \frac{1}{K-1}\left(N-(J+1)-\frac{1}{2K}\right) + \frac{1}{2K\left(\frac{K-1}{K}\right)^{N-(J+1)}}\end{aligned}$$

that is, if

$$1 \geq \frac{1}{K-1}\left(N-2-\frac{1}{2K}\right) + \frac{1}{2K\left(\frac{K-1}{K}\right)^{N-2}}$$

Under our assumption that $K \geq N+1$, this is true provided

$$1 \geq \frac{1}{N}(N-2) + \frac{1}{2(N+1)\left(\frac{N}{N+1}\right)^{N-2}}$$

which we can easily verify is always satisfied.

This shows that expression (A.11) takes the highest value when $J = 1$ and thus the seller's payoff for any $J \geq 1$ is bounded above by

$$\pi_{B_1}^{m_1^L}(J) \leq A(N-1) \equiv \tfrac{1}{K}\left(1 - \left(\tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-2} + \left(\tfrac{K-1}{K}\right)\left(1 - \left(\tfrac{K-1}{K}\right)^{N-2}\right)\right)\right) \tag{A.12}$$
$$+ (N-2)\min\left\{c, \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1} + \frac{1}{K^2}\left(\frac{K-1}{K}\right)\frac{1}{2}\left(1 - \left(\frac{K-1}{K}\right)^{N-(J+1)}\right)\right\} - c$$

We still need to check whether such upper bound $A(N-1)$ is bigger or smaller than $\pi_{B_1}^{m_1^L}(N)$. Using (A.10), if the seller sells information to all other buyers, the price at which he gains the object lowers to 0 (as this is the second highest bid when no buyer hears $m_1^L = \theta_i$). On the other hand, a buyer's willingness to pay for information when all other buyers purchase information is 0. Hence we have:

$$\pi_{B_1}^{m_1^L}(N) = \frac{1}{K}\left(1 - \left(\left(\frac{K-1}{K}\right)\left(1 - \left(\frac{K-1}{K}\right)^{N-1}\right)\right)\right) - c$$

Then, using again the fact that $c \geq \frac{1}{K}\left(\frac{K-1}{K}\right)^2 > \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1}$,

$$\begin{aligned}
\pi_{B_1}^{m_1^L}(N) - A(N-1) &= \tfrac{1}{K}\left(\tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-1} + \tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-2}\right) \\
&- (N-2)\min\left\{c, \tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-1} + \tfrac{1}{K^2}\left(\tfrac{K-1}{K}\right)\tfrac{1}{2}\left(1 - \left(\tfrac{K-1}{K}\right)^{N-(J+1)}\right)\right\} \\
&= \tfrac{1}{K}\left(\tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-1} + \tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-2}\right) \\
&- (N-2)\left(\tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-1} + \tfrac{1}{K^2}\left(\tfrac{K-1}{K}\right)\tfrac{1}{2}\left(1 - \left(\tfrac{K-1}{K}\right)^{N-(J+1)}\right)\right) \\
&\leq \tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-1} - (N-2)\tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-1}
\end{aligned}$$

Since $N \geq 4$, the last term is clearly negative so that we have $\pi_{B_1}^{m_1^L}(N) < A(N-1)$. This implies that the highest value of c for which B_1 gets informed is smaller or equal to

$$\begin{aligned}
c^* &= \frac{1}{K}\left(1 - \left(\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-2} + \left(\frac{K-1}{K}\right)\left(1 - \left(\frac{K-1}{K}\right)^{N-2}\right)\right)\right) \\
&\quad + (N-2)\left(\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1} + \frac{1}{2K^2}\left(\frac{K-1}{K}\right)\left(1 - \left(\frac{K-1}{K}\right)^{N-2}\right)\right)
\end{aligned}$$

