Thompson Aggregators, Scott Continuous Koopmans Operators, and Least Fixed Point Theory*

Robert A. Becker
Department of Economics
Indiana University
Bloomington, IN 47405
USA

Juan Pablo Rincón-Zapatero
Departamento de Economia
Universidad Carlos III de Madrid
28903 Getafe (Madrid)
Spain

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Abstract

We reconsider the theory of Thompson aggregators proposed by Marinacci and Montrucchio [31]. We prove the existence of a Least Fixed Point (LFP) solution to the Koopmans equation. It is a recursive utility function. Our proof turns on demonstrating the Koopmans operator is a Scott continuous function when its domain is an order bounded subset of a space of bounded functions defined on the commodity space. Kleene's Fixed Point Theorem yields the construction of the LFP by an iterative procedure. We argue the LFP solution is the Koopmans equation's principal solution. It is constructed by an iterative procedure requiring less information (according to an information ordering) than approximations for any other fixed point. Additional distinctions between the LFP and GFP (Greatest Fixed Point) are presented. A general selection criterion for multiple solutions for functional equations and recursive methods is proposed.

JEL Codes: D10, D15, D50,E21

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We dedicate our paper to the memory of our friend and colleague, Carine Nourry, whose substantial contributions to growth theory and macroeconomic fluctuations, legacy as founding director of the Graduate Program at the Aix-Marseille School of Economics, and personal courage, inspire us.

1 Introduction

Recursive utility theory aims to describe classes of stationary intertemporal utility functions that are tractable in an array of capital theoretic and macrodynamic applications. Optimal growth models (e.g. Beals and Koopmans [7]) have been the main application of recursive utility objective functions. The underlying frameworks for deterministic recursive utility theories is cast within discrete time infinite horizon models. The goal is to describe utility functions with many of the attractive properties of stationary additive utility functions with exponential discounting. The set of available consumption sequences are a subset of the set of all real-valued sequences. This subset is the recursive utility function's domain.

Contemporary recursive utility function research focuses on proving a recursive utility function solves a particular functional equation, the *Koopmans Equation*. A solution is found as a fixed point of the equation's corresponding nonlinear operator, the *Koopmans operator*. The decision maker may be the planner of optimal growth theory or an infinitely-lived household in general equilibrium models. This agent has an underlying intertemporal preference ordering over a sequence of alternative consumption sequences with generic element $C = \{c_1, c_2, \ldots, c_t, \ldots\}$ where c_t is the time-dated consumption at time $t = 1, 2, \ldots$. Koopmans ([24], [25]) proposed an axiomatic structure for a consumer's stationary preference ordering and deduced that a utility representation had a recursive property: the utility function, evaluated at C, is $U(C) = W(c_1, U(SC))$ for some function $y = W(x, y')$, where S is the shift operator $SC = \{c_2, c_3, \ldots\}$ and $y = W(c_1, y')$ *aggregates* current consumption, c_1, and future utility, y', into a current utility value, y. The function W is the **aggregator**. Here U is a recursive utility function and the Koopmans equation is $U(C) = W(c_1, U(SC))$ for each C.

Lucas and Stokey [28] turned Koopmans axiomatic theory around by taking the aggregator as the building block or primitive concept defined on the real variables x and y, a set of possible utility functions (or, simply the *utility space*) on the available consumption sequences, and defined the Koopmans equation as before, but now treating W as given and U as the unknown. Put differently, given an aggregator and a utility space, the problem is to find a U satisfying the condition $U(C) = W(c_1, U(SC))$ for each C. If the Koopmans equation has a solution in the utility space, then that function is a recursive utility function that represents some preference ordering over the available consumption sequences. Of course, additional restrictions on the aggregator are essential to show the Koopmans equation has a solution. Lucas and Stokey [28] in fact proposed sufficient conditions for the solution's existence and uniqueness within the class of functions in their model's utility space. Their proof verified that the Koopmans operator is a contraction map on that utility space, which is the Banach space of bounded real-valued functions defined on the set of all bounded nonnegative real-valued sequences endowed with the sup-norm. The Koopmans operator (belonging to W), denoted by T_W is a self-map on the utility space defined for a given U in the utility space, and for each available C, by the formula:

$$(T_W U)(C) = W(c_1, U(SC)).$$

If this operator admits a fixed point: $U(C) = T_W U(C)$ for each C, then that utility function is a recursive utility function. Their definition of the Koopmans operator depends on the joint properties of the aggregator and the underlying utility space specification. The Thompson aggregator case dictates a different utility space underlies the Koopmans operator's definition.

