Existence and Uniqueness of Solutions to the Bellman Equation in Stochastic Dynamic Programming*

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Abstract

In this paper we develop a framework to analyze stochastic dynamic optimization problems in discrete time. We obtain new results about the existence and uniqueness of solutions to the Bellman equation through a notion of Banach contractions that generalizes known results for Banach and local contractions. We apply the results obtained to an endogenous growth model and compare our approach with other well known methods, such as the weighted contraction method, countable local contractions and the Q-transform.

*This paper is based on my working papers Rincón-Zapatero (2019, 2022). I acknowledge of the referees and associate editors of this and of another journal for their insightful comments, that significantly improved the exposition, leading to the current version. The usual caveat applies.

Keywords— Stochastic dynamic programming, Bellman equation, contraction mapping, weighted contraction, local contraction, Q-transform, endogenous growth

1 Introduction

Stochastic dynamic programming incorporates uncertain events into a suitable framework to find optimal policies. A useful approach for showing the existence of optimal stationary plans is to prove that the dynamic programming equation admits a unique solution —the value function— in a suitable space of functions. See Blackwell (1965), Maitra (1968), Furukawa (1972), Bertsekas and Shreve (1978), Stokey *et al.* (1989), Hernández-Lerma and Lasserre (1999), or Bäuerle and Rieder (2011), where this problem is analyzed in detail. Also, there is a large amount of literature that applies stochastic dynamic programming to economics. Brock and Mirman (1972), Mirman and Zilcha (1975), Donaldson and Mehra (1983), Danthine and Donaldson (1981), Majumdar *et al.* (1989), Hopenhayn and Prescott (1992) or Mitra (1998) are only a few of the many relevant papers that have contributed to developing this field of research. Olson and Roy (2006) makes a review of the contributions to the stochastic optimal growth model.

Many dynamic programs have both unbounded rewards and unbounded shocks, which cannot be handled by the theory initiated by Blackwell (1965), based on the properties of monotonicity and discount of the dynamic programming operator. In general, a problem with unbounded utility cannot be transformed into an equivalent bounded problem, since the optimal policies of both models will differ due to the dynamic structure. Also, imposing artificial bounds to deal with a compact shock space may be incompatible with modelling uncertainty by means of a first order stochastic process. Think, for instance, of the simple random walk. It takes every integer with positive probability.

Weighted contractions, (Wessels (1977), Stokey *et al.* (1989), Boyd (1990), Hernández-Lerma and Lasserre (1999), or Bäuerle and Rieder (2011)), countable local contractions, (Matkowski and Nowak (2011), Jaśkiewicz and Nowak (2011), Balbus *et al.* (2018)) and the recent Q–transform due to Ma *et al.* (2022) are useful approaches to deal with stochastic programs with unbounded rewards and shocks.

The weighted norm approach needs to identify a suitable bounding function. This is not immediate in some models. Our paper provides sufficient conditions which do not need a bounding function. In fact, we show a one-sector optimal growth model with linear technology and strictly increasing and concave utility function which does not admit a bounding function, but our approach applies.

Countable local contractions[1] imposes, roughly speaking, that the conditional probability measures defined by the transition kernel have bounded support. This is due to the need of constructing a countable family of increasing compact sets covering the state space. We dispense with this assumption. The same growth model described above serves to show that this method gives a more restricted condition to the discount factor.

The Q–transform consists in taking conditional expectations at both sides of the Bellman equation that, in some models, converts an unbounded dynamic program into a bounded one. This is a similar idea to what we do to obtain the Companion Operator Parameter L, see Proposition 3.3 below, but the purpose is different, as we work with the original Bellman operator. The Q–transform deals with a transformed operator. For unbounded from above rewards, the Q–transform is the same that weighted contraction, but for unbounded from below rewards it may take advantage of the averaging operation to obtain a bounded program. We present a quadratic example where it is not possible to apply the Q–transform, nor the countable contraction approach, but our results show that the Bellman equation defines a contraction mapping.[2]

Our aim is to develop a new framework to study programs with unbounded rewards

[1] The local contraction approach generalizes the Banach contraction principle for function spaces whose topology is defined by a family of seminorms. Hadžić and Stanković (1970) is one of the first papers dealing with this extension. Rincón-Zapatero and Rodríguez-Palmero (2003, 2007, 2009), independently, introduced different hypotheses and applied the results to the deterministic Bellman and Koopmans equations. Martins da Rocha and Vailakis (2010) extended the theory to the case of an uncountable family of seminorms.

[2] It is worth noting that the results in Ma *et al.* (2022) and Jaśkiewicz and Nowak (2011) may be applied to unbounded from below programs where the utility function may take the value $-\infty$ on the state space; this is beyond the scope of this paper.

and/or unbounded shocks by extending the local contraction method developed in Rincón-Zapatero and Rodríguez-Palmero (2003, 2009) and Martins da Rocha and Vailakis (2010) for deterministic programs to the stochastic setting, while preserving the monotonicity of the Bellman operator. To this end, we define a suitable space of functions and a suitable family of seminorms. The seminorms combine the usual supremum norm in the endogenous variables with an L^1 norm in the exogenous variables, and define a complete space of functions — a Carathéodory function space —.

To work within this framework, we need to extend the notion of contraction mapping, by considering the contraction parameter(s) in the local contraction definition as an operator acting on the family of seminorms. This operator is what we call the companion operator associated with the contraction mapping[3]. The theory we develop is quite general and could be applied to other equilibrium problems in economics beyond stochastic dynamic programming.

Our framework allows us to relax continuity of the period utility function with respect to the exogenous variable. This is important since Feller continuity of the Markov chain is not enough to preserve continuity when the space of shocks is not compact. Also, the L^1 type norm defined on shocks makes it possible to obtain less restrictive bounds on the discount factor than with other known methods, as it is demonstrated in the paper.

Our approach is designed for problems where utility functions may be unbounded from below, but not taking the value $-\infty$ on the state space. This is restrictive, as it takes out of consideration important problems in economics. Nevertheless, we still get new insights in better–behaved models.[4]

[3]This idea is not new. Kozlov *et al.* (2010) developed a fixed point theorem in locally convex spaces whose topology is given by a family of seminorms. However, the results obtained depend on the companion contraction parameter operator being linear, precluding application to the dynamic programming equation, since it genuinely demands a nonlinear companion contraction parameter, due to the presence of a maximization operation in the definition of the Bellman equation.

[4]To adapt our results to this class of models it may be promising to work with pseudometrics, instead of seminorms, as in Rincón-Zapatero and Rodríguez-Palmero (2003); this paper does not explore this issue. Also, the theory we develop applies only to problems where uncertainty is exogenous. However, to extend the approach to models where actions affect uncertainty is possible, and

The paper is organized as follows. Section 2 develops a theory of contraction mappings on topological spaces whose topology is given by a family of pseudometrics that makes it Hausdorff and sequentially complete. The contraction parameter is given by an operator acting on pseudometrics. Section 3 applies the results of Section 2 to the stochastic dynamic programming equation for models with shocks driven by an exogenous Markov chain. The main assumption used to obtain our results states that today's conditional expectation of the utility function is bounded by the present value of tomorrow's conditional expectation, in such a way that the resulting infinite sum of all expected values is finite. In Section 4 we study a model of endogenous growth, allowing for correlated and unbounded shocks. Section 5 makes a comparison of our results with those obtained with weighted contractions, countable local contractions, and the Q–transform addressed above. Section 6 concludes. Appendixes A and B contain the proofs not in the main text of Section 2 and 3, respectively. Appendix C provides a pure currency model where the value function is discontinuous with respect to the shock variable, showing in a simple economic model the well known fact that Feller continuity of the Markov chain is not enough to preserve continuity when the shock space is not compact.

2 A general class of Banach contractions

Let $(E, \mathcal{D})$ be a topological space, where E is a set whose topology is generated by a saturated family of pseudometrics $\mathcal{D} = \{d_a\}_{a \in A}$, with A an arbitrary index set. Since the family $\mathcal{D}$ is saturated, the topology it generates is Hausdorff[5]. We suppose that $(E, \mathcal{D})$ is

in fact Rincón-Zapatero (2022) drafts how it could be done.

[5]A pseudometric $d : E \times E \to \mathbb{R}_+$ is a function satisfying $d(x, y) \geq 0$, $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in E$, but $d(x, y) = 0$ does not imply $x = y$. The family $\mathcal{D}$ of pseudometrics is saturated if $d_a(x, y) = 0$ for all $a \in A$ implies $x = y$. Sometimes, the pseudometrics are defined through seminorms p_a, $a \in A$, by $d_a(x, y) = p_a(x - y)$, where now E is a real vector space. A seminorm is a function $p : E \to \mathbb{R}_+$ that satisfies all the axioms to be a norm, except that $p(x) = 0$ does not imply that x is the null vector of E. If the family of seminorms is saturated, then the topology defined by the family is Hausdorff and the space E is a locally convex space. See Willard (1970) for further details.

sequentially complete: if $\{x_n\}$ is a sequence in E which is Cauchy with respect to all $d_a \in \mathcal{D}$, that is, if $d_a(x_n, x_m) \to 0$ as $n, m \to \infty$, then there is $x \in E$ such that $d_a(x_n, x) \to 0$ as $n \to \infty$ for all $a \in A$.

Given a sequentially complete subset $F \subseteq E$, we study the existence and uniqueness of a fixed point of a mapping $T : F \to E$.

Let $\mathbb{R}^A$ be the set of functions $d : A \to \mathbb{R}_+$ and let $\mathbb{R}^A_+$ be the non–negative cone of $\mathbb{R}^A$. On this set we consider the order it generates, that is, for two elements $d, d' \in \mathbb{R}^A_+$, we say that $d \leq d'$ if and only if $d(a) \leq d'(a)$ for all $a \in A$. The family $\mathcal{D}$ can be embedded into $\mathbb{R}^A_+$, since that, for $x, y \in E$ given, the mapping $a \mapsto d_a(x, y)$ defines a function in $\mathbb{R}^A_+$, that we denote $d^{x,y}(a) := d_a(x, y)$. In general, for a given subset $F \subseteq E$, we let $D(F)$ be the set of functions in $\mathbb{R}^A_+$ which are generated by pairs $x, y \in F$, that is

$$D(F) := \{d : A \to \mathbb{R}_+ \ : \ d = d^{x,y} \text{ for some } x, y \in F\}.$$

Definition 2.1. *Let $F \subseteq E$. The mapping $T : F \to E$ is an L-local contraction on F with contraction operator parameter L (COP, for short), if there are a set $C \subseteq \mathbb{R}^A_+$ such that $D(F) \subseteq C$, and an operator $L : C \to \mathbb{R}^A_+$, such that*

$$d_a(Tx, Ty) \leq (Ld^{x,y})(a),$$

for all $x, y \in F$ and for all $a \in A$.

Note that the inequality above can be rewritten $d^{Tx,Ty} \leq Ld^{x,y}$, that is, as an order relation in the space $\mathbb{R}^A_+$. The definition of L–contractions for mappings $T : F \longrightarrow E$, not imposing $T : F \longrightarrow F$, will facilitate the definition of the COP parameter L of the Bellman operator in Section 3. Of course, the property $T : F \longrightarrow F$ is fundamental for Theorem 3.5 below, and will be checked carefully in Section 3.

The following two examples show that the operator L is a generalization of the concept of contraction parameter of a (local) contraction mapping.

Example 2.2 (Banach contractions)**.** *In the classical Banach's Theorem, E is endowed with a complete metric d, so the index set A is a singleton, $\mathcal{D} = \{d\}$, and T is a contraction of constant parameter β, with $0 < \beta < 1$: $d(Tx, Ty) \leq \beta d(x, y)$, for any $x, y \in E$. The COP is $L = \beta I$, where I is the identity map in $\mathbb{R}_+$.*

A generalization of the Banach contraction concept is provided in Wong (1968), where it is considered $T : E \longrightarrow E$ for which there is a function $L : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying

$$d(Tx, Ty) \leq L(d(x, y)), \tag{2.1}$$

for all $x, y \in E$. Note that our definition is an extension of this concept to topological spaces whose topology is given by a family of pseudometrics.

Example 2.3 (k–local contractions)**.** *Suppose that $A = \mathbb{N}$ is countable. In Rincón-Zapatero and Rodríguez-Palmero (2003, 2007), we introduced the concept of k–local contraction in the study of the deterministic Bellman and Koopmans equations, respectively. A k–local contraction on F, $k = 0, 1, 2, \ldots$, is a mapping $T : F \subseteq E \longrightarrow E$ satisfying*

$$d_j(Tx, Ty) \leq \beta_j d_{j+k}(x, y)$$

for some fixed sequence of numbers $\{\beta_j\}_{j\in\mathbb{N}}$ with $0 < \beta_j < 1$, and for all $x, y \in F$. If we let $s = \mathbb{R}^{\mathbb{N}}$ be the set of real sequences and s^+ be the subset of s of nonnegative sequences, then the COP associated with T is the linear operator $L : s^+ \longrightarrow s^+$ acting on sequences given by

$$L(d_1, d_2, \ldots, d_j, \ldots) = (\beta_1 d_{1+k}, \beta_2 d_{2+k}, \ldots, \beta_j d_{j+k}, \ldots),$$

where $k \geq 0$ is fixed.

Suppose that A is uncountable and let a mapping $\alpha : A \longrightarrow A$. Martins da Rocha and Vailakis (2010) worked with the following generalization of the countable class above: $T : E \longrightarrow E$ is an α-local contraction if there exists a function $\beta : A \longrightarrow [0, 1)$ such that

$$d_a(Tx, Ty) \leq \beta(a) d_{\alpha(a)}(x, y).$$

The COP L acts on functions $d : A \longrightarrow \mathbb{R}_+$ by translation in the independent variable by α, and a multiplication by β, that is, $(Ld)(a) = \beta(a)d(\alpha(a))$. It turns out that L is also a linear mapping, as in the countable case above.

In what follows, we use the standard notation for successive iterations of the operators T and L. For instance, L^0 is the identity operator on C, $L^1 = L$, and for $t \geq 2$, $L^t = L \circ L^{t-1}$. We impose to C, L and T the assumptions (I) to (VI) listed below. The assumptions (I) to

(V) concern the behavior of L on the set C. Assumption (VI) links directly the operators T and L.

(I) *$D(F) \subseteq C$ (hence the null function $0 \in C$). For all $d, d' \in C$, the sum $d+d' \in C$, and any bounded subset of C is countable chain complete*[6]*. Moreover, if $d' \in C$, $d \in \mathbb{R}_+^A$ and $d \leq d'$, then $d \in C$.*

(II) *$L(C) \subseteq C$; $L0 = 0$.*

(III) *L is monotone: for all $d, d' \in C$ with $d \leq d'$, $Ld \leq Ld'$.*

(IV) *L is subadditive: for any $d, d' \in C$*

$$L(d + d') \leq Ld + Ld'.$$

(V) *L is upper semicontinuous sup-preserving*[7]*: for any bounded countable chain in C, $d_1 \leq d_2 \leq \cdots \leq d_t \leq \cdots$,*

$$L \sup_t d_t \leq \sup_t Ld_t.$$

(VI) *There are $x_0 \in F$ and $r_0 \in C$ with $d_a(x_0, Tx_0) \leq r_0(a)$ and*

$$R_0(a) := \sum_{t=0}^{\infty} L^t r_0(a) < \infty,$$

for all $a \in A$.