Recalling that the threshold for information acquisition to take place in the case of a seller of information of unknown type is

$$\begin{aligned} c^I &\equiv \frac{1}{K}\left(\frac{K-1}{K}\right)+(N-2)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1} \\ &= \frac{1}{K}\left(1-\left(\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-2}+\frac{1}{K}\left(1-\left(\frac{K-1}{K}\right)^{N-2}\right)\right)\right)+(N-2)\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1}, \end{aligned}$$

we have

$$\begin{aligned} c^I-c^* &= \frac{1}{K}\left(1-\left(\frac{K-1}{K}\right)^{N-2}\right)\left(\frac{K-1}{K}-\frac{1}{K}\right)-(N-2)\frac{1}{K^2}\left(\frac{K-1}{K}\right) \\ &= \frac{K-2}{K^2}\left(1-\left(\frac{K-1}{K}\right)^{N-2}\right)-(N-2)\frac{1}{2K^2}\left(\frac{K-1}{K}\right)\left(1-\left(\frac{K-1}{K}\right)^{N-2}\right) \\ &= \frac{1}{K^2}\left(1-\left(\frac{K-1}{K}\right)^{N-2}\right)\left((K-2)-\frac{N-2}{2}\left(\frac{K-1}{K}\right)\right). \end{aligned}$$

Since

$$K-2-\frac{N-2}{2}\left(\frac{K-1}{K}\right)\geq K-2-\frac{N-2}{2}=\frac{K-4+K-(N-2)}{2},$$

under our assumption that $K\geq N\geq 4$, it follows that $c^I>c^*$. We conclude that the incentives to become informed are lower for buyer B_1, whose type is known by others, than for any other buyer whose type is unknown, even when the alternative message strategy (A.10) is adopted . Hence the efficiency properties of the equilibria where B_1 is the seller of information are also lower.

5. The owner of the object as seller of information (Section 6.5)

Suppose information can be sold by the owner of the object and that, in addition, he can secretly learn the buyers' types by paying a cost c_B.

If the seller of information is the owner of the object and he is informed about buyers' types, he will face a conflict of interest in his reporting strategy. If in fact he finds that no more than one of the buyers likes the object but at least two other buyers like a different variety of the object, i.e. say $v=\theta_i$ only for $i=1$ but $\theta_2=\theta_3\neq\theta_1$, then he prefers to report θ_2 rather than the truth. Reporting θ_2 allows him to increase buyers' bids for the object and thus the price at which the object is sold in the auction. A similar situation arises if the owner finds that no buyer likes the object, while in the other possible events he is willing to report the truth. This incentives to lie also mean that the seller is willing to learn the buyers' types provided the cost c_B is not too high.

PROPOSITION **5** *When the owner of the object is the only seller of information,*

1. *If the cost c_B of secretly acquiring information about buyers' types is sufficiently low,*[40]

$$\left(\frac{K-1}{K}\right)^{N-J+1}+\left(\frac{K-1}{K}\right)^{N-J}\left(\frac{N-J+K-1}{K}-\frac{(K-2)\cdot ... \cdot(K-N+J+1)}{(K-1)^{N-J-2}}\right)\geq c_B, \tag{A.13}$$

   there is no equilibrium where the object is always allocated to the buyer who values it most with probability one.

2. *If c_B is above the threshold in (A.13) and the seller does not acquire information about buyers' types, the allocation is ex post efficient but there is underinvestment in information: for values of c lying in the following, non empty interval:*

$$\left(\frac{K-1}{K}\right)\left(1-\left(\frac{K-1}{K}\right)^{N-2}\right)\leq c\leq\left(\frac{K-1}{K}\right)\left(1-\left(\frac{K-1}{K}\right)^{N-1}\right) \tag{A.14}$$

   information acquisition is socially efficient, but does not take place in equilibrium.

Proof of Proposition 5 We characterize first the unique candidate equilibrium where the owner of the object is uninformed about buyers' types. In this situation the owner can never increase his auction revenue by lying, his report is then always truthful. As argued in the proof of Proposition 2, the maximal revenue from the sale of truthful information obtains when information is sold to all buyers except one ($J = 1$) and is given by (23). In this case the owner's revenue from the auction is either 0 (if no buyer who purchases information likes the object), $1/K$ (if exactly one buyer who purchases information likes the object), and 1 otherwise. The total payoff of the owner when $J = 1$ is then:

$$\begin{aligned}&(N-1)\min\left\{c,\tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-1}\right\}+\left(1-\left(\tfrac{K-1}{K}\right)^{N-1}-(N-1)\tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-2}\right)+\\&+\tfrac{1}{K}\left((N-1)\tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-2}\right)-c=1-\left(\tfrac{K-1}{K}\right)^{N-1}-c-(N-1)\max\left\{\tfrac{1}{K}\left(\tfrac{K-1}{K}\right)^{N-1}-c,0\right\}.\end{aligned} \tag{A.15}$$