They prove the Koopmans operator satisfies Blackwell's sufficient condition for an operator to be a contraction mapping.[1] This includes showing the Koopmans operator is a monotone operator, that is $U \leq V$ (pointwise) implies $T_W U \leq T_W V$ (pointwise). The Contraction Mapping Theorem then yields $T_W^N \theta$ sup-norm converges to U, and consequently, for each C,

$$U(C) = \lim_{N\to\infty} T_W^N \theta(C) = \lim_{N\to\infty} W(c_1, W(c_2, \ldots, W(c_{N,}, 0))).$$

Here $T_W^N \theta$ is the N^{th} iterate of $T_W \theta$ according to the formula: $T_W^N \theta = T_W \left(T_W^{N-1}\theta\right)$ for $N \geq 1$,with $\theta(C) = 0$ for each C, the zero function, and $T_W^0 \theta \equiv \theta$.

The proof that the Koopmans operator is a contraction mapping depends on their assumption that the aggregator satisfies a global Lipschitz condition in its second argument and that Lipschitz constant is smaller than one. The aggregators that satisfy this type of Lipschitz condition are classified now as **Blackwell aggregators**. Several papers extend their approach to other aggregator specifications. Boyd [12] and Becker and Boyd [8] discuss many extensions in the Blackwell family. A number of papers published after Becker and Boyd's monograph extended the Blackwell model in novel ways where the aggregator's global Lipschitz condition fails and the Koopmans operator is not a contraction map (see Rincón-Zapatero and Rodriguez-Palmero ([33], [34]), Le Van and Vailakis [27], and Martins-da-Rocha and Vailakis ([29], [30])).

Marinacci and Montrucchio [31] proposed aggregators that did not fit into the previous literature. They named these examples as members of the **Thompson aggregator** class. For example, the KDW aggregator presented in Section 2 may fail to be a Blackwell aggregator for some interesting economic parameterizations. It is a member of their Thompson class in those situations. They proposed new methods for solving the corresponding Koopmans equation for a given Thompson aggregator. They built on the observation that the Koopmans operator is, in many cases, easily shown to be a monotone operator. Moreover, in their setup, this operator acts on a complete lattice of utility functions. The

[1] Becker and Boyd ([8], p. 48) prove a generalized Blackwell theorem for Riesz spaces of the type appearing in our paper.

Tarski Fixed Point Theorem's (Tarski FPT) conditions hold and extremal fixed points exist (and, may be distinct). One extremal fixed point is the smallest, or **Least Fixed Point (LFP)** and other is the largest, or **Greatest Fixed Point (GFP)**. The Tarski FP and the existence of extremal solutions to the Koopmans equation is a non-constructive result. We show in our **Constructive Recovery Theorem (CRT)** (Section 3) these extremal fixed points exist using successive approximations as the **Tarski-Kantorovich Fixed Point Theorem's (TK FPT)** conditions hold (see our Mathematical Appendix for the exact form of this result as used here). Our proof turns on verifying an order continuity property holds. It is a purely order theoretic condition and connects to fundamental results obtained by Kantorovich [21]. Details are developed in Section 3. The LFP is found by the iteration yielding the sequence $\left\{T_W^N\theta\right\}$, just as in Lucas and Stokey's work [28]. This iteration indexed on the natural numbers can fail to yield the operator's LFP without order continuity. Marinacci and Montrucchio [31] did not verify this order continuity property in proposing qualitative properties of the LFP and GFP solutions to the Koopmans equation. Our CRT resolves fills this gap and yields qualitative features of the extremal solutions, such as semi-continuity and concavity properties.