Since $L^t r_0 \in C$, for all $t = 0, 1, \ldots$, and the countable chain $\{r_0, r_0 + Lr_0, \ldots, r_0 + Lr_0 + \cdots + L^t r_0, \cdots\}$ is bounded in C by (VI), R_0 is in C by assumption (I).

For $F \subseteq E$, $x_0 \in F$, and $m \in \mathbb{R}_+^A$, let the set

$$V_F(x_0, m) = \{x \in F \,:\, d_a(x_0, x) \leq m(a),\ \forall a \in A\}. \tag{2.2}$$

[6] A subset $S \subseteq C$ is bounded with respect to the order inherited from $\mathbb{R}^A$ if there is $d' \in C$ such that $d \leq d'$ for all $d \in S$. The bounded subset S is countably chain complete if for any countably chain $d_1 \leq d_2 \leq \cdots d_t \leq \cdots$ in S, $\sup_{t \in \mathbb{N}} d_t \in S$.

[7] For instance, the sup-preserving property, $L(\sup_t d_t) = \sup_t Ld_t$, plays a prominent role in the Fixed Point Theorem of Kantorovich-Tarski. In our context, it can be weakened to a kind of upper semicontinuity.

When E is a metric space, that is, when A is a singleton, the pseudometric is a metric, and $V_F(x_0, m)$ is simply the intersection with F of the closed ball centered at x_0 and radius m.

Let $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and set $J := A \times \mathbb{N}_0$. Consider next the family of pseudometrics $\Delta := (\delta_j)_{j \in J}$ on F where

$$\delta_{a,t}(x, y) = L^t(d_a^{x,y}).$$

Assumptions (I)-(IV) imply that δ_j is a pseudometric while a straightforward computation shows that

$$\delta_{a,t}(Tx, Ty) = L^t(d_a(Tx, Ty) \leq L^{t+1})d_a(x, y)) = \delta_{a,t+1}(Tx, Ty).$$

Now let the map $r : J \to J$ be defined by $r(a, t) = (a, t + 1)$. Then T is a local contraction with respect to (Δ, r).[8]

Theorem 2.4. *Let $(E, \mathcal{D})$ be a Hausdorff and sequentially complete topological space. Let $T : F \to F$ be an L-local contraction on the sequentially complete subset $F \subseteq E$ and let $x_0 \in F$ be such that (I)–(VI) hold true. Suppose that the family of pseudometrics Δ defined above is saturated and the space (E, Δ) is complete. Then there is a unique fixed point $x^* \in V_F(x_0, R_0)$ of T, which is the limit of any iterating sequence $y_{t+1} = Ty_t$, $t = 0, 1, 2, \ldots$, where $y_0 = x \in V_F(x_0, R_0)$ is arbitrary.*

Proof. The result follows from (Martins da Rocha and Vailakis, 2010, Theorem 2.1), with $K = V_F(x_0, R_0)$. See Lemma A.1 in Appendix A. $\square$

[8]This construction was shown to the author by a referee. It allows us to apply directly (Martins da Rocha and Vailakis, 2010, Theorem 2.1) to obtain existence and uniqueness of the fixed point. However, in some cases, the family Δ could be not saturated or not defining a sequentially complete topology. An example within the dynamic programming class is as follows. Suppose a deterministic problem with $X = \mathbb{R}_+$ and $\Gamma(x) = \{x + 1\}$. Then, for all compact sets K of X and all functions $p = p^f$, with f continuous, $Lp(K) = \beta \max_{x \in K} p(\Gamma(x)) = \beta p_{K+1}$ and $L^t p(K) = \beta^t p_{K+t}$, where $K + t = \{x + t : x \in K\}$, for all $t = 1, 2, \ldots$. It is clear that $\{\delta_{K,t}\}$ is not a saturated family of seminorms. Supposing that f and g are continuous functions on $\mathbb{R}_+$, such that $f \neq g$ in $[0, 1]$ and $f = g$ in $(1, \infty)$; yet, $\delta_{K,t}(f - g) = \beta^t p_{K+t}(f - g) = 0$ for all K and all $t = 1, 2, \ldots$. A direct approach, without the use of Δ, is shown in Rincón-Zapatero (2022). The applications we study in further sections have Δ both saturated and complete.

The next result is a corollary to the above theorem that provides conditions for the uniqueness of the fixed point in F and not only in $V_F(x_0, R_0)$.

When T is indeed an L-local contraction on the whole E, this result provides global uniqueness of the fixed point on E.

Corollary 2.5. *Let $(E, \mathcal{D})$ be a Hausdorff, sequentially complete space. Let $T : F \to F$ be an L-local contraction on the sequentially complete subset $F \subseteq E$ and let $x_0 \in F$ be such that (I)–(VI) hold true. Suppose that for all $x \in F$ there exists $r_0 \in C$ satisfying (VI) such that $x \in V_F(x_0, R_0)$, where $R_0 = \sum_{t=0}^{\infty} L^t r_0$. Then there is a unique fixed point of T in F and convergence to the fixed point of successive iterations of T is attained from any $x \in F$.*

Next we establish a useful sufficient condition for (VI). Note that the Bellman operator satisfies the extra condition imposed on L.

Proposition 2.6. *Let $(E, \mathcal{D})$ be a Hausdorff and sequentially complete topological space. Let $T : F \longrightarrow F$ be an L–local contraction on $F \subseteq E$, with COP L satisfying (I) to (V) and $L(\alpha d) \leq \alpha L d$, for all $d \in C$, for all $\alpha \in [0, 1]$. Let $x_0 \in F$, for which there is $t_0 \in \{0, 1, 2, \ldots\}$, $s \in C$, and $\theta \in [0, 1)$ such that*

$$L^{t_0} d_0 \leq s \quad \text{and} \quad Ls \leq \theta s,$$

where $d_0(a) = d_a(x_0, Tx_0)$. Then (VI) holds with $r_0 = d_0$.

3 Stochastic Dynamic Programming and Bellman Equation

Consider a dynamic programming model $(X, Z, \Gamma, Q, U, \beta)$, where $X \times Z$ is the set of possible states of the system, Γ is a correspondence that assigns a nonempty set $\Gamma(x, z)$ of feasible actions to each state (x, z) and Q is the transition function, which associates a conditional probability distribution $Q(z, \cdot)$ on Z to each $z \in Z$. Hence, the law of motion is assumed to be a first-order Markov process, which could be degenerate, giving rise to a deterministic model. We will use indistinctly the notation $Q_z(\cdot) = Q(z, \cdot)$; the function U is the one–period return function, defined on the graph of Γ, $\Omega = \{(x, y, z) : (x, z) \in X \times Z, y \in \Gamma(x, z)\}$, and β is a discount factor.

Starting at some state (x_0, z_0), the agent chooses an action $x_1 \in \Gamma(x_0, z_0)$, obtaining a return of $U(x_0, x_1, z_0)$ and the system moves to the next state (x_1, z_1), which is drawn according to the probability distribution $Q(z_0, \cdot)$. Iteration of this process yields a random sequence $(x_0, z_0, x_1, z_1, \ldots)$ and a total discounted return $\sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}, z_t)$. A history of length t is $z^t = (z_0, z_1, \ldots, z_t)$. Let Z^t be the set of all histories of length t, and let $\mathcal{Z}^t = \mathcal{Z} \times \cdots \times \mathcal{Z}$ (t times), where $\mathcal{Z}$ is the Borel σ-algebra of Z. A (feasible) plan π is a constant value $\pi_0 \in X$ and a sequence of measurable functions $\pi_t : Z^t \longrightarrow X$, such that $\pi_t(z^t) \in \Gamma(\pi_{t-1}(z^{t-1}), z_t)$, for all $t = 1, 2, \ldots$. Denote by $\Pi(x_0, z_0)$ the set of all feasible plans starting at the state (x_0, z_0). Any feasible plan $\pi \in \Pi(x_0, z_0)$, along with the transition function Q, defines a distribution $\mathbb{P}^{\pi,(x_0,z_0)}$ on all possible futures of the system $\{(x_t, z_t)\}_{t=1}^{\infty}$, as well as the expected total discounted utility

$$u(\pi, x_0, z_0) = \mathbb{E}^{\pi,(x_0,z_0)} \left(\sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}, z_t) \right).$$

The expectation $\mathbb{E}^{\pi,(x_0,z_0)}$ is taken with respect to the distribution $\mathbb{P}^{\pi,(x_0,z_0)}$. The problem is then to find a plan $\pi \in \Pi(x_0, z_0)$ such that $u(\pi, (x_0, z_0)) \geq u(\widehat{\pi}, (x_0, z_0))$ for all $\widehat{\pi} \in \Pi(x_0, z_0)$, for all $(x_0, z_0) \in X \times Z$. The value function of the problem is $v(x_0, z_0) = \sup_{\pi \in \Pi(x_0,z_0)} u(\pi, (x_0, z_0))$.

Consider the functional equation corresponding to the above dynamic programming problem as stated in Stokey *et al.* (1989). For $x \in X$, $z \in Z$

$$v(x, z) = \sup_{y \in \Gamma(x,z)} \left\{ U(x, y, z) + \beta \int_Z v(y, z') Q(z, dz') \right\}. \tag{3.1}$$

A solution of the Bellman equation satisfying additional assumptions is the value function of the infinite programming problem. This is the content of Theorem 3.5 below, whose proof needs the notion of the probability measure μ^t defined on the sequence space of shocks $(Z^t, \mathcal{Z}^t)$ for finite $t = 1, 2, \ldots$, where

$$(Z^t, \mathcal{Z}^t) = (Z \times \cdots \times Z, \mathcal{Z} \times \cdots \times \mathcal{Z}) \quad (t \text{ times})$$

and where $\mathcal{Z}$ is defined in (B1) below. For any rectangle $B = A_1 \times \cdots \times A_t \in \mathcal{Z}^t$, μ^t is defined by

$$\mu^t(z_0, B) = \int_{A_1} \cdots \int_{A_{t-1}} \int_{A_t} Q_{z_{t-1}}(dz_t) Q_{z_{t-2}}(dz_{t-1}) \cdots Q_{z_0}(dz_1),$$

and by the Hahn Extension Theorems, $\mu^t(z_0, \cdot)$ has a unique extension to a probability measure on all of $\mathcal{Z}^t$. We omit the details, which can be found in Stokey *et al.* (1989), Section 8.2, whose presentation we follow closely.

Defining the Bellman operator in a suitable function space E, such that for $f \in E$

$$(Tf)(x, z) = \sup_{y \in \Gamma(x,z)} \left\{ U(x, y, z) + \beta \int_Z f(y, z') Q(z, dz') \right\},$$

the Bellman functional equation (3.1) is a fixed point problem for T. This fixed point problem is completely understood for the case where U is bounded. There are now also different approaches for some special cases for unbounded U and unbounded shock space. [9] It is worth mentioning the constant returns to scale model in Stokey *et al.* (1989) and in Álvarez and Stokey (1998), the logarithmic and the quadratic parametric examples analyzed in Stokey *et al.* (1989), and the weighted norm approach in Boyd (1990), Hernández-Lerma and Lasserre (1999) or Bäuerle and Rieder (2011). Matkowski and Nowak (2011) and Jaśkiewicz and Nowak (2011) make a nice translation of the approach initiated by Rincón-Zapatero and Rodríguez-Palmero (2003) for deterministic programs to the stochastic case.

We now impose the standing hypotheses. Most of them are taken from Stokey *et al.* (1989), but there are essential differences, as we admit an unbounded utility U and an unbounded shock space Z.

(B1) $X \subseteq \mathbb{R}^l$, $Z \subseteq \mathbb{R}^k$ are Borel sets, with Borel σ-algebra $\mathcal{X}$ and $\mathcal{Z}$, respectively. The set X is endowed with the Euclidean topology.

(B2) $0 < \beta < 1$.

(B3) $Q : Z \times \mathcal{Z} \to [0, 1]$ satisfies

[9] Allowing for a non–compact shock space is important for a qualitative analysis of models, see for instance Binder and Pesaran (1999) and Stachurski (2002), and more recently, Ma and Stachurski (2019).

(a) for each $z \in Z$, $Q(z, \cdot)$ is a probability measure on $(Z, \mathcal{Z})$; and

(b) for each $B \in \mathcal{Z}$, $Q(\cdot, B)$ is a Borel measurable function.

(B4) The correspondence $\Gamma : X \times Z \longrightarrow 2^X$ is nonempty, compact-valued and continuous; its graph is denoted Ω.

(B5) $U : \Omega \longrightarrow \mathbb{R}$ is a Carathéodory function, that is, it satisfies

(a) for each $(x, y) \in X \times X$, the function of z

$$U(x, y, \cdot) : \{z \in Z \, : \, (x, y, z) \in \Omega\} \longrightarrow \mathbb{R},$$

is Borel measurable;

(b) for each $z \in Z$, the function of (x, y)

$$U(\cdot, \cdot, z) : \{(x, y) \in X \times X \, : \, (x, y, z) \in \Omega\} \longrightarrow \mathbb{R},$$

is continuous.

The reason for working with Carathéodory functions instead of continuous functions in the three variables (x, y, z) is twofold. On the one hand, the Markov operator

$$(\mathrm{M}f)(x, z) := \int_Z f(x, z')Q(z, dz'), \tag{3.2}$$

does not preserve continuity of f, if f is continuous but not bounded, as the simple currency model in Appendix C shows. Our approach dispenses with the assumption of strong Feller continuity of Q, which means that the mapping $z \mapsto \int_Z f(z')Q_z(dz')$ is continuous for all bounded and measurable f. Continuity plays a major role in the application of the Banach Contraction Theorem and the weighted approach, see e.g. Stokey *et al.* (1989), Boyd (1990), Hernández-Lerma and Lasserre (1999), or Bäuerle and Rieder (2011). On the other hand, the Bellman operator is well defined for the class of Carathéodory functions in the unbounded case, while working with the supremum norm is not possible. A direct attack of the Bellman equation in the space of (x, z)–continuous functions does not work for unbounded functions and/or unbounded shock space: known theorems on local contractions—with a countable or uncountable index set —are not suitable, due to the averaging operation involved in the computation of conditional expectations. For this reason we are going to use L^1-type seminorms, whose precise definition is given below.

We now describe the function space. For each $z \in Z$, let $L^1(Z, \mathcal{Z}, Q_z)$ be the space of Borel measurable functions[10] $g : Z \longrightarrow \mathbb{R}$ such that $\int_Z |g(z')|Q_z(dz') < \infty$. In what follows, we let $\mathcal{K}$ be the family of all compact subsets of X. Consider the space $E := \mathcal{L}^1(Z; C(X))$, formed by Carathéodory functions $f : X \times Z \longrightarrow \mathbb{R}$ such that the function $z' \mapsto \max_{x\in K} |f_x(z')|$ is in $L^1(Z, \mathcal{Z}, Q_z)$, for all compact sets $K \in \mathcal{K}$, and all $z \in Z$. Define

$$p_{K,z}(f) := \int_Z \max_{x\in K} |f(x, z')|Q_z(dz').$$

The proof of the following result can be found in Rincón-Zapatero (2022).