It is fairly straightforward to verify that the total payoff of the owner when $J = 2$ has the same value while it is strictly smaller for any $J > 2$. When $J = 0$, since the price at which

[40] The expression in square brackets indeed describes the probability of the event that only one buyer likes the object while at least two other buyers like the same variety: $\min_{J\in\{1,2\}}\{\Pr(\exists\ i,j\in(1,..N-J)\text{ such that }\theta_i=\theta_j\wedge v=\theta_k\text{ for at most one }k\in(1,..,N-J))\}$

information is sold is zero, the owner's payoff is simply his auction revenue and is equal to 0 if no buyer, or exactly one buyer, likes the object, and 1 otherwise:

$$1 - \left(\frac{K-1}{K}\right)^{N} - N\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1} - c$$

It is then immediate to see that

$$1-\left(\frac{K-1}{K}\right)^{N-1}-(N-1)\max\left\{\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1}-c,0\right\} > 1-\left(\frac{K-1}{K}\right)^{N}-\frac{N}{K}\left(\frac{K-1}{K}\right)^{N-1}$$

always holds. Thus, the owner's payoff is maximized by setting the price of information so that $J = 1$ or $J = 2$.

We show next that in this candidate equilibrium, when the cost c_B is sufficiently low, the owner of the object has a profitable deviation, consisting in secretly acquiring information about the buyers' types. By so doing, the owner will not affect the beliefs of the buyers (since the deviation is not observed by them), and hence their willingness to pay for information, but adopting the following reporting strategy allows him to affect the price in the auction in his favor:

$i)$ if there is at least one pair $i, j \in (1, ..N - J)$ such that $\theta_i = \theta_j$ (i.e. there is a 'tie') :

$$m_S = \begin{cases} v \text{ if } v = \theta_i = \theta_j \text{ for some } i, j \in (1, ..N - J), \\ \theta_i \neq v \text{ if } v = \theta_k \text{ for at most one } k \in (1, .., N - J) \end{cases}$$

$ii)$ if $\theta_i \neq \theta_j$ for all $i, j \in (1, .., N - J)$ (there are no 'ties'):

$$m_S = \begin{cases} v \text{ if } v = \theta_i \text{ for some } i \in (1, ..N - J), \\ \theta_i \neq v \text{ if } v \neq \theta_i \text{ for all } i = 1, .., N - J \end{cases} \tag{A.16}$$

That is, the seller will lie in two cases: (i) when only one buyer likes the object but there are two or more buyers who like the same variety of the object; (ii) when none of the buyers likes the object. These lies allow to raise the auction revenue, in the first case from $1/K$ to 1, in the second one (when $J = 1$) from 0 to $1/K$. Letting

$$P_1^J \ : \ = \Pr\left(\exists\ i, j \in (1, ..N - J) \text{ such that } \theta_i = \theta_j \wedge v = \theta_k \text{ for at most one } k \in (1, .., N - J)\right)$$
$$P_2 \ : \ = \Pr\left(\theta_i \neq \theta_j \text{ for all } i, j \in (1, .., N - 1) \wedge v \neq \theta_i \text{ for all } i = 1, .., N - 1\right)$$

for $J = 1, 2, ..$ the expected gain from the deviation is $P_1^1\left(1 - \frac{1}{K}\right) + P_2\frac{1}{K}$ when $J = 1$ and $P_1^2\left(1 - \frac{1}{K}\right)$ when $J = 2$. Both expressions, when (A.13) holds, are greater than c_B, so the deviation is profitable. There is then no equilibrium, under (A.13), where the owner of the object does not acquire information about buyers' types. Since when he does acquire such information his report is not always truthful, for the same reasons as above, and truthfulness is needed for allocational efficiency, this establishes Part 1. of the proposition.