Kantorovich [21] and Marinacci and Montrucchio ([31], p. 1785) indicate the LFP is the equation's **principal solution.** Our CRT successive approximation argument of the LFP states that the sequence of iterates $\left\{T_W^N\theta\right\}$ is nondecreasing (pointwise) with each iteration and converges pointwise to the supremum of that sequence, denoted by U_∞, which is the LFP. This monotonicity property of successive approximations is consistent with an interpretation of a theoretical computational procedure where more information about the LFP recursive utility function is added in each successive iterative step. The idea is that the successive approximation procedure starts at the zero function where there is NO information about any function in the utility space means it is an "approximation" for any possible utility function. The first iteration yields $(T_W\theta)(C) = W(c_1, 0) \geq 0$. This means we have an approximate utility value for the infinite horizon if positive consumption is limited to a single period only. In this manner, we know more about a prospective LFP utility function than with the uninformative zero function input. In the second iteration, $\left(T_W^2\theta\right)(C) = W(c_1, W(c_2, 0))$ follows. There is more information about the LFP in the sense that two consumption periods have been inputted instead of just one period, as in the first step. At this point we see (by the monotonicity property of the aggregator):

$$0 \leq (T_W\theta)(C) = W(c_1, 0) \leq \left(T_W^2\theta\right)(C) = W(c_1, W(c_2, 0)),$$

and this is a better approximation to the LFP solution than obtained in the first iteration. This is the information order interpretation of successive approximation derived from the theoretical computer science literature; it is developed in detail in Section 3 and used to motivate our interest in the Scott topology. This consistency of successive approximations initiated by inputting the zero function with the information ordering is an important property of the LFP construction. This is the basis for our first argument supporting the LFP as the Koopmans equation's principal solution.

We combine the order theoretic ideas in our working paper [9] derived from Kantorovich [21] with topological order continuity ideas due to Scott [35] in order to amplify our defense of the LFP as the principal solution. This is undertaken in Section 4. First, a constructive existence argument facilitated by verifying the operator's Scott topological order continuity property is sufficient to prove the LFP exists. Our LFP construction applies **Kleene's Fixed Point Theorem (Kleene FPT)** (see the Appendix)**.** It hypothesizes a Scott continuous operator on a complete lattice of functions and constructs the LFP by a sequence of successive approximations. Our second theme is the LFP differs from the GFP based on an information ordering or theoretical computation perspective. We argue the LFP's construction and approximation requires less information according to the information ordering than any other solution. It also differs in a qualitative way when viewed through the Scott topology's properties. The Kleene FPT argument shows the sequence of approximations to the LFP is eventually in each Scott neighborhood of the LFP. This cannot be said within the CRT's order convergence framework as it lacks any notion of a neighborhood of the LFP. That is, order convergence of the successive approximations alone cannot inform us the sequence of iterations is eventually "close" to the LFP.

This paper's main result is the **Least Fixed Point Existence and Construction Theorem** where the Koopmans operator is a Scott continuous self-map on an order bounded subset of the space of possible utility functions. This topological continuity notion is closely related to order continuity, but has subtle differences owing to its topological setup. The Scott continuity property implies a *monotone sup – preservation property* obtains for an *isotone* (or, *nondecreasing*) *net*. In particular, this property obtains for the successive approximation sequence based on iteration of the Koopmans operator with initial seed the zero function, which Scott converges to its supremum.[2]

Scott [35] introduced his eponymous topology within the context of recursive function theory and theories of computation within computer science. The use of Scott's topological ideas (as distinct from non-topological order convergence notions) is uncommon in the economics literature. Vassilakis [38] is exceptional on that score. He builds on Scott's ideas and subsequent developments in logic and computer science to untangle a major conceptual issue arising in game theory whenever infinite regress arguments arise when postulating an equilibrium solution concept. He links computability ideas to approximation ideas. There is a similarity between his approximation ideas (and those in the computer science literature) analogous to the links made here upon iterating the Koopmans operator. However, we do not make the deeper connections to computability theory addressed in Vassilakis' economic applications.