Proposition 3.1. *Suppose that the family $\mathcal{P} := \{p_{K,z}\}_{K\in\mathcal{K},z\in Z}$ is saturated. Then the space $E = \mathcal{L}^1(Z; C(X))$ with the topology generated by $\mathcal{P}$ is a locally convex complete space.*

In particular, this proposition states that E is sequentially complete. In all the applications we study in further sections, the family $\mathcal{P}$ is saturated.

In the notation of Section 2, the index set of the family of seminorms is $A = \mathcal{K}\times Z$. Given a solution $f \in \mathcal{L}^1(Z; C(X))$ of (3.1), define the policy correspondence $G^f : X \times Z \to 2^X$ by

$$G^f(x, z) = \{y \in \Gamma(x, z) \,:\, f(x, z) = U(x, y, z) + \beta \mathrm{M}f(y, z)\},$$

where M was defined in (3.2). This is the optimal policy correspondence, denoted simply by Γ^*, when f is the value function, v.

Remember from Section 2, that for a subset $F \subseteq E$, the set $D(F)$ is in this context

$$D(F) = \{p : \mathcal{K} \times Z \to \mathbb{R}_+ \,:\, p(K, z) = p_{K,z}(f) \text{ for some } f \in F\}.$$

Notation 3.2. *Where needed, we will use p^f to denote the element of $D(F)$ which is obtained from $f \in F$, that is $\mathrm{p}^f(K, z) = p_{K,z}(f)$. Also*

$$\psi(x, z) \equiv \max_{y\in\Gamma(x,z)} U(x, y, z) = T0(x, z)$$

[10]It is well known that $L^1(Z, \mathcal{Z}, Q_z)$ consists of equivalence classes rather than functions, identifying functions that are equal Q_z–almost everywhere.

while, for $p : \mathcal{K} \times Z \longmapsto \mathbb{R}_+$, the function $p[\Gamma] : X \times Z \longmapsto \mathbb{R}_+$ is defined by $p[\Gamma](x,z) = p(\Gamma(x,z),z)$, that is, it is the function of (x,z) obtained through p, when the compact sets K equal $\Gamma(x,z)$, for $x \in X$, $z \in Z$.

The next result shows that T is an L-local contraction, and gives the expression of L: Given $p : \mathcal{K} \times Z \mapsto \mathbb{R}_+$ for which $p[\Gamma] \in \mathcal{L}^1(Z; C(X))$, the operator L computes the seminorm of the function $p[\Gamma]$, that is, $(Lp)(K,z) = \beta p_{K,z}(p[\Gamma])$. Note that L is *nonlinear*. The expanded definition of the operator L is the expression (3.3) below.

Proposition 3.3. *Let the Bellman operator $T : F \longrightarrow E$, where $F \subseteq \mathcal{L}^1(Z; C(X))$, such that for all $p \in D(F)$, $p[\Gamma] \in \mathcal{L}^1(Z; C(X))$. Then, T is an L-local contraction on F with COP $L : D(F) \longrightarrow \mathbb{R}_+^{\mathcal{K} \times Z}$ given by*

$$(Lp)(K,z) = \beta \int_Z \max_{x \in K} p(\Gamma(x,z'),z') Q_z(dz'), \tag{3.3}$$

for all $K \in \mathcal{K}$ and $z \in Z$.

Proof. Following Blackwell (1965), we exploit the fact that T is monotone, in conjunction with the properties of the seminorms $p_{K,z}$. Let $f, g \in E$ and let $x \in X$, $K \in \mathcal{K}$ and $z \in Z$. Let $y \in \Gamma(x,z)$ and $z' \in Z$ arbitrary. Then $f(y,z') \leq g(y,z') + |f(y,z') - g(y,z')|$ implies $f(y,z') \leq g(y,z') + \max_{y \in \Gamma(x,z)} |f(y,z') - g(y,z')|$ and then, by monotonicity and linearity of the integral,

$$\int_Z f(y,z') Q_z(dz') \leq \int_Z g(y,z') Q_z(dz') + \int_Z \max_{y \in \Gamma(x,z)} |f(y,z') - g(y,z')| Q_z(dz').$$

We are allowed to take the integral by Lemma B.1. The inequality is maintained after multiplying by β and adding $U(x,y,z)$ to both sides. Then, by taking the maximum in $y \in \Gamma(x,z)$ to both sides, we have

$$\begin{aligned}(Tf)(x,z) &\leq (Tg)(x,z) + \beta \max_{y \in \Gamma(x,z)} \int_Z \max_{y \in \Gamma(x,z)} |f(y,z') - g(y,z')| Q_z(dz') \\ &= (Tg)(x,z) + \beta \int_Z \max_{y \in \Gamma(x,z)} |f(y,z') - g(y,z')| Q_z(dz') \\ &= (Tg)(x,z) + \beta p_{\Gamma(x,z),z}(f - g).\end{aligned}$$

Exchanging the roles of f and g, we have

$$|(Tf)(x,z) - (Tg)(x,z)| \leq \beta p_{\Gamma(x,z),z}(f-g). \tag{3.4}$$

It is convenient to write this inequality with the dummy variable z' instead of z. Now, taking the maximum in $x \in K$ and averaging with respect to the measure Q_z, we obtain

$$\int_Z \max_{x\in K} |(Tf)(x,z') - (Tg)(x,z')| Q_z(dz') \leq \beta \int_Z \max_{x\in K} p^{f-g}(\Gamma(x,z'),z') Q_z(dz').$$

Taking the maximum with respect to $x \in K$ in (3.4), we get $p_{K,z}(Tf - Tg) \leq (Lp^{f-g})(K,z)$, for all $K \in \mathcal{K}$, $z \in Z$, where L is the operator defined in (3.3). □

One of the difficulties in applying contraction techniques to the dynamic programming equation, when the return function and/or the space of shocks is unbounded, is the selection of a suitable space of functions where the Bellman operator is a selfmap. Assumption (B6) below provides a scheme to construct such a space along the lines of assumption (VI) in Section 2. This is in the same spirit of Assumption 9.3 in Stokey *et al.* (1989), pp. 248-249. This assumption is not about bounding the one-shot utility function U along any policy path by a function that depends only on time and the initial state, but about bounding its expected value with respect to the initial state. This is an important difference, as it allows us to deal with an unbounded space of shocks.

(B6) There is a collection of nonnegative functions $\{l_t\}_{t=0}^{\infty} \in \mathcal{L}^1(Z; C(X))$, such that for all $x \in X$, for all $z \in Z$

$$l_0(x,z) \geq \max(\psi(x,z), 0);$$
$$l_{t+1}(x,z) \geq \beta \int_Z \max\nolimits_{y\in\Gamma(x,z)} l_t(y,z') Q_z(dz'), \quad \text{for all } t = 0, 1, \ldots,$$

and the series $w := \sum_{t=0}^{\infty} l_t$ is unconditionally convergent, that is,

$$R_0(K,z) := \sum_{t=0}^{\infty} p_{K,z}(l_t) < \infty,$$

for all $K \in \mathcal{K}$, for all $z \in Z$.

Now we consider a suitable set C where L is defined.

$$C = \Big\{ p : \mathcal{K} \times Z \longmapsto \mathbb{R}_+ \, : \, p(K,z) \leq cR_0(K,z) \text{ for some } c > 0, \\ \text{and } p[\Gamma] \in \mathcal{L}^1(Z, C(X)) \Big\}. \tag{3.5}$$

As it is proved in Lemma B.4, C is not empty, as it contains the images of $V(0, R_0)$ by the family of seminorms $\mathcal{P}$.

Theorem 3.5 below is a fixed point theorem for the Bellman operator. We state a previous lemma.

Lemma 3.4. *Let assumptions (B1) to (B6) hold. Then T and L with C defined in* (3.5), *satisfy (I) to (VI).*

Theorem 3.5. *Let assumptions (B1) to (B6) hold. The following is true.*

(a) The Bellman equation admits a unique solution v^* in $V(0, R_0)$ and for all $v_0 \in V(0, R_0)$, $T^n v_0 \to v^*$ as $n \to \infty$, that is, $p_{K,z}(T^n v_0 - v^*) \to 0$, for all $K \in \mathcal{K}$ and $z \in Z$.

(b) The fixed point v^* coincides with the value function, $v = v^*$ and for all $z \in Z$ the optimal policy correspondence $\Gamma^*(\cdot, z) : X \to 2^X$ is non-empty, compact valued and upper hemicontinuous.

Proof. (a) T is an L–contraction by Proposition 3.3 and all the assumptions of Theorem 2.4 hold true by Lemma 3.4. Hence T admits a unique fixed point v^* in $V(0, R_0)$ and the successive iterations of T starting at any $v_0 \in V(0, R_0)$ converges to the v^* as $n \to \infty$ in the topology generated by the seminorms.

(b) To see that v^* is the value function of the problem, we invoke Theorem 9.2 in Stokey *et al.* (1989). Recall that, for any function F that is $\mu^t(z_0, \cdot)$-integrable, its conditional expectation can be expressed as

$$\begin{aligned} \mathrm{E}_{z_0}(F) &:= \int_{Z^t} F(z^t) \mu^t(z_0, dz^t) \\ &= \int_{Z^{t-1}} \left[\int_Z F(z^{t-1}, z_t) Q_{z_{t-1}}(dz_t) \right] \mu^{t-1}(z_0, dz^{t-1}) \\ &= \int_Z \left[\int_{Z^{t-1}} F(z_1, z_2^t) \mu^{t-1}(z_1, dz_2^t) \right] Q_{z_0}(dz_1). \end{aligned}$$

The assumptions of Theorem 9.2 in Stokey *et al.* (1989) are: (i) Γ is non-empty valued, with a measurable graph and admits a measurable selection; (ii) for each (x_0, z_0) and each feasible plan π from (x_0, z_0), $U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t)$ is $\mu^t(z_0, \cdot)$-integrable, $t = 1, 2, \ldots$, and the limit

$$U(x_0, \pi_0, z_0) + \lim_{n\to\infty} \sum_{t=1}^{n} \int_{Z^t} \beta^t U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t)\mu^t(z_0, dz^t) \tag{3.6}$$

exists; and (iii) $\lim_{t\to\infty} \int_{Z^t} \beta^t v^*(\pi_{t-1}(z^{t-1}), z_t)\mu^t(z_0, dz^t) = 0$.

(i) is implied by (B5) and (ii) is implied by (B6), since $|U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t)|$ is clearly measurable, given that U is a Carathéodory function. Moreover, since l_0 in (B6) is in $\mathrm{Ca}(X \times Z)$, we can apply Fubini's Theorem so that $l_0(\pi_1(z^1), z_2)$ is $\mu^2(z_0, \cdot)$-integrable and

$$\begin{aligned}
\int_{Z^2} l_0(\pi_1(z^1), z_2)\mu^2(z_0, dz^2) &= \int_{Z^1} \left(\int_Z l_0(\pi_1(z^1), z_2) Q_{z_1}(dz_2) \right) \mu^1(z_0, dz^1) \\
&\leq \int_{Z^1} \frac{1}{\beta} l_1(\pi_0(z_0), z_1)\mu^1(z_0, dz^1) \\
&\leq \frac{1}{\beta^2} l_2(x_0, z_0).
\end{aligned}$$

Both inequalities are due to assumption (B6). By induction, we get that $l(\pi_{t-1}(z^{t-1}), z_t)$ is $\mu^t(z_0, \cdot)$–integrable and

$$\int_{Z^t} l_0(\pi_{t-1}(z^{t-1}), z_t)\mu^t(z_0, dz^t) \leq \frac{1}{\beta^t} l_t(x_0, z_0).$$

Since $|U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t)| \leq l_0(\pi_{t-1}(z^{t-1}), z_t)$, the first part of (ii) is proved. Indeed, this estimate provides the bound

$$\begin{aligned}
&|U(x_0, \pi_0(z_0), z_0)| + \sum_{t=1}^{n} \int_{Z^t} \beta^t |U(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t)|\mu^t(z_0, dz^t) \\
&\quad \leq |U(x_0, \pi_0(z_0), z_0)| + \sum_{t=1}^{n} l_t(x_0, z_0) \leq w_0(x_0, z_0),
\end{aligned}$$

hence the second part of (ii) also holds, that is, the limit (3.6) is finite. Moreover, since the above inequality holds for any $\pi \in \Pi(x_0, z_0)$, it shows that the n–th iteration of T on the null function as the initial seed satisfies $|T^n 0(x_0, z_0)| \leq w_0(x_0, z_0)$. Hence, since $\int_Z |T^n 0(x_0, z_1) - v^*(x_0, z_1)| Q_{z_0}(dz_1)$ tends to 0 as $n \to \infty$, by part (a) above, we obtain the bound

$$\int_Z |v^*(x_0, z_1)| Q_{z_0}(dz_1) \leq \int_Z w_0(x_0, z_1) Q_{z_0}(dz_1). \tag{3.7}$$

This inequality will be used to show (iii). First, we claim that for any t, for any $\pi \in \Pi(x_0, z_0)$,

$$\int_{Z^t} \beta^t w_0(\pi_{t-1}(z^{t-1}), z_t)\mu^t(z_0, dz^t) \leq \sum_{s=t}^{\infty} l_s(x_0, z_0).$$

To prove it, we employ mathematical induction. Let $t = 1$. Then, by assumption (B6)

$$\begin{aligned}
\int_Z \beta w_0(\pi_0(z_0), z_1)\mu^1(z_0, dz^1) &= \int_Z \beta \sum_{t=0}^{\infty} l_t(\pi_0(z_0), z_1)Q_{z_0}(dz_1) \\
&= \sum_{t=0}^{\infty} \beta \int_Z l_t(\pi_0(z_0), z_1)Q_{z_0}(dz_1) \\
&\leq \sum_{t=0}^{\infty} l_{t+1}(x_0, z_0).
\end{aligned}$$

The exchange of the integral and infinite sum is possible by the Monotone Convergence Theorem. Suppose that the property is true for t and let us prove it for $t+1$. Then it will hold for any t. Note

$$\begin{aligned}
&\int_{Z^{t+1}} \beta^{t+1} w_0(\pi_t(z^t), z_{t+1})\mu^{t+1}(z_0, dz^{t+1}) \\
&\qquad = \int_Z \left(\beta \int_{Z^t} \beta^t w_0(\pi_{t-1}(z^{t-1}), z_t)\mu^t(z_0, dz^t) \right) Q_{z_0}(dz_1) \\
&\qquad \leq \int_Z \beta \sum_{s=t}^{\infty} l_s(\pi_0(z_0), z_1)Q_{z_0}(dz_1) \\
&\qquad \leq \sum_{s=t+1}^{\infty} l_s(x_0, z_0),
\end{aligned}$$

again by the Monotone Convergence Theorem, and where we have used Fubini's Theorem and the induction hypothesis. This and (3.7) imply (iii), since the series w_0 converges. Thus, v^* is the value function. The claims about Γ^* are immediate from the Theorem of the Maximum of Bergé and the Measurable Maximum Theorem, see Aliprantis and Border (1999). $\square$

The following result provides a sufficient condition for (B6).