To establish Part 2., consider again the candidate equilibrium where the owner is not informed about buyers' types and investigate for which values of c the owner's payoff is indeed greater in this situation (where its value is given by (A.15)) than if he stays uninformed, in which case his payoff is $1/K$:

$$1-\left(\frac{K-1}{K}\right)^{N-1}-c-(N-1)\max\left\{\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1}-c,0\right\}\geq\frac{1}{K}. \qquad \text{(A.17)}$$

Next, we compare the threshold for efficient information acquisition, given by (3), with the one implicitly defined by (A.17) for information to be acquired in equilibrium and verify the latter is strictly smaller. First, when $c\geq\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1}$ this is true if

$$\left(\frac{K-1}{K}\right)\left(1-\left(\frac{K-1}{K}\right)^{N-2}\right)<\left(\frac{K-1}{K}\right)\left(1-\left(\frac{K-1}{K}\right)^{N-1}\right),$$

always satisfied. When $c<\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1}$ it is true if

$$\left(\frac{K-1}{K}\right)\left(1-\left(\frac{K-1}{K}\right)^{N-2}\right)-(N-1)\left(\frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1}-c\right)<\left(\frac{K-1}{K}\right)\left(1-\left(\frac{K-1}{K}\right)^{N-1}\right)$$

which holds *a fortiori*.■

It is interesting to notice that the payoff of the owner of the object is always lower in an equilibrium where he acquires information about buyers' types (as in 1. above) than in one where he does not acquire such information (as in 2.). In the second case, since the seller is uninformed about buyers' preferences, he cannot increase his payoff by lying and is then willing to report truthfully the variety of the object, like the uninterested trader. Hence the information sold is of the highest quality and the allocation is ex post efficient. On the other hand, in the first case buyers anticipate that the seller's report will be noisy and lead to the possibility of an ex post misallocation of the object so that the total surplus will be lower. This, together with the lower informational content of the reports sent, adversely affect the buyers' willingness to pay for information, as well as the auction revenue when more than one buyer happens to like the object. We show in the proof of Part 1. of Proposition 5 that altogether such negative effects prevail over the positive one on the auction revenue when buyers of information do not like the object.

Why is it then the case that, when the cost c_B of secretly acquiring information about buyers' types is not too high, in all equilibria the seller acquires this information? This is because in a candidate equilibrium where the seller does not acquire information about buyers' types. he would always have an incentive to deviate and secretly acquire it. By doing so, his revenue from the sale of information would not be affected, while his revenue from the auction would increase; as long as the cost c_B is low enough such deviation is profitable.[41] As a consequence, in equilibrium the seller acquires such information.

The result in Part 2. of Proposition 5, that when the owner of the object is uninformed about buyers' types there is still inefficiency in information acquisition (even though the allocation is ex post efficient), can be easily understood by an argument similar to that in Section 3.2. As when information is sold by a potential buyer of the good, since allocations are always ex post efficient whenever information is acquired, the sum of the changes in the payoff of all traders between the situation with and without information acquisition equals the change in total welfare, $W_1 - W_0$. The payoff of buyers who purchase information is again zero in both cases (for c below but close to the threshold for information to be acquired), hence the change in total welfare equals the change in the payoff of the seller, $\Delta\pi_S^S$, plus the payoff of the buyer (B_N) who remains uninformed.[42] That is[43]:

$$W_1 - W_0 = \pi_{B_N}^S + \Delta\pi_S^S. \tag{A.18}$$

Underinvestment obtains whenever $\pi_{B_N}^S > 0$. Since the uninformed buyer gets the same payoff as when information is sold by a potential buyer, i.e. $\pi_{B_N}^S = \frac{1}{K}\left(\frac{K-1}{K}\right)^{N-1}$, the result follows. Notice that the source of the inefficiency is here only the informational *free riding* of the uninformed buyer, not the *rent dissipation* by the indirectly informed buyers, in contrast to what we found in Section 3.2 when the seller of information is a buyer.