Section 2 reviews Thompson aggregators, sets up the underlying commodity

[2] Actually, there are many Scott limits! But the principal Scott limit is the LFP as shown in Section 4. The principal Scott limit is the one preserved in monotonic sup-preservation with Scott limits.

space and the vector space of possible utility functions from which the functional equation's solutions are sought. Section 3 reviews order convergence and continuity. Our CRT result appears there as well. We highlight the Koopmans operator's order continuity structure abstracted in the Scott continuity environment next. Section 4 presents basic Scott topology convergence and continuity concepts followed by an argument supporting the existence and construction of the LFP as the equation's principal solution. We also argue why the Scott continuity property further distinguishes the LFP from the GFP and why this distinction is a selection criterion identifying the LFP as the Koopmans equation's principal solution. The last section concludes the paper. A Mathematical Appendix reviews basic material on partially ordered sets and lattices as well as states the TK FPT and Kleene FPT as used in this paper. Basic Riesz space concepts are also in that appendix. We follow the Riesz space conventions and definitions in Aliprantis and Border [2] unless otherwise noted.

2 Thompson Aggregators and the Utility Function Space

2.1 Concave Thompson Aggregators

We introduce the defining properties of concave Thompson aggregators. We modify Marinacci and Montrucchio's [31] continuity axiom and concavity restrictions in order to implement our constructive recovery program.[3]

An **aggregator** is a function of two variables, (x, y), where x denotes current consumption and y denotes a future utility value. Current consumption is always nonnegative. **Thompson aggregators** are nonnegative functions and $y \geq 0$ is assumed. Formally, $W : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ = [0, \infty)$ is an aggregator, written as $W(x, y) \geq 0$ when $(x, y) \geq (0, 0)$. This aggregator function is also assumed to be *jointly continuous*, *monotone*, and *concave* on $\mathbb{R}_+ \times \mathbb{R}_+$. *Additionally*, $W(x, y) = y$ has, for each x, at least one solution $y \geq 0$ and $W(x, 0) > 0$ also holds for each $x > 0$.

Marinacci and Montrucchio [31] impose two additional conditions.

(M1) *W is $\gamma-$ subhomogeneous — there is some $\gamma > 0$ such that:*

$$W(\mu^\gamma x, \mu y) \geq \mu W(x, y)$$

for each $\mu \in (0, 1]$ and each $(x, y) \in \mathbb{R}^2_+$.

(M2) *W satisfies the **MM-Limit Condition**: for a given $\alpha \geq 1$ and $\gamma > 0$* (from $(M1)$)*,*

$$\lim_{t \to \infty} \frac{W(1, t)}{t} < \alpha^{-1/\gamma}, \qquad (1)$$

with $t > 0$.

[3] Our concavity assumption is stronger than Marinacchi and Montrucchio's [31] assumption the aggregator is concave at $y = 0$ for each $x \geq 0$. Bloise and Vailakis [11] use similar language for their dynamic programming theory.

The parameter α in $(M2)$ is the economy's maximum possible consumption growth factor. Condition $(M2)$ turns out to be an important **joint restriction** on the preferences embodied in the aggregator function as well as on the underlying commodity space, as might arise from properties of technologies in production economies and/or endowments in exchange economies. Joint restrictions of this type routinely appear in treatments of the Blackwell aggregator class. We list satisfaction of the MM Limit Condition as an explicit axiom that might, or might not, obtain for a particular aggregator in order to emphasize that some restrictions may apply on the underlying model's deep preference and technology parameters.

An aggregator satisfying the above conditions, along with $(M1)$ and $(M2)$, is said to be a **concave Thompson aggregator**. One possible **CES aggregator** is defined by the formula:

$$W(x, y) = \frac{1}{2}\sqrt{x} + \frac{1}{2}\sqrt{y} \tag{2}$$

satisfies $(M1)$ and $(M2)$ and is a concave Thompson aggregator . A routine calculations shows it does **not** satisfy a Lipschitz condition in $y \geq 0$ given x since $\sup_{y \geq 0} (\partial W / \partial y) = +\infty$.