Proposition 3.6. *Let assumptions (B1) to (B5) hold. Suppose that there is $l_0 \in \mathcal{L}^1(Z; C(X))$ with $|\psi| \leq l$, $\alpha \geq 0$ such that $\alpha\beta < 1$, and*

$$\int_Z \max_{y \in \Gamma(x,z)} l_0(y, z')Q_z(dz') \leq \alpha l_0(x, z),$$

for all $x \in X$, $z \in Z$. Then (B6) holds, with $R_0(K, z) = p_{K,z}(l_0)/(1 - \alpha\beta)$.

Proof. Choose $l_t = (\alpha\beta)^t l_0$, for $t = 0, 1, \ldots$. Then

$$\beta \int_Z \max_{y \in \Gamma(x,z)} l_t(y, z')Q_z(dz') = \beta(\alpha\beta)^t \int_Z \max_{y \in \Gamma(x,z)} l_0(y, z')Q_z(dz')$$
$$\leq (\alpha\beta)^{t+1} l_0(x, z) = l_{t+1}(x_0, z_0).$$

Hence, $w(x_0, z_0) = l_0(x_0, z_0)/(1 - \alpha\beta)$ and $R_0(K, z) = p_{K,z}(l_0)/(1 - \alpha\beta)$, for $K \in \mathcal{K}$ and $z \in Z$. □

3.1 Sharper Estimates

Many interesting problems have utility functions which are unbounded from below[11] In this case, the estimates given in assumption (B6) are not efficient. A generalization of (B6) allows us to construct sharper estimates in the form of an order interval of functions that is mapped into itself by the operator T.

(B6)' There are two collections of functions $k_t, l_t : X \times Z \to \mathbb{R}$, with $k_t, l_t \in \mathcal{L}^1(Z; C(X))$, for all $t = 0, 1, \ldots$, satisfying that for all $x \in X$, all $z \in Z$, there exists $\overline{y}(x, z) \in \Gamma(x, z)$ such that

$$k_0(x, z) \leq \min(U(x, \overline{y}(x, z), z), 0);$$

$$k_{t+1}(x, z) \leq \beta \int_Z k_t(\overline{y}(x, z), z')Q_z(dz');$$

$$l_0(x, z) \geq \max(\psi(x, z), 0);$$

$$l_{t+1}(x, z) \geq \int_Z \max_{y \in \Gamma(x,z)} l_t(y, z')Q_z(dz'), \quad \text{all } t = 1, 2, \ldots;$$

and the series

$$u_t(x, z) := \sum_{s=t}^{\infty} k_s(x, z) \quad \text{and} \quad w_t(x, z) := \sum_{s=t}^{\infty} l_s(x, z)$$

are unconditionally convergent, for all $t = 0, 1, 2, \ldots$.

[11] We refer here to problems where the utility function is not bounded from below, but never takes the value $-\infty$.

Let $I_{u_0,w_0} = \{f \in \mathcal{L}^1(Z; C(X)) : u_0 \leq f \leq w_0\}$. The following result states Theorem 3.5 in the more restricted space of functions $V_{I_{u_0,w_0}}$, thus providing a method to get the contraction property of the Bellman operator in an unbounded from below case where U never takes the value $-\infty$ on Ω. In the following theorem, we let $\widetilde{l}_t = \max\{|k_t|, |l_t|\}$ for all $t = 0, 1, \ldots$ and $\widetilde{R}_0(K, z) = \sum_{t=0}^{\infty} p_{K,z}(\widetilde{l}_t)$ for all $K \in \mathcal{K}$ and $z \in Z$.

Theorem 3.7. *Suppose that (B1)–(B5) and (B6') hold. Then Theorem 3.5 holds with* $V_{I_{u_0,w_0}}(0, \widetilde{R}_0)$*, replacing* $V(0, \widetilde{R}_0)$.

Proof. By Lemma B.5, T is a self map on I_{u_0,w_0}. Notice that $|\psi(x, z)| \leq \widetilde{l}_0(x, z)$. Hence, (B6) holds true for $\widetilde{l}_t$ by definition of $\widetilde{l}_t$ and Theorem 3.5 applies in $V_{I_{u_0,w_0}}(0, \widetilde{R}_0)$. □

In what follows, to simplify the exposition, we introduce the following notation: for a function $f \in \mathrm{Ca}(X \times Z)$

$$\widehat{f}(x, z, z') = \max_{y \in \Gamma(x,z)} f(y, z'). \tag{3.8}$$

4 Application to endogenous growth

Endogenous growth models have become fundamental to understand economic growth. Several contributions consider an unbounded shock space, like Stachurski (2002), Kamihigashi (2007), Matkowski and Nowak (2011) or Baüerle and Jaśkiewicz (2018). I consider here the stochastic endogenous growth model studied in Jones *et al.* (2005), which is described as follows. The preferences of the agent over random consumption sequences are given by

$$\max\ \mathrm{E} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} \upsilon(\ell_t)}{1-\sigma}, \tag{4.1}$$

subject to

$$c_t + k_{t+1} + h_{t+1} \leq z_t A k_t^{\alpha} (n_t h_t)^{1-\alpha} + (1 - \delta_k) k_t + (1 - \delta_h) h_t, \tag{4.2}$$

$$\ell_t + n_t \leq 1, \tag{4.3}$$

$$c_t, k_t, h_t, \ell_t, n_t \geq 0 \tag{4.4}$$

for all $t = 0, 1, \ldots$, with k_0 and h_0 given. Here, $\{z_t\}$ is a Markov stochastic process with transition probability $Q_z(\cdot)$ and $Z = (0, \infty)$; c_t is consumption; ℓ_t is leisure; n_t is hours spent working; k_t and h_t are the stock of physical and human capital, respectively; δ_k and δ_h are the depreciation rates on physical and human capital, respectively; and υ is a continuous function on $(0,1]$, strictly increasing. The usual non-negativity constraints on consumption, investment, leisure and hours worked apply. If we let $k' = k_{t+1}$, $h' = h_{t+1}$ and $k = k_t$, $h = h_t$, $c = c_t$, $n = n_t$ and $\ell = \ell_t$, then the feasible correspondence is

$$\Gamma(k,h,z) = \Big\{(k',h') \;:\; \text{There are } c,n,\ell \text{ such that (4.2)–(4.4) hold}\Big\},$$

and the utility function is $U(c,\ell) = c^{1-\sigma}\upsilon(\ell)/(1-\sigma)$. Regarding the function υ, we consider $\upsilon(\ell) = \ell^{\psi(1-\sigma)}$. The endogenous state space is $X = [0,\infty)\times[0,\infty)$ and the family of compact sets $\mathcal{K}$ is formed by compact sets in the product space $\mathbb{R}_+ \times \mathbb{R}_+$. The Markov chain is given by the log–log process

$$\ln z_{t+1} = \rho \ln z_t + \ln w_{t+1}, \tag{4.5}$$

with $\rho \geq 0$ and where the w's are i.i.d., with support in $W \subseteq (0,\infty)$. Let μ be the distribution measure of the w's. Note that $\rho = 0$ corresponds to shocks z_t that are i.i.d.. Jones *et al.* (2005) suppose that $z_t = \exp\big(\zeta_t - (1/2)\sigma_\epsilon^2/(1-\rho^2)\big)$, where $\zeta_{t+1} = \rho\zeta_t + \epsilon_{t+1}$ and the ϵ's are i.i.d., normal with mean 0 and variance σ_ϵ^2. This corresponds to (4.5) with $w_{t+1} = \exp\big(\epsilon_{t+1} - (1/2)\sigma_\epsilon^2/(1+\rho)\big)$. We do not restrict ϵ to be normally distributed.

To shorten notation, we will use along this subsection the following definitions.

$$\begin{aligned}
\gamma &= \alpha^\alpha(1-\alpha)^{1-\alpha},\\
\delta &= \min\{\delta_k,\delta_h\},\\
\nu &= (1-\delta)^{1-\sigma},\\
\mathcal{E} &= \prod_{s=0}^{\infty} \mathrm{E}w^{\rho^s(1-\sigma)}.
\end{aligned}$$

Along this section, convergence means convergence with respect to the seminorms $p_{K,z}$, $K \in \mathcal{K}$, $z \in Z$. We only assume that the expectation $\mathrm{E}w$ is finite.

Proposition 4.1. *Consider the endogenous growth model described in* (4.1)–(4.5) *with* $0 \leq \sigma < 1$ *and* $0 \leq \rho < 1$. *If*

$$\beta\left(A\gamma\left(\mathrm{E}w\right)^{1/(1-\rho)}\right)^{1-\sigma} + \nu < 1, \tag{4.6}$$

then the associated Bellman equation admits a unique solution, v^*, in the set $V(0, R_0)$ defined in (2.2) *given by*

$$V(0, R_0) = \left\{f \in \mathcal{L}^1(Z; C(X)) \, : \, p_{K,z}(f) \leq R_0(K, z), \quad \forall (K, z) \in \mathcal{K} \times Z\right\},$$

where, for $K \in \mathcal{K}$ and $z \in Z$, $R_0(K, z) = p_{K,z}(\sum_{t=0}^{\infty} l_t)$ and the family $\{l_t\}_{t=0}^{\infty}$ is given by (4.7) *and* (4.9) *in the proof below. Moreover, v^* is the value function v and $T^n v_0$ converges to v as $n \to \infty$ for all initial guesses $v_0 \in V(0, R_0)$. Finally, the optimal policies are continuous functions of (k, h).*

Proof. We check all the hypotheses of Theorem 3.5. It is clear that (B1)–(B5) are fulfilled. We focus on (B6) and define $g(k, h, z) = Azk^{\alpha}h^{1-\alpha} + (1 - \delta)(k + h)$. Since $0 \leq \sigma < 1$, both U and v are bounded from below by zero, and v is bounded above by 1. By the definition of δ, we have $zAk^{\alpha}(nh)^{1-\alpha} + (1 - \delta_k)k + (1 - \delta_h)h \leq g(h, k, z)$. Then

$$\psi(k, h, z) = \max_{(k',h',c,n,\ell)\in\Gamma(k,h,z)} u(c, \ell) \leq \frac{1}{1 - \sigma} g(k, h, z)^{1-\sigma} \equiv l_0(k, h, z). \tag{4.7}$$

According to (3.8), let

$$\widehat{l}_0(k, h, z, z') = \max_{(k',h',c,n,\ell)\in\Gamma(k,h,z)} l_0(k', h', z').$$

Let us determine a bound for $\widehat{l}_0(k', h', z, z')$. To this end, consider the Lagrange problem

$$\begin{aligned} \max \ & l_0(k', h', z'), \\ \text{s. t.: } & k' + h' \leq g(k, h, z), \\ & k', h' \geq 0, \end{aligned} \tag{4.8}$$

and notice that its feasible set is larger than $\Gamma(k, h, z)$. The constraint is binding at the optimal solution, which is $k' = \alpha g(k, h, z)$, $h' = (1 - \alpha)g(k, h, z)$. Substituting this into the objective function of (4.8), we find its optimal value, which is

$$\left(Az'\alpha^{\alpha}(1 - \alpha)^{1-\alpha} + (1 - \delta)\right)^{1-\sigma} l_0(k, h, z) \leq \left(A^{1-\sigma}(z')^{1-\sigma}\gamma^{1-\sigma} + \nu\right) l_0(k, h, z),$$

where the inequality is due to the function $c \mapsto c^{1-\sigma}$ being subaditive, as it is concave and null at zero. Computing the conditional expectation of the right hand side of the above inequality, we have

$$\int_Z \widehat{l}_0(k', h', z, z')Q_z(dz') \leq \left(A^{1-\sigma}z^{\rho(1-\sigma)}\mathrm{E}w^{1-\sigma}\gamma^{1-\sigma} + \nu\right) l_0(k, h, z).$$

Thus, we define $l_1(k,h,z) = \beta \left(A^{1-\sigma} z^{\rho(1-\sigma)} \mathrm{E}w^{1-\sigma}\gamma^{1-\sigma} + \nu\right) l_0(k,h,z)$.

To calculate $\widehat{l}_1(k',h',z,z')$, we note that we face the same Lagrange problem (4.8) above, modulo the "constant" factor $\beta \left(A^{1-\sigma}(z')^{\rho(1-\sigma)} \mathrm{E}w^{1-\sigma}\gamma^{1-\sigma} + \nu\right)$. Thus we will find

$$\widehat{l}_1(k',h',z,z') \le \left(A^{1-\sigma}(z')^{\rho(1-\sigma)} \mathrm{E}w^{1-\sigma}\gamma^{1-\sigma} + \nu\right) l_1(k,h,z)$$

and then

$$\int_Z \widehat{l}_1(k',h',z,z') Q_z(dz') \le \left(A^{1-\sigma} z^{\rho^2(1-\sigma)} \mathrm{E}w^{1-\sigma} \mathrm{E}w^{\rho(1-\sigma)}\gamma^{1-\sigma} + \nu\right) l_1(k,h,z).$$

Now define $l_2(k,h,z) = \beta \left(A^{1-\sigma} z^{\rho^2(1-\sigma)} \mathrm{E}w^{1-\sigma} \mathrm{E}w^{\rho(1-\sigma)}\gamma^{1-\sigma} + \nu\right) l_1(k,h,z)$. Using induction, it can be proved exactly the same as for the cases $t=1$ and $t=2$, that the family of functions $\{l_t\}_{t=0}^{\infty}$ given by

$$l_{t+1}(k,h,z) = \beta \left(A^{1-\sigma} z^{\rho^{t+1}(1-\sigma)} \mathrm{E}w^{1-\sigma} \mathrm{E}w^{\rho(1-\sigma)} \cdots \mathrm{E}w^{\rho^t(1-\sigma)}\gamma^{1-\sigma} + \nu\right) l_t(k,h,z), \quad (4.9)$$

for all $t = 0,1,\ldots$, satisfies the inequalities demanded in (B6). Regarding the series $w_0(k,h,z) = \sum_{t=0}^{\infty} l_t(k,h,z)$, note that the ratio

$$\frac{l_{t+1}(k,h,z)}{l_t(k,h,z)} = \beta A^{1-\sigma} z^{\rho^{t+1}(1-\sigma)} \mathrm{E}w^{1-\sigma} \mathrm{E}w^{\rho(1-\sigma)} \cdots \mathrm{E}w^{\rho^t(1-\sigma)}\gamma^{1-\sigma} + \nu,$$

converges to $\beta A^{1-\sigma}\mathcal{E}\gamma^{1-\sigma} + \nu$ as $t \to \infty$, which is smaller than one by the assumption of the theorem, since by Jensen's inequality $\mathcal{E} \le (\mathrm{E}w)^{(1-\sigma)/(1-\rho)}$. Thus, by the ratio test, the series converges pointwise. It is easy to see that inequality (4.6) guarantees unconditional convergence as well, thus (B6) is fulfilled. To finish the proof, Theorem 3.5 shows that the optimal policy correspondence is non–empty and upper hemi–continuous. The Convex Maximum Theorem of Bergé assures that the value function is concave and since the utility function is strictly concave, the optimal policies are unique, so continuous with respect to (k,h).