6. Vertical differentiation of the information (Section 6.4)

Consider the following extension of the model. Suppose the good not only comes in one of the K types we described, but also in one of 2 quality levels, H (High) and L (Low). Formally, the true characteristic of the object is now $v = (k, q) \in \mathcal{S} \equiv \mathcal{K} \times \{H, L\}$. Suppose,

[41] The fact that, as argued above, the payoff of the owner of the object is higher when he does not know the buyers' types shows that he would benefit by committing not to seek such information but he can only do this effectively when the cost c_B is high enough.

[42] As shown in the proof of Proposition 5 in the Appendix, the seller's optimal choice is always to sell information to all buyers but one.

[43] We use a superscript S to denote variables at equilibria where the owner of the object is the seller of information.

in addition, that buyers are also of two types: while all buyers only care for one, randomly drawn variety, some of them are sensitive to quality (Se) and others are insensitive to quality (In). An In buyer has a constant valuation of 1 for the object of the variety he likes. A Se buyer values the object of the variety he likes V if it is of H quality, and 0 if it is of L quality. Let us assume for simplicity that H and L have identical probabilities for each variety of object, and that buyers have identical probabilities to be of type Se and In. Also, consider again the case where the set of available messages to the seller of information is again given by the set of direct messages plus the empty message, so that a generic message m is now a pair (k, q), where $k \in \mathcal{K}$, $q \in \{H, L\}$, or the message $m = 0$.

We note first that in this environment we may have ex-post inefficiencies due to misrepresentation of information also when a potential buyer of the object is the provider of information, something that could not happen when information was only about a horizontal variety. Consider in fact a situation where a buyer of a type $B_i \in In$ is the seller of information. Then in the event where the true quality of the object is given by $q = H$ and the true variety by $k = \theta_i = \theta_j$ for some $B_j \in Se$, the allocation of the object is inefficient at any equilibrium where B_i adopts a strategy analogous to (1) and tells the truth whenever the true variety k of the object does not coincide with his own type. When the two coincide, the seller of information will lie to buyers telling them the object 'is not right for them'. This allows the seller to gain the object when he likes it but now there may be some other buyer who values it more. Think of a movie producer who does not value quality of acting, since his viewership only values good looks. If he is the provider of information his reports about the quality of acting will sometimes be misleading, and hence a producer willing to pay a lot for an actor with both good looks and high-level acting may end up without his services.

We show next that, when the seller of information is the owner of the object, there is always misrepresentation of information and thus ex-post misallocation of the good with positive probability. In particular, if in equilibrium he tells the truth about the variety k of the object, he would not reveal any relevant information over the quality of the object:

Proposition **6** *Suppose the set of possible characteristics of the object is given by $\mathcal{S}$ and the seller of information is the owner of the object. Then at any equilibrium where, for every report m sent by the seller we have $\Pr(k = i|m) = 1$ for some $i \in \mathcal{K}$, it must be that if $V > 2$ the Se type buyers who like variety i bid more for the object than the In type buyers, whatever is the true quality q, and vice versa if $V < 2$.*

Hence we can never have at the same time perfect revelation both of the true variety and the true quality of the object.[44] It follows that the equilibrium allocation of the object is ex post inefficient, which in turn implies that not all social surplus can be appropriated by the seller and so that there will be underinvestment in information acquisition. The source of the inefficiency is that when the seller of information is the owner of the object he faces a conflict of interest, analogous to the one we found in Section 6.5 when the cost c_B for secretly acquiring information about the buyers' types is low enough: the seller may want to lie to exaggerate how much buyers like the object so as to increase his revenue from the sale of the object. However now the inefficiency arises whatever is c_B.

Thus when uncertainty has also a vertical differentiation element, misrepresentation and allocational inefficiency occur both when the seller of information is a potential buyer and when he is the owner of the object. The only situation where the report sent is always truthful and the equilibrium allocation remains ex post efficient is the one where the seller of information is a disinterested trader. Hence when the vertical differentiation element is sufficiently important (e.g., neither the set of buyers of type Se nor that of type In are 'too small') welfare may now be higher in this case.

[44] It should be clear that both are needed for allocational efficiency.