$\square$

5 Comparison with other approaches

In this section we compare our results with those obtained by other methods: the already classical weighted norm approach, the one based on countable local contractions and the recent Q–transform. In each case, we give a brief description of the method and then, we work through model examples showing the differences with our method. A word of caution is needed here. As said in the Introduction, we study dynamic problems where the actions do not influence the evolution of uncertainty: it is exogenous. The present paper's aim is to take advantage of the special structure of the state space of the kind of problems we analyze, to obtain further insights.

5.1 The weighted norm approach

In the weighted contraction approach—see Boyd (1990), Becker and Boyd (1997) for the deterministic case and Hernández-Lerma and Lasserre (1999), Jaśkiewicz and Nowak (2011) and Bäuerle and Rieder (2011) for the stochastic case, it is postulated the existence of a continuous function $\varphi : X \times Z :\longrightarrow \mathbb{R}_{++}$, called bounding or weighing function, such that there exist nonnegative constants M and α such that for all $x \in X$, $z \in Z$, $y \in \Gamma(x, z)$

(W1) $|U(x, y, z)| \leq M\varphi(x, z)$ and

(W2) $\int_Z \varphi(y, z')Q_z(dz') \leq \alpha\varphi(x, z)$.

Given such a function φ, the following Banach space is considered

$$C_\varphi = \left\{ f : X \times Z \longrightarrow \mathbb{R} \text{ continuous } : \sup_{(x,z)\in X\times Z} \frac{|f(x,z)|}{\varphi(x,z)} < \infty \right\},$$

where the norm is defined by $\|f\|_\varphi = \sup_{(x,z)\in X\times Z} (|f(x, z)|/\varphi(x, z))$.

Consider the following result, that can be found in (Hernández-Lerma and Lasserre, 1999, Section 8.3), or in (Bäuerle and Rieder, 2011, Theorem WSN, p. 208).

Theorem 5.1. *Let φ be a continuous function satisfying (W1)-(W2) above, such that*

$$\alpha\beta < 1. \tag{5.1}$$

If

(WC1) Γ is nonempty and continuous and $\Gamma(x, z)$ is compact valued for all $(x, z) \in X \times Z$,

(WC2) U is continuous

(WC3) $(x, z) \longmapsto \int_Z \varphi(x, z')Q_z(dz')$ is continuous,

then (a) the value function is the unique continuous solution to the Bellman equation in C_φ, (b) value function iteration converges from any initial $f \in C_\varphi$, (c) at least one optimal policy exists and (d) that policy maximizes the right hand side of the Bellman equation.

The conditions of Theorem 5.1 are stricter than those of Theorem 3.5. To see this, we can take $l_0 = \varphi$ in Proposition 3.6 above to construct the family $\{l_t\}_{t=0}^\infty$ such that Theorem 3.5 applies. The opposite is not true, that is, Theorem 3.5 is not contained in Theorem 5.1. First, Theorem 3.5 does not impose continuity in both variables (x, z), as required in (WC3). We show in Appendix C a simple pure currency model where the value function is discontinuous with respect to the exogenous variable and where no bounding function φ may satisfy (WC3). Second, even if (WC3) is fulfilled, the uniqueness of the solution to the Bellman equation is given in a larger space, since $\mathrm{Ca}(X \times Z)$ contains $C_\varphi(X \times Z)$, for any bounding function φ. Third and last, more importantly beyond the issues of continuity just discussed, there are bounded from below models for which a bounding function cannot exist but Theorem 3.5 is applicable. Since that, in principle, there are infinitely many candidates for bounding functions, to find a model with returns bounded from below, for which there is no a suitable bounding function is not an easy task. We will use the result below, which states a necessary condition for the existence of a bounding function φ.

For a function f depending on the variables (x, z), remind the notation $\widehat{f}(x, z, z') = \max_{y \in \Gamma(x,z) \in X \times Z} f(y, z')$. Also, define $\binom{0}{0} = 1$.

Proposition 5.2. *Let there be a dynamic programming problem $(X, Z, \Gamma, Q, U, \beta)$ for which there is a continuous bounding function φ satisfying the conditions of Theorem 5.1. Define the family of functions $\{l_t\}_{t=0}^\infty$, with $l_0 \in C_\varphi(X \times Z)$ and $l_{t+1} = \beta \int_Z \widehat{l}_t Q_z$. Then, there is a constant M such that*

$$\sum_{t=0}^{\infty} \binom{t+r}{r} l_t \leq \frac{M}{(1-\alpha\beta)^{r+1}} \varphi, \quad \textit{for all } r = 0, 1, \ldots. \tag{5.2}$$

Proof. We often eliminate the arguments (x, z) in what follows to simplify notation. Note that $l_1 = \beta \int_Z \widehat{l}_0 Q_z \leq M\beta \int_Z \widehat{\varphi} Q_z \leq M(\alpha\beta)\varphi$ for some constant M and $\alpha\beta < 1$, since $l_0 \in C_\varphi(X \times Z)$ and φ is a suitable bounding function. By induction, $l_t \leq M(\alpha\beta)^t \varphi$, for all t. Hence, $w_0 = \sum_{t=0}^\infty l_t \leq M\varphi/(1-\alpha\beta)$ is finite and

$$\int_Z \widehat{w}_0 Q_z \leq \frac{M}{(1-\alpha\beta)} \int_Z \widehat{\varphi} Q_z \leq M \frac{\alpha}{(1-\alpha\beta)} \varphi.$$

Let $w_1 = \sum_{s=1}^{\infty} l_s$. Then, as in the proof of Theorem 3.5, we have

$$\int_Z \widehat{w_0} Q_z = \int_Z \sum_{t=0}^{\infty} \widehat{l}_t Q_z = \sum_{t=0}^{\infty} \int_Z \widehat{l}_t Q_z = \frac{1}{\beta} \sum_{t=0}^{\infty} l_{t+1} = \frac{1}{\beta} w_1.$$

Thus, we have obtained $w_1(x,z) \leq M\varphi(x,z)\alpha\beta/(1-\alpha\beta)$. In general, the following inequality holds

$$w_t(x,z) \leq M \frac{(\alpha\beta)^t}{1-\alpha\beta} \varphi(x,z), \quad \text{for all } t = 0, 1, \ldots, \tag{5.3}$$

where $w_t = \sum_{s=t}^{\infty} l_s$. To show this, we use the Principle of Induction. The cases $t = 0, 1$ have just been proved. Suppose that it is true for t. Then

$$\int_Z \widehat{w}_t Q_z \leq M \frac{(\alpha\beta)^t}{1-\alpha\beta} \int_Z \widehat{\varphi} Q_z \leq M \frac{(\alpha\beta)^t}{1-\alpha\beta} \alpha\varphi,$$

and on the other hand, as in the computation above

$$\int_Z \widehat{w_t} Q_z = \int_Z \sum_{s=t}^{\infty} \widehat{l}_s Q_z = \sum_{s=t}^{\infty} \int_Z \widehat{l}_s Q_z = \frac{1}{\beta} \sum_{s=t}^{\infty} l_{s+1} = \frac{1}{\beta} w_{t+1},$$

thus we get the inequality sought. Adding (5.3) from $t = 0$ to $t = \infty$, we obtain[12]

$$\sum_{t=0}^{\infty} (t+1)\ell_t(x,z) \leq \frac{M}{(1-\alpha\beta)^2} \varphi(x,z) < \infty. \tag{5.4}$$

From (5.4), replacing x by x' and z by z' and integrating with respect to Q_z in both sides of the inequality and using the properties of φ, we have

$$\int_Z \sum_{t=0}^{\infty} (t+1)\widehat{l}_t Q_z = \sum_{t=0}^{\infty} (t+1) \int_Z \widehat{l}_t Q_z = \frac{1}{\beta} \sum_{t=0}^{\infty} (t+1) l_{t+1} \leq M \frac{\alpha}{(1-\alpha\beta)^2} \varphi.$$

Thus, $\sum_{t=0}^{\infty} (t+1) l_{t+1} \leq M\varphi\alpha\beta/(1-\alpha\beta)^2$, which is (5.2) with $r = 1$. Repeating the scheme above s steps, we get $\sum_{t=0}^{\infty} (t+1) l_{t+s} \leq M\varphi(\alpha\beta)^s/(1-\alpha\beta)^2$. Adding in s again as in (5.4), we obtain

$$\sum_{t=0}^{\infty} \binom{t+2}{2} l_t = \sum_{t=0}^{\infty} \frac{(t+2)(t+1)}{2} l_t \leq \frac{M}{(1-\alpha\beta)^3} \varphi.$$

By induction, and using the same arguments as for the cases $r = 1$ and $r = 2$, it can be proved that for any $r \geq 0$

$$\sum_{t=0}^{\infty} \binom{t+r}{r} l_t \leq \frac{M}{(1-\alpha\beta)^{r+1}} \varphi.$$

$\square$

[12] $w_0 + w_1 + w_2 + \cdots = (l_0 + l_1 + l_2 + \cdots) + (l_1 + l_2 + \cdots) + (l_2 + \cdots) + \cdots = l_0 + 2l_1 + 3l_2 + \cdots$.

This result will be used to show that the weighted norm approach cannot be applied to the following simple growth model with a linear technology, multiplicative shocks and a bounded from below, strictly increasing and strictly concave, felicity function (but our approach works.)

Example 5.3. Consider an optimal growth model which Bellman equation is

$$v(k,z) = \max_{k' \in [0,zk]} \left(U(k,k',z) + \beta \int_Z v(k',z') Q_z(dz') \right),$$

where $k \in [0,\infty)$, $Z = \{1,g\}$, with $g > 1$, $Q_z := Q$ is given by $Q(g) = p > 0$, $Q(1) = q > 0$, with $p + q = 1$ and the utility function is $U(k,k',z) = u(zk - k')$, where

$$u(c) = \frac{1+c}{3+\ln^2(1+c)} - \frac{1}{3}.$$

The function u is nonnegative, continuous, unbounded, strictly increasing and strictly concave on $[0,\infty)$, with $u(0) = 0$. [13]

The discount factor is taken to be $\beta \le 1/(gp+q) < 1$. Clearly, any solution of the Bellman equation has $v(0,z) = 0$, for all k, z. Let $k > 0$ and $z \in \{1,g\}$. With a view to use the necessary condition in Proposition 5.2, let us take $\ell_0(k,z) = \max_{k' \in [0,zk]} U(k,k',z) = u(zk)$. It is clear that $l_0 \le M\varphi$ for $M = 1$, since $l_0 = \psi \le \varphi$ by definition of the bounding function φ. Let

$$\begin{aligned} \ell_1(k,z) &= \beta \int_Z \max_{k' \in [0,zk]} \ell_0(k',z') Q(dz') = \beta(pu(gzk) + qu(zk)), \\ \ell_2(k,z) &= \beta \int_Z \max_{k' \in [0,zk]} \ell_1(k',z') Q(dz') = \beta^2(p^2u(g^2zk) + pqu(gzk) + qpu(gzk) + q^2u(zk)) \\ &\vdots \end{aligned}$$

In general, $\ell_t(k,z) = \beta^t \sum_{s=0}^t \binom{t}{s} p^s q^{t-s} u(g^s zk)$, for all t, which follows by induction. We have

$$\sum_{t=0}^{\infty} l_t(k,z) = \sum_{t=0}^{\infty} \sum_{s=0}^{t} \binom{t}{s} (\beta p)^s (\beta q)^{t-s} u(g^s zk) = \sum_{s=0}^{\infty} (\beta p)^s u(g^s zk) \sum_{t=s}^{\infty} \binom{t}{s} (\beta q)^{t-s},$$

[13] Letting $c > 0$, denote $x = \ln(1+c) > 0$. Then $u'(c) = (3 + x^2 - 2x)/(3+x^2)^2 > 0$ and $u''(c) = -2e^{-x}((x-1)^3 + 2)/(3+x^2)^3 < 0$. Also note that u is unbounded but $u'(c) \to 0$ as $c \to \infty$. For another example where there is no suitable bounding function φ whatever the value of the discount factor β, see Rincón-Zapatero (2022).

where the change of order summation is admissible since the double series is of positive terms. Now,

$$\sum_{t=s}^{\infty}\binom{t}{s}(\beta q)^{t-s} = \frac{1}{s!}\sum_{t=s}^{\infty} t(t-1)\cdots(t-s+1)(\beta q)^{t-s}.$$

The series at the right hand side is the value of the s-th derivative of the power series $\sum_{t=0}^{\infty} x^t = 1/(1-x)$, evaluated at $x = \beta q$, hence $\sum_{t=s}^{\infty} t(t-1)\cdots(t-s+1)(\beta q)^{t-s}$ equals $(d^s/dx^s)\ (1/(1-x))|_{x=\beta q} = s!/(1-\beta q)^{s+1}$. Thus

$$\sum_{t=0}^{\infty} l_t(k,z) = \frac{1}{(1-\beta q)}\sum_{s=0}^{\infty}\left(\frac{\beta p}{1-\beta q}\right)^s u(g^s zk). \tag{5.5}$$

Let us see that this series converges for all $k > 0$ and $z \in Z$. Note that

$$u(g^s zk) = \frac{1}{3+\ln^2(1+g^s zk)} + \frac{g^s zk}{3+\ln^2(1+g^s zk)} - \frac{1}{3}$$

Thus, the series (5.5) decomposes into the sum of three series. The first and the third series are

$$\frac{1}{(1-\beta q)}\sum_{s=0}^{\infty}\left(\frac{\beta p}{1-\beta q}\right)^s \frac{1}{3+\ln^2(1+g^s zk)} \quad \text{and} \quad \frac{-1}{3(1-\beta q)}\sum_{s=0}^{\infty}\left(\frac{\beta p}{1-\beta q}\right)^s,$$

respectively, which are convergent, since $\beta p < 1-\beta q$ (use the ratio test), and the series in the middle is

$$\frac{1}{(1-\beta q)}\sum_{s=0}^{\infty}\left(\frac{\beta pg}{1-\beta q}\right)^s \frac{zk}{3+\ln^2(1+g^s zk)}.$$

Obviously, this series converges when $\beta < 1/(gp+q)$. When $\beta = 1/(gp+q)$, $(\beta pg)/(1-q\beta) = 1$, and the series reduces to $(1-q\beta)^{-1} zk \sum_{s=0}^{\infty} 1/(3+\ln^2(1+g^s zk))$, which is convergent.[14] A similar and straightforward argument shows that the series $\sum_{t=0}^{\infty} p_{K,z}(l_t)$ converges for all compact set K of $[0,\infty)$ and $z \in Z$.[15] Hence, all conditions of Theorem 3.5 are fulfilled

[14]Since

$$\lim_{s\to\infty}\frac{3+\ln^2(1+g^s zk)}{3+\ln^2(g^s zk)} = 1,$$

by the Limit Comparison Test, the series has the same character than

$$\sum_{s=0}^{\infty}\frac{1}{3+\ln^2(g^s zk)} = \sum_{s=0}^{\infty}\frac{1}{3+(s\ln g+\ln(zk))^2},$$

which is convergent for all $k > 0$ and $z \in Z$.

[15]Since $p_{K,z}(l_t) = \int_Z \max_{k\in K} l_t(k,z')Q(dz')$ and u is increasing, the problematic part in $\sum_{t=0}^{\infty} p_{K,z}(l_t)$, which is $(1-q\beta)^{-1} za \sum_{t=0}^{\infty} 1/(3+\ln^2(1+g^t za))$, is also convergent, where $a = \max K$ and $K \neq \{0\}$ is a compact set of $[0,\infty)$. If the compact set is $K = \{0\}$, then $p_{K,z}(l_t) = 0$.

and this growth model admits a solution. Now we argue by contradiction, assuming that a bounding function φ satisfying the assumptions of Theorem 5.1 exists when $\beta = 1/(gp + q)$. Then, by Proposition 5.2, there are constants M and $\alpha \geq 0$ such that $\alpha\beta < 1$ and $\sum_{t=0}^{\infty} \binom{t+r}{r} l_t(k,z) \leq \alpha\beta M\varphi(k,z)/(1-\alpha\beta)^2 < \infty$, for all $k > 0$, $z \in Z$, for all $r \geq 0$. This is clearly impossible: for instance, for $r = 1$, the lefthand series is

$$\sum_{t=0}^{\infty}(t+1)l_t(k,z) = \frac{zk}{(1-q\beta)}\sum_{t=0}^{\infty}\frac{t+1}{3+\ln^2(1+g^t zk)},$$

which diverges, attaining a contradiction.[16] Thus, a function φ satisfying the assumptions of Theorem 5.1 cannot exist.

This example can be extended to $u(c) = (1+c)/(b+\ln^a(1+c)) - 1/b$, with $a > 1$ and $b \geq \max(1+a, (1-a)^{(1-a)})$. These inequalities guarantee that u is strictly increasing and strictly concave. The series (5.5) converges since $a > 1$ with the same condition for β, $\beta \leq 1/(gp+q)$, but the series

$$\sum_{t=0}^{\infty}\binom{t+r}{r}l_{t+1}(k,z) = \frac{zk}{r!(1-q\beta)}\sum_{t=0}^{\infty}\frac{(t+r)(t+r-1)\cdots(t+1)}{b+\ln^a(1+g^t zk)}$$

diverges for all positive integer $r \geq a-1$, since the numerator is a polynomial of degree r and the series then has the same character than $\sum_{t=1}^{\infty} 1/t^{a-r}$, which is convergent if and only if $a - r > 1$.

5.2 Countable local contractions

Matkowski and Nowak (2011) and Jaśkiewicz and Nowak (2011) extend the (countable) local contraction approach developed in Rincón-Zapatero and Rodríguez-Palmero (2003) from the deterministic case to the stochastic case.[17] In this section we describe the method

[16] Note that

$$\frac{\frac{t+1}{3+\ln^2(1+g^t zk)}}{\frac{1}{t+1}} = \frac{(t+1)^2}{3+(t\ln g+\ln(zk))^2}\,\frac{3+(t\ln g+\ln(zk))^2}{3+\ln^2(1+g^t zk)} \to \frac{1}{\ln^2 g}\cdot 1,$$

as $t \to \infty$, thus by the Limit Comparison Test, the series has the same character than the harmonic series $\sum_{t=0}^{\infty} 1/(t+1)$.

[17] Our approach can be considered a way to extend the (uncountable) local contraction method in Martins da Rocha and Vailakis (2010) from the deterministic to a stochastic setting. The words

and point out some of the features that may restrict its applicability to stochastic dynamic programs where the Markov chain is exogenous.

The following are the assumptions in (Jaśkiewicz and Nowak, 2011, Section 3), adapted to our setting:

(W) The technological correspondence Γ is upper semicontinuous, $\Gamma(x,z)$ is compact for each $(x,z) \in X \times Z$, $U : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous and for any continuous function $f : X \times Z \longrightarrow \mathbb{R}$

$$(x,z) \mapsto \int_Z f(x,z')Q_z(dz') \tag{5.6}$$

is continuous.

(C1) There exists a sequence $\{X_j\}_{j=0}^{\infty}$ of non-empty Borel subsets of $X \times Z$ such that $X_j \subseteq X_{j+1}$ for all $j \geq 0$ and $X \times Z = \bigcup_{j=0}^{\infty} \text{Int} X_j$.

(C2) Letting $m_j = \sup_{(x,z)\in X_j} \sup_{y\in\Gamma(x,z)} \max(U(x,y,z),0)$, for all $j \geq 0$, assume that $m_0 > 0$, $m_j < 0$ for every $j \geq 0$ and $\beta \lim_{j\to\infty} m_{j+1}/m_j < 1$.

(C3) For each $j \geq 0$ and $(x,z) \in X_j$, $y \in \Gamma(x,z)$, we have $Q_z(Z_{j+1}) = 1$, where $Z_{j+1} = \{z' \in Z : (y,z) \in X_{j+1}\}$.

Then, (Jaśkiewicz and Nowak, 2011, Theorem 1) establishes the existence of a solution to the Bellman equation in a suitable class of upper semicontinuous functions, which is the value function, and the existence of optimal policies. We remit the reader to the paper to take care of the details. Here we want only to make a comparison with our approach.

Assumption (W) is more general than ours, since it does not require continuity, except (5.6), which is stronger (we only require continuity with respect to x, not with respect to the exogenous state variable z). (C3) basically implies that Z is bounded or that the probability measures Q_z have bounded support. To see this in an example, consider the classical growth model with $X = [0,\infty)$, $Z = (0,\infty)$, $\Gamma(x,z) = [0, zf(k)]$, with f a typical production function, continuous, strictly increasing and unbounded. Let u be a continuous utility function on consumption which is nonnegative and unbounded, and let $U(k,k',z) = u(zf(k) - k')$, with $k' \in \Gamma(x,z)$.

Suppose, as we did in the Endogenous Growth Model of Section 4, that the shocks evolve

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"countable" and "uncountable" refer to the cardinality of the family of seminorms used to describe the topology of the function space where a given operator is defined.

as in (4.5), $\ln z_{t+1} = \rho \ln z_t + \ln w_{t+1}$, where the domain of the iid shocks w_t is $W \subseteq (0, \infty)$. Let us see that trying to construct a suitable increasing family of compact sets covering the state space such that (C1)–(C3) holds leads to a contradiction. Let $X_j = K_j \times Z_j$, where $K_j \subseteq X$ and $Z_j \subseteq Z$ (neither K_j nor Z_j have to be compact sets). Then, by (C2), $m_j = \sup_{(k,z)\in K_j\times Z_j} u(zf(k))$, and for this number to be finite, both K_j and Z_j have to be bounded. Then, consider (C3):

$$1 = Q_z(Z_{j+1}) = \int_W I_{Z_{j+1}}(z^\rho w)\mu(dw),$$

for all $z \in Z_j$, where I_B denotes the indicator function of the Borel set B. But this is clearly impossible if w has not bounded support, since for any $z > 0$, the range of $z^\rho w$ is $(0, \infty)$. For instance, if w has density e^{-w} in $W = (0, \infty)$ and we suppose that $Z_j = (0, z_j]$, with $z_j > 0$, then

$$Q_{z_j}(Z_{j+1}) = \int_{(0,\infty)} I_{Z_{j+1}}(z_j^\rho w)e^{-w}dw = \int_{(0,z_{j+1}/z_j^\rho]} e^{-w}dw = 1 - e^{-z_{j+1}/z_j^\rho} < 1.$$

Thus, (C3) does not hold.

Beyond the constraint (C3), conditions (C1) and (C2) may also be limiting. To illustrate this, consider again the growth model studied in Example 5.3. Note that Assumption (C1) to (C3) become:

(C1) $[0, \infty) \times \{1, g\} = \bigcup_{j=0}^{\infty} \mathrm{Int} X_j$, where $X_j = [0, g^j] \times \{1, g\}$ is a suitable selection now.

(C2) If $m_j = \sup_{(k,z)\in X_j} \sup_{k'\in[0,zk]} U(k, k', z) = u(g^{j+1})$, for $j = 1, 2, \ldots$, then it must be

$$\beta \lim_{j\to\infty} \frac{m_{j+1}}{m_j} < 1.$$

(C3) It is fulfilled trivially with the selection of the family $\{X_j\}$ made in C1 (this is possible since Z is bounded).

Note that $\beta \lim_{j\to\infty} m_{j+1}/m_j = \beta g < 1$, which implies $\beta < 1/g$. Our method works for $\beta \leq 1/(gp + q)$ as shown above (and the weighted contraction for $\beta < 1/(pg + q)$, but not for $\beta = 1/(gp + q)$.) Since $pg + q \leq g$, because $g > 1$ and $pg + q$ is a convex combination of g and 1, we have $\frac{1}{g} < \frac{1}{gp+q}$, as soon as $0 < p < 1$, thus $\beta < 1/g$ is more restrictive for β than $\beta \leq 1/(gp + q)$.

5.3 Q–transform

The Q–transform is a novel method due to Ma *et al.* (2022), and constitutes an improvement of the weighted norm approach in certain cases. The Q–transform applies to rather general dynamic problems. Roughly speaking, it consists in taking conditional expectations at both sides of the Bellman equation that, in some models, converts an unbounded dynamic program into a bounded one. In our context, the transformed Bellman equation would be ($\mathrm{E}_z(\{\bullet\}) = \int_Z \bullet\, Q_z(dz'))$)

$$g(x,y,z) = \beta \mathrm{E}_z \sup_{y' \in \Gamma(y,z')} \left\{ U(y,y',z') + g(y,y',z') \right\}.$$

There are the connections: $x \in X$, $z, z' \in Z$, $y \in \Gamma(x,z)$, $y' \in \Gamma(y,z')$. Also, since that v satisfies the Bellman equation

$$g(x,y,z) = \beta \mathrm{E}_z v(y,z'), \quad y \in \Gamma(x,z).$$

Defining as usual $\psi(x,z) = \sup_{y \in \Gamma(x,z)} U(x,y,z)$ and $\chi(y,z) = \mathrm{E}_z \psi(y,z')$, the assumptions about these functions are (see (Ma *et al.*, 2022, Assumption 5.1)): (1) there is a weighing function φ and there exist constants $M \geq 0$, $\alpha \geq 0$, $\alpha\beta < 1$ such that $\psi(x,z) \leq M\varphi(x,z)$ and $\mathrm{E}_z \varphi(y,z') \leq \alpha \varphi(x,z)$, for all $x \in X$, $z \in Z$ and $y \in \Gamma(x,z)$, and (2) $\chi(y,z)$ is bounded below for $y \in \Gamma(x,z)$, for all $x \in X$, $z \in Z$.

Condition (1) cares for unbounded from above growth, and it is the same that the weighted norm approach. Thus, the Q–transform cannot be applied to Example 5.3 above for a discount factor $\beta = 1/(pg+q)$. To show that there are models for which condition (2) fails, but that can be worked with Theorem 3.7, consider a modification of the linear quadratic example in (Stokey *et al.*, 1989, p. 277) with $X = \mathbb{R}$, $Z = [0,\infty)$, $U(x,y,z) = zx - bx^2 - c(y-x)^2$, with $b, c > 0$. Here, $zx - bx^2$ is a firm's net revenue when its capital stock is x, and $c(y-x)^2$ is the cost of changing the capital stock from x to y. Suppose that $z_{t+1} = \rho z_t + w_t$, where $\rho > 0$ and $\{w_t\}$ is a sequence of independent and identically distributed random variables with $w_t \geq 0$, with finite mean, $\mathrm{E}w$, second order moment, $\mathrm{E}w^2$, and domain $W \subseteq \mathbb{R}_+$. Also, given x, the next capital stock may vary in $[-(1+r)|x|, (1+r)|x|]$, with $r > 0$.

Given $(x,z) \in X \times Z$, we have $\psi(x,z) = zx - bx^2$ and $\mathrm{E}_z \psi(y,z') = (\rho z + \mathrm{E}w)y - by^2$, for $y \in [-(1+r)|x|, (1+r)|x|]$. Clearly, $\chi(y,z) = \mathrm{E}_z \psi(y,z')$ is unbounded from below, that is

$$\inf_{(x,z)\in X\times Z}\ \inf_{y\in[-(1+r)|x|,(1+r)|x|]}\chi(y,z)=-\infty,$$

thus the Q–transform cannot be applied.

Let us check that (B6)' is fulfilled for suitable values of the discount factor β. This example will help us to show the flexibility of the method of averaging with respect to the exogenous variable. Again, this is possible due to the special structure of the problems we analyze, where actions do not "choose" probability measures.

To find a suitable family $\{k_t\}$, choose $\overline{y}(x,z)=x$, which is an admissible policy, and let $k_0(x,z)=U(x,\overline{y}(x,z),z)=zx-bx^2$, $k_1(x,z)=\beta\mathrm{E}_z k_0(\overline{y}(x,z),z')=\beta((\rho z+\mathrm{E}w)x-bx^2)$ and, in general, $k_t=\beta^t((\rho^t z+(1+\rho+\cdots+\rho^{t-1}\mathrm{E}w)x-bx^2)$. Clearly, $\Sigma_{t=0}^{\infty}k_t(x,z)$ converges unconditionally if $\beta\rho<1$.

Regarding the family $\{l_t\}$, we between discriminate two cases.

(a) Suppose that $\rho<1+r$, that is, the trend growth of the shock is smaller than the endogenous variable growth.

Note that $zx-bx^2\le z^2/4b$, thus $\psi(x,z)\le z^2/4b$, for all $x,z\ge 0$. Take $l_0(z)=z^2/4b$, which is independent of x, hence

$$\mathrm{E}_z(l_0(z'))=\frac{1}{4b}\int_W(\rho z+w)^2dw\equiv\frac{1}{4b}\chi(z).$$

Let $l_1(z)=\beta\chi(z)/4b$. Now, $\int_W(\rho z+w)^2dw=\rho^2z^2+2z\mathrm{E}w+\mathrm{E}w^2$, thus

$$\mathrm{E}_z(l_1(z'))=\beta\frac{1}{4b}\left(\rho^2\int_W(\rho z+w)^2dw+2(\rho z+\mathrm{E}w)\mathrm{E}w+\mathrm{E}w^2\right),$$

thus take $l_2(z)=\beta^2\left(\rho^2\chi(z)+2(\rho z+\mathrm{E}w)\mathrm{E}w+\mathrm{E}w^2\right)/4b$. By an inductive argument, it is possible to prove that

$$l_{t+1}(z)=\frac{1}{4b}\beta^t\left((\rho^2)^t\chi(z)+2z\mathrm{E}w\sum_{s=t+1}^{2t}\rho^s+2(\mathrm{E}w)^2\sum_{s=1}^{2t-1}\rho^s+\mathrm{E}w^2\sum_{s=0}^{t}(\rho^2)^s\right),$$

satisfies $l_{t+1}=\beta\mathrm{E}_z l_t$. Taking β such that $\beta\rho^2<1$, the series $\sum_{t=0}^{\infty}l_t(z)$ converges for all z, and clearly, $\sum_{t=0}^{\infty}p_z(l_t)$ converges as well.

(b) Suppose that $\rho\ge 1+r$.

Choose now $l_0(x,z) = zx \geq \psi(x,z)$. Let

$$l_1(x,z) = \beta \mathrm{E}_z(\max_{y\in\Gamma(x,z)} l_0(y,z')) = \beta(\rho z + \mathrm{E}w)|x|$$

and, in general

$$l_t(x,z) = \beta^t(\rho^t z + (1+\rho+\cdots+\rho^{t-1})\mathrm{E}w)(1+r)^t|x|.$$

The condition $\beta\rho(1+r) < 1$ implies that the sum $\sum_{t=0}^{\infty} l_t(x,z)$ converges unconditionally, since $p_{K,z}(l_t) = \beta^t(\rho^t(\rho z+\mathrm{E}w)+(1+\rho+\cdots+\rho^{t-1})\mathrm{E}w)(1+r)^t(\max_{x\in K}|x|)$, where K is any compact subset of $\mathbb{R}$, and then $\sum_{t=0}^{\infty} p_{K,z}(l_t)$ is clearly convergent.

Thus, collecting all the constraints above for β, we have proved that Theorem 3.7 applies if $\beta(1+r)\min(\rho, 1+r) < 1$.

Let us try the weighted norm approach. Since costs are quadratic, we may consider $\varphi(x,z) = (z+m)|x| + |x|^2$, with $m \geq 0$ being a suitable constant. Then $|U(x,y,z)| \leq (1+b)\varphi(x,z)$, for all $x \in X$ and all $z \in Z$. Since $\mathrm{E}_z\varphi(y,z') = (\rho z + \mathrm{E}w + m)|y| + |y|^2$, for $y \in [-(1+r)|x|, (1+r)|x|]$, condition (5.1) becomes to

$$\beta(1+r) \sup_{(x,z)\in X\times Z} \frac{(\rho z + \mathrm{E}w + m) + (1+r)|x|}{z+m+|x|} < 1.$$

Independently of the value of m, if $\rho > 1+r$, then the expression above is decreasing in $|x|$, for all $z > (rm - \mathrm{E}w)/(\rho - 1 - r)$, thus for z in this region, the supremum is attained at $x = 0$, and its value is ρ. On the other hand, if $\rho < 1+r$, then the expression is increasing in $|x|$, for all $z > (\mathrm{E}w - rm)/(r+1-\rho)$, thus for z in this region the supremum is attained at ∞, and its value is $1+r$. For $\rho = 1+r$ the supremum is $\rho = 1+r$. Thus, the inequality (5.1) is $\beta(1+r)\max(\rho, 1+r) < 1$, which is obviously stronger for β than $\beta(1+r)\min(\rho, 1+r) < 1$.

To close this example, note that (Jaśkiewicz and Nowak, 2011, Theorem 1) cannot be applied if the shocks w_t do not have compact support, by the same reasons adduced in Section 5.2.

6 Conclusions

We have developed a framework to analyze stochastic dynamic problems with unbounded rewards and shocks, where the reward function does not take $-\infty$ as a value and the shocks are exogenous. We obtain new existence and uniqueness results of solutions to the Bellman equation through the use of a notion of Banach contraction which generalizes Banach and local contractions. We introduce seminorms that give a different treatment to the endogenous state variable and the exogenous one. While a supremum norm on arbitrary compact sets is considered in the former variable, an L^1 type norm is considered in the latter variable. Putting together this definition with the aforementioned generalization of the local contraction concept, we are able to maintain the monotonicity (in a mild sense) of the Bellman operator, thus proving that it is essentially a contractive operator.

We provide a method to check the hypotheses needed to apply the approach, based on assumptions (B6) or (B6)', that can be used straightforwardly to analyze a variety of models. We apply these results to an Endogenous Growth model with bounded from below rewards. We compare our method with the weighted contraction approach, the countable local contraction approach of Jaśkiewicz and Nowak (2011) and the Q–transform method of Ma *et al.* (2022), showing instances where the method developed in this paper works, but the aforementioned methods fail. In particular, we show a neoclassical stochastic growth model with bounded from below rewards and a discount factor for which a weighing function cannot exist, but our approach can be used.

The paper could be extended in several ways: (1) to deal with problems where the reward function does take $-\infty$ as a value; a possible direction would be to substitute the seminorms by pseudometrics that mix conditional expectation with distances *á la* Thompson metric, as in Rincón-Zapatero and Rodríguez-Palmero (2003) and Martins da Rocha and Vailakis (2010) for deterministic problems; (2) to analyze problems where uncertainty is not exogenous; this way has been initiated in Rincón-Zapatero (2022); and (3), to work with non additive preferences; possibly the method could be extended to the case of Blackwell aggregators, which satisfy a global Lipschitz condition with respect to future expected utility, of Lipschitz constant smaller than one, but it would be harder to consider Thompson aggregators as those introduced in Marinacci and Montrucchio (2010). However, Bloise and Vailakis (2018) and Bloise, *et al.* (2021) have already found a fruitful path to study stochastic dynamic programs with recursive utility given by Thompson aggregators by means of Tarski's Fixed Point Theorem.

A Proofs of Section 2

Lemma A.1. *Let $T : F \longrightarrow F$ be an L-local contraction on $F \subseteq E$ and let $x_0 \in F$ be such that (I)–(VI) hold true for a suitable $r_0 \in C$. Let R_0 be defined as in (VI). Then*

(a) $T : V_F(x_0, R_0) \longrightarrow V_F(x_0, R_0)$.

(b) For any $a \in A$, $\lim_{t\to\infty}(L^t R_0)(a) = 0$.

Proof. Due to the subhomogeneity of L for finite sums, $L(r_0 + Lr_0 + \cdots + L^T r_0) \leq Lr_0 + \cdots + L^{T+1}r_0 \leq R_0$, for all finite T. Letting $T \to \infty$, we obtain $r_0 + LR_0 \leq R_0$. Let $x \in V_F(x_0, R_0)$, so $d_a(x_0, x) \leq R_0(a)$ for all $a \in A$. By the triangle inequality and since T is an L–local contraction

$$\begin{aligned} d_a(x_0, Tx) &\leq d_a(x_0, Tx_0) + d_a(Tx_0, Tx) \\ &\leq d_0(a) + (Ld_a)(x_0, x) \\ &\leq d_0(a) + (LR_0)(a) \\ &\leq R_0(a). \end{aligned}$$

This proves (a). To show (b), note that, by the same arguments used to prove (a), for $L^t R_0 \leq L^t r_0 + L^{t+1} r_0 + \cdots$, for all $t = 0, 1, \ldots$. Then $L^t R_0(a)$ is bounded by the remainder of the convergent series $R_0(a)$, thus it converges to 0 as $t \to \infty$, for all $a \in A$. □

Proof of Corollary 2.5. By Theorem 2.4, T admits a unique fixed point x^* in $V_F(x_0, R_0)$, where $R_0^* = \sum_{t=0}^{\infty} L^t r_0^*$, for any $r_0^* \in C$ for which R_0^* is a convergent series. Suppose, by contradiction, that T admits another fixed point $x^{**} \neq x^*$ in F. By assumption, there is $r_0^{**} \in C$ such that $x^{**} \in V_F(x_0, R_0^{**})$, where $R_0^{**} = \sum_{t=0}^{\infty} L^t r_0^{**}$ is finite. Let $r_0 = \max\{r_0^*, r_0^{**}\}$ and $R_0 = \sum_{t=0}^{\infty} L^t r_0$. Then R_0 is finite and $R_0^*, r_0^{**} \leq R_0$. Moreover, both $x^*, x^{**} \in V_F(x_0, R_0)$, thus by Theorem 2.4, $x^* = x^{**}$. The convergence of iterating sequences is also an immediate consequence of Theorem 2.4. $\square$

PROOF OF PROPOSITION 2.6. Note that $\sum_{t=t_0}^{\infty} L^t d_0 \leq s + Ls + L^2 s + \cdots \leq (1 + \theta + \theta^2 + \cdots)s = \frac{1}{1-\theta}s$. Hence, $\sum_{t=0}^{\infty} L^t d_0 = \sum_{t=0}^{t_0-1} L^t d_0 + \sum_{t=t_0}^{\infty} L^t d_0 \leq \sum_{t=0}^{t_0-1} L^t d_0 + \frac{1}{1-\theta}s$ is finite for all $a \in A$.

B Proofs of Section 3

A function $f : X \times Z \longrightarrow \mathbb{R}$ is a Carathéodory function on $X \times Z$ if it satisfies

1. for each $x \in X$, the function $f_x := f(x, \cdot) : Z \longrightarrow \mathbb{R}$ is Borel measurable;

2. for each $z \in Z$, the function $f^z := f(\cdot, z) : X \longrightarrow \mathbb{R}$ is continuous.

Under our assumptions, a Carathéodory function is jointly measurable in $X \times Z$, see Aliprantis and Border (1999), Lemma 4.50. Also, a function that is Carathéodory on $X \times Z$ is obviously Carathéodory on $A \times Z$ for all $A \subseteq X$. Let us denote by $\mathrm{Ca}(A \times Z)$ the set of all Carathéodory functions on $A \times Z$.

Given $f \in \mathrm{Ca}(X \times Z)$, we denote

$$\widehat{f}(x, z) := \max_{y \in \Gamma(x,z)} f(y, z), \quad \widehat{|f|}(x, z) := \max_{y \in \Gamma(x,z)} |f(y, z)|.$$

We will make use of the following lemma in the main text and along this appendix.

Lemma B.1. *(1) For all $f \in \mathrm{Ca}(X \times Z)$, both $\widehat{f}$, $\widehat{|f|} \in \mathrm{Ca}(X \times Z)$.*

(2) For all $f \in \mathcal{L}^1(Z; C(X))$, both $\widehat{f}$, $\widehat{|f|} \in L^1(Z, \mathcal{Z}, Q_z)$, for all $z \in Z$.

Proof. (1) Given the assumption made about the continuity of Γ, by the Bergé Theorem of the Maximum, the map $x \mapsto \widehat{f}(x, z)$ is continuous, for any $z \in Z$ fixed, and by the Measurable Theorem of the Maximum, $z \mapsto \widehat{f}(x, z)$ is Borel measurable; thus, $\widehat{f}$ is a Carathéodory function on $X \times Z$. Obviously, the same is true for $\widehat{|f|}$.

(2) Since $p_{K,z}(f) < \infty$ for all $K \in \mathcal{K}$ and all $z \in Z$, and $\Gamma(x, z)$ is a compact set for any $x \in X$, $z \in Z$, then $\int_Z \widehat{|f|}(y, z') Q_z(dz') = p_{\Gamma(x,z),z}(f) < \infty$. Obviously, the same is true for $\widehat{f}$. $\square$

Proof of Lemma 3.4. We organize the proof in several previous lemmas.

Lemma B.2. *Let assumptions (B1) to (B6) to hold. Then*

1. $\sum_{t=0}^{\infty} L^t p^{l_0} < \infty$;

2. $R_0[\Gamma] \in \mathcal{L}^1(Z; C(X))$ *and* $p^{l_0} + LR_0 \leq R_0$.

Proof. Given $x \in X$ and $z \in Z$, $\beta p^{l_t}[\Gamma](x, z) = \beta \int_Z \max_{y \in \Gamma(x,z)} l_t(y, z') Q_z(dz') \leq l_{t+1}(x, z)$, hence $p^{l_t}[\Gamma] \in \mathcal{L}^1(Z, C(X))$ and then $Lp^{l_t}(K, z) = \beta p_{K,z}(p^{l_t}[\Gamma]) \leq p_{K,z}(l_{t+1})$, for all $t = 0, 1, \ldots$. Thus, $L^t p^{l_0} \leq L^{t-1} p^{l_1} \leq \cdots \leq p^{l_t}$. By (B6), the series $\sum_{t=0}^{\infty} p^{l_t}(K, z)$ converges for all $K \in \mathcal{K}$ and $z \in Z$, thus $\sum_{t=0}^{\infty} L^t p^{l_0}$ converges. To conclude the proof, by the triangle inequality

$$\begin{aligned} p_{K,z}(p^{l_0}[\Gamma] + \cdots + p^{l_t}[\Gamma]) &\leq p_{K,z}(p^{l_0}[\Gamma]) + \cdots + p_{K,z}(p^{l_0}[\Gamma]) \\ &\leq (p_{K,z}(l_1) + \cdots + p_{K,z}(l_{t+1}). \end{aligned}$$

Letting $t \to \infty$ and adding $p_{K,z}(\psi)$ to both sides of the above inequality, we have $p_{K,z}(\psi) + p_{K,z}(R_0[\Gamma]) \leq R_0(K, z)$, showing at the same time that $R_0[\Gamma] \in \mathcal{L}^1(Z; C(X))$. □

Lemma B.3. *Let assumptions (B1) to (B6) hold. Then* $f \in V(0, R_0)$ *implies* $Tf \in \mathcal{L}^1(Z; C(X))$.

Proof. Let $f \in \mathcal{L}^1(Z; C(X))$. We use the notation f_x and f^z introduced above at the beginning of this section. The function f_x is Borel measurable for all $x \in X$ and Q_z–integrable for any $z \in Z$. Thus, f_x can be written as the difference of two positive, Q_z–integrable functions, $f_x = f_x^+ - f_x^-$, where $f_x^+ = \max(f_x, 0)$ and $f_x^- = \max(-f_x, 0)$. Applying Theorem 8.1 in Stokey *et al.* (1989), both Mf_x^+ and Mf_x^- are Borel measurable. Since $(Mf)_x = M(f_x) = M(f_x^+) - M(f_x^-)$, $(Mf)_x$ is measurable for any $x \in X$. To see that $(Mf)^z$ is continuous, consider a sequence $\{x_n\}$ in X that converges to $x \in X$. Then the sequence and its limit form the compact set $K = \{x_n\} \cup \{x\}$. Let $f_n := f_{x_n}$, for $n \geq 1$. For all $z' \in Z$, $f_n(z') \to f_x(z')$ as $n \to \infty$, since f is continuous in x. Moreover, $|f^{z'}| \leq \sup_{x \in K} |f^{z'}(x)|$, and $z' \mapsto \sup_{x \in K} |f^{z'}(x)|$ is Q_z–integrable by definition of $\mathcal{L}^1(Z; C(X))$, thus by the Lebesgue dominated convergence theorem

$$(Mf)(x_n, z) = \int_Z f_n(z') Q_z(dz') \to \int_Z f_x(z') Q_z(dz') = (Mf)(x, z),$$

thus $(Mf)^z$ is continuous. Hence, Mf is a Carathéodory function and thus $U(x,y,z) + \beta Mf(y,z)$ is continuous in (x,y) for all z, and it is Borel measurable in z for all (x,y). By the Bergé Maximum Theorem, the function Tf is thus continuous in x for all z, and by the Measurable Maximum Theorem, it is Borel measurable for any x. In short, the function

$$(x,z) \mapsto Tf(x,z) = \max_{y\in\Gamma(x,z)} (U(x,y,z) + \beta Mf(y,z)),$$

is a Carathéodory function. Moreover, if $f \in F$ and $x \in X$, $z \in Z$

$$\begin{aligned}
|Tf(x,z)| &\le |\max_{y\in\Gamma(x,z)} U(x,y,z)| + \beta \max_{y\in\Gamma(x,z)} \int_Z \max_{y\in\Gamma(x,z)} |f(y,z')|Q_z(dz') \\
&\le l_0(x,z) + \beta \int_Z \max_{y\in\Gamma(x,z)} w(y,z')Q_z(dz') \\
&\le l_0(x,z) + \beta p_{\Gamma(x,z),z}(f).
\end{aligned}$$

Since $\Gamma(x,z) \in \mathcal{K}$, for $f \in V(0,R_0)$, we have $p_{\Gamma(x,z),z}(f) \le R_0[\Gamma](x,z)$. By Lemma B.2, $p_{K,z}(l_0) + \beta p_{K,z}(R_0[\Gamma]) \le R_0(K,z)$. Hence $p_{K,z}(Tf) \le R_0(K,z)$. This proves that $Tf \in V(0,R_0)$, and hence that $Tf \in \mathcal{L}^1(Z,C(X))$. □

Lemma B.4. *Let assumptions (B1) to (B6) hold. Then $D(V(0,R_0)) \subseteq C$.*

Proof. Since $f \in V(0,R_0)$, $p^f \le R_0$, hence we can take $c=1$. Also, $p^f \in \mathrm{Ca}(X\times Z)$, since $p^f[\Gamma](x,z) = \int_Z \max_{y\in\Gamma(x,z)} |f(y,z')|Q_z(dz')$ is continuous in x and Borel measurable in z, by Lemma B.1. Moreover, $p^f[\Gamma] \le R_0[\Gamma]$ implies $p_{K,z}(p^f[\Gamma]) \le p_{K,z}(R_0[\Gamma]) \le \frac{1}{\beta}R_0(K,z)$, by Lemma B.2. Hence, $p^f[\Gamma] \in \mathcal{L}^1(Z,C(X))$. □

Now, we are in position to prove Lemma 3.4. First, let us see that $L : C \longrightarrow C$. Let $p \in C$; by the definition of the operator L and Lemma B.2

$$Lp(K,z) = \beta p_{K,z}(p[\Gamma]) \le \beta p_{K,z}(cR_0[\Gamma]) \le cR_1(K,z) \le cR_0(K,z),$$

and so, $Lp[\Gamma] \le cR_0[\Gamma]$ and $Lp[\Gamma] \in \mathcal{L}^1(Z,C(X))$. Second, we prove that the assumptions (I) to (VI) are fulfilled. Regarding (I), note that $p+q \in C$ if $p,q \in C$, trivially, as well it is also immediate that if $p' \in C$ and $p \le p'$, then $p \in C$. On the other hand, if a countable chain of partial sums $p_0, p_0+p_1, p_0+p_1+p_2, \ldots$, is bounded by an element P in C, then the infinite sum, $p := \sum_{n=0}^{\infty} p_n$, is well defined and $p \le P \le cR_0$ for some constant c. Moreover, since $p[\Gamma] \leq cR_0[\Gamma]$ and $R_0[\Gamma] \in \mathcal{L}^1(Z; C(X))$ by Lemma B.2, the Monotone Convergence Theorem implies that $p[\Gamma](x, \cdot)$ is Q_z–integrable for all $z \in Z$ and all $x \in X$. On the other hand, each function $p_i[\Gamma](\cdot, z)$ is continuous in x, for all $i = 1, 2, \ldots$. By the Weierstrass M test, the function $p[\Gamma](\cdot, z)$ is also continuous in x for all $z \in Z$. These two observations imply that $p[\Gamma] \in \mathcal{L}^1(Z; C(X))$. (II) is trivial; (III) holds, since the integral is monotone, and regarding (IV), it holds true, since for all $p, q \in C$, $p_{K,z}(p[\Gamma]+q[\Gamma]) \leq p_{K,z}(p[\Gamma])+p_{K,z}(q[\Gamma])$ by definition of the seminorms $p_{K,z}$, hence

$$\begin{aligned} L(p+q)(K,z) &= p_{K,z}(p[\Gamma] + q[\Gamma]) \\ &\leq p_{K,z}(p[\Gamma]) + p_{K,z}(q[\Gamma]) \\ &= Lp(K,z) + Lq(K,z). \end{aligned}$$

L is clearly sup-preserving in C by the Monotone Convergence Theorem, hence (V) also holds. Finally, (VI) is implied by Lemma B.2 and Lemma B.3. □

Let the two families of functions $\{u_t\}_{t=0}^{\infty}$ and $\{l_t\}_{t=0}^{\infty}$ defined in (B6)'. Let $I_{u_0,w_0} = \{f \in \mathcal{L}^1(Z; C(X)) \,:\, u_0 \leq f \leq w_0\}$.

Lemma B.5. $T : I_{u_0,w_0} \longrightarrow I_{u_0,w_0}$.

Proof. We prove that for all $x \in X$, $z \in Z$ and $t = 0, 1, \ldots$

$$\beta \int_Z u_t(\overline{y}(x,z), z') Q_z(dz') \geq u_{t+1}(x,z)$$

and

$$\beta \int_Z \max_{y \in \Gamma(x,z)} w_t(y, z') Q_z(dz') \leq w_{t+1}(x,z).$$

In fact

$$\begin{aligned} \beta \int_Z u_t(\overline{y}(x,z), z') Q_z(dz') &= \int_Z \sum_{s \geq t} k_s(\overline{y}(x,z), z') Q_z(dz') \\ &= \beta \sum_{s \geq t} \int_Z k_s(\overline{y}(x,z), z') Q_z(dz') \\ &= \beta \sum_{s \geq t} k_{s+1}(x,z) \\ &= u_{t+1}(x,z), \end{aligned}$$

where the exchange of integral and summatory is due to the Monotone Convergence Theorem. In the same way

$$\begin{aligned}
\beta \int_Z \max_{y \in \Gamma(x,z)} w_t(y, z') Q_z(dz') &\leq \beta \int_Z \max_{y \in \Gamma(x,z)} \sum_{s \geq t} l_s(y, z') Q_z(dz') \\
&\leq \sum_{s \geq t} \beta \int_Z \max_{y \in \Gamma(x,z)} l_s(y, z') Q_z(dz') \\
&\leq \sum_{s \geq t} l_{s+1}(x, z) \\
&= \sum_{s \geq t+1} l_s(x, z) \\
&= w_{t+1}(x, z).
\end{aligned}$$

Let $f \in I_{u_0,w_0}$. That Tf is in $\mathcal{L}^1(Z, C(X))$ is proved as in Lemma B.3 above. Let us show first that $Tf \geq u_0$. For $x \in X$, and $z \in Z$, to simplify notation, let $\overline{y} = \overline{y}(x, z)$. Then

$$\begin{aligned}
Tf(x, z) &\geq U(x, \overline{y}_0, z)| + \beta \int_Z |f(\overline{y}, z')| Q_z(dz') \\
&\geq k_0(x, z) + \beta \int_Z u_0(\overline{y}, z')| Q_z(dz') \\
&\geq k_0(x, z) + \beta u_1(x, z) \\
&= u_0(x, z).
\end{aligned}$$

Also

$$\begin{aligned}
Tf(x, z) &\leq \max_{y \in \Gamma(x,z)} U(x, y, z) + \beta \max_{y \in \Gamma(x,z)} \int_Z \max_{y \in \Gamma(x,z)} f(y, z') Q_z(dz') \\
&\leq l_0(x, z) + \beta \int_Z \max_{y \in \Gamma(x,z)} f(y, z') Q_z(dz') \\
&\leq l_0(x, z) + \beta \int_Z \max_{y \in \Gamma(x,z)} w_0(y, z') Q_z(dz') \\
&\leq l_0(x, z) + w_1(x, z) \\
&= w_0(x, z).
\end{aligned}$$

$\square$

C Continuity of the Markov operator

The issue of continuity of the value function in the unbounded case (and unbounded space of shocks) is not an easy one. The translation of Lemma 12.14 in Stokey *et al.* (1989) to this case is not straightforward, even if the Markov chain is strong Feller continuous. Recall that Q has the weak (strong) Feller property if M maps bounded continuous functions (*resp.* bounded measurable functions) on Z into bounded continuous functions. To see the kind of problems that may emerge for unbounded functions, consider the following example, adapted from Stoyanov (2013). Let $Z = [0, \infty)$ and let the transition function $Q : Z \times \mathcal{Z} \longrightarrow \mathbb{R}$ be defined as follows:

$$Q(z, B) = \begin{cases} \delta_0(B), & \text{if } z = 0; \\ \int_B dF_z(z'), & \text{if } z > 0, \end{cases} \tag{C.1}$$

where δ_0 is the Dirac measure at the point 0, that is, $\delta_0(B) = 1$ if $0 \in B$ and $\delta_0(B) = 0$ otherwise, and for $0 < z \leq 1$

$$F_z(z') = \begin{cases} 0, & \text{if } z' = 0; \\ z'z^2 + 1 - z, & \text{if } 0 < z' \leq \frac{1}{z}; \\ 1, & \text{if } z' > \frac{1}{z}. \end{cases} \quad ,$$

Finally, for $z \geq 1$, $Q(z, B) = \lambda(B \cap [0, 1])$, where λ denotes the Lebesgue measure of $\mathbb{R}$.

Note that, for $0 < z < 1$, F_z is a distribution function: it is nondecreasing, continuous except at 0, where the right sided limit exists, $0 \leq F_z \leq 1$, and

$$\int dF(z') = (z'z^2 + 1 - z - 0)|_{z'=0} + \int_0^{\frac{1}{z}} z^2 dz' = 1 - z + z = 1.$$

Moreover, it is clear that $Q(\cdot, B)$ is Borel measurable. Thus, Q is a transition function. Let $f(y, z) = f(z)$ be independent of y and continuous in z. Then Mf is well defined in this particular example and depends only on z, with $(Mf)(0) = \int f(z')Q(0, dz') = f(0)$. For $0 < z < 1$ we have

$$\begin{aligned}(Mf)(z) = \int f(z')Q(z, dz') &= \int f(z')dF_z(z') \\ &= f(0)(z'z^2 + 1 - z - 0)|_{z'=0} + \int_0^{\frac{1}{z}} f(z')z^2 dz' = f(0)(1 - z) + z^2 \int_0^{\frac{1}{z}} f(z')dz'.\end{aligned}$$

Since f is continuous, Mf is continuous for $0 < z < 1$. For $z \geq 1$, Mf is constant and given by

$$(Mf)(z) = \int f(z')Q(z, dz') = \int_{[0,1]} f(z')dz'.$$

Now, if f is bounded, there is $k > 0$ such that $-k \leq f \leq k$, hence $-kz \leq z^2 \int_0^{\frac{1}{z}} f(z')dz' \leq kz$, thus $z^2 \int_0^{\frac{1}{z}} f(z')dz'$ tends to 0 as $z \to 0^+$, hence $Mf(z) \to f(0) = Mf(0)$, when $z \to 0^+$, thus Mf is continuous at 0. On the other hand, $f(0)(1-z) + z^2 \int_0^{\frac{1}{z}} f(z')dz'$ tends to $\int_0^1 f(z')dz' = Mf(1)$ as $z \to 1^-$, thus Mf is continuous at 1. Thus Mf is continuous and hence Q is strong Feller continuous. However, considering the unbounded function $g(z) = z$, we have $Mg(0) = g(0) = 0$ and $Mg(z) = 1/2$ for $z > 0$, thus Mg is discontinuous at 0.

Consider now the following simple pure currency model with linear utility, where agents' preferences are subject to random shocks. See Stokey *et al.* (1989) for further details about this model. Let the utility the utility $u(c, z) = (1+z)c$ depend on consumption c and shock z and let $\Gamma(m) = [0, m+y]$, where $m \geq 0$, $y > 0$ is a constant, $X = \mathbb{R}_+$, $Z = [0, \infty]$, and let a discount factor β such that $\beta < 2/3$. The dynamic programming equation is

$$v(m, z) = \max_{m' \in [0, m+y]} \Big\{ (1+z)(m+y-m') + \beta \int_{[0,\infty)} v(m', z')Q_z(dz') \Big\}.$$

The random shocks are assumed to be governed by the Markov chain Q described in (C.1). We are simply interested in showing that the value function is not jointly continuous in (m, z). It is easily checked that

$$v(m, z) = \begin{cases} m + y + y\frac{\beta}{1-\beta}, & \text{if } z = 0; \\ (1+z)(m+y) + \frac{3}{2}y\frac{\beta}{1-\beta}, & \text{if } z > 0, \end{cases}$$

is a solution in the class $\mathrm{Ca}(\mathbb{R}_+ \times \mathbb{R}_+)$, which coincides with the value function, and it is not continuous in z.

To prove that (B6) is fulfilled, take $l_0(m, z) = \psi(m, z) = (1+z)(m+y)$. Now, noticing that $\int_Z z'Q_z(dz') = 1/2$ and recalling that Q_0 is the Dirac measure at 0, it is easy to compute, where $m' = m + y$

$$\int_Z \ell_0(m', z')Q_z(dz') = \beta \begin{cases} m + 2y, & \text{if } z = 0; \\ (3/2)\,(m+2y), & \text{if } z > 0. \end{cases}$$

Thus, $l_1(m,z) = \beta(m+2y)$, iz $z=0$, and $l_1(m,z) = \beta\,(3/2)\,(m+2y)$, for $z>0$. In general, $l_t(m,z) = \beta^t(3/2)(m+(t+1)y)$, for all $t \geq 1$. Clearly, $\sum_{t=0}^{\infty} l_t$ is unconditionally convergent for all $\beta < 1$, thus assumption (B6) holds. Given β, $\varphi(m,z) = (1+az)(m+y)$, with $a \geq 1$ is a bound of $U(m,m',z)$. However, $\int_Z \varphi(m',z')\,Q_0(dz') = \varphi(m',0) = m+y$ and $\int_Z \varphi(m',z')\,Q_z(dz') = (1+(a/2))(m'+by)$, if $z>0$. Thus $(m,z) \mapsto \int_Z \varphi(m',z')\,Q_z(dz')$ is discontinuous at $z=0$ and then (WC3) in Theorem 5.1 is not fulfilled. Any other majorant function of the selected φ will suffer the same problem.

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