Koopmans, Diamond, and Williamson [26] introduced the **KDW aggregator**. It is defined on $\mathbb{R}_+ \times \mathbb{R}_+$ by the formula

$$W(x, y) = \frac{\delta}{d} \ln\left(1 + ax^b + dy\right)$$

where $a, b, d, \delta > 0$. This aggregator satisfies $(M1)$ with $\gamma = b^{-1}$ and also satisfies $(M2)$. Becker and Rincón-Zapatero [9] verify these conditions obtain in this case. The KDW aggregator is an example of a $\gamma-$subhomogeneous aggregator that is NOT a homogeneous aggregator as is the CES example. The KDW specification also illustrates why *(M2)* only requires $\gamma > 0$.

The KDW aggregator always satisfies a Lipschitz condition in its second argument. It is a Blackwell aggregator when $0 < \delta < 1$, but it is not a Blackwell aggregator whenever $\delta \geq 1$. If $0 < b < 1$, then the KDW aggregator is concave in x for each y, then W is also concave in (x, y) and a concave Thompson aggregator since *(M1)* holds as $\gamma = b^{-1} > 1$. It is interesting to note that $(M1)$ applies to both current consumption and future utility arguments, whereas the question of discounting or not is a property of the future utility argument alone as well as parameter δ's magnitude.

The KDW aggregator satisfies $(M2)$. That is, the limit $L = 0$ in (1). Here, just notice for $x = 1$,

$$\frac{W(1, y)}{y} = \frac{\ln(1 + a + dy)}{y} \to 0 \text{ as } y \to \infty$$

for **any** $a, d \geq 0$. In this case, $(M2)$ holds for *any* $a \geq 1$.

Additional Thompson aggregator examples may be found in Marinacci and Montrucchio [31] and Bloise and Vailakis [11].

2.1.1 The Commodity Space

Recursive utility functions are defined on commodity spaces which are subsets of sequence spaces. The underlying commodity space is the positive cone of a principal ideal of the Riesz space of all real-valued sequences, s, with the usual coordinatewise partial order and corresponding definitions of sup and inf (coordinatewise). A generic point $C = \{c_t\}$ is a sequence indexed on the natural numbers. The positive cone $s^+ = \{C \in s : C \geq 0\}$ where 0 is the constant sequence of zeroes. The inequality $C \geq 0$ means $c_t \geq 0$ for each t. Reserve θ for the real-valued zero function, $\theta(C) = 0$, defined on this space. Define the absolute value of $C \in s, |C|$, by the formula: $|C| = \sup\{C, (-C)\}$. This holds coordinatewise: $|C| = \{|c_1|, |c_2|, \ldots\}$. The space s is a Dedekind complete Riesz space. That is, each order bounded set in s has a sup (join) and inf (meet). An order interval in s is a set of the form $\langle C_*, C^* \rangle = \{C : C_* \leq C \leq C^*\}$. An order bounded subset of the commodity space is a subset of an order interval.

A point in s^+ is strictly positive if each coordinate is positive. Given a strictly positive vector $\omega \in s^+$, define the **principal ideal** in s:

$$A_\omega = \{C \in s : |C| \leq \lambda\omega \text{ for some scalar } \lambda > 0\}.$$

The positive cone of A_ω is (with the induced partial order inherited from s):

$$A_\omega^+ = \{C \in A_\omega : C \geq 0\}.$$

This is the commodity space in the anticipated economic applications. It is a Dedekind complete Riesz space in its induced partial order.

Growth models have principal ideal commodity spaces with $\alpha \geq 1$. There are two interesting economic examples: first, set $\omega = (1, 1, \ldots)$, and $A_\omega = \ell_\infty$, the vector space of all bounded real-valued sequences. This arises in models where feasible consumption and capital accumulation programs are bounded due to constraints on initial resources and the productive technology. For example, there is a maximum sustainable capital stock as in the neoclassical one-sector model. The second case arises when the productive technology admits a sustainable, constant, maximum growth rate where $\omega = (\alpha, \alpha^2, \ldots)$ for $\alpha > 1$. The maximum growth rate is $\alpha - 1 > 0$. The linear one-sector model with production function $f(k) = \alpha k$, $\alpha > 1$, illustrates this case. Note $l_\infty \subset A_\omega \subset s$ when $\alpha > 1$. For each $C \in A_\omega$,

$$\|C\|_\infty = \inf\{\lambda > 0 : |C| \leq \lambda\omega\}$$

defines a lattice norm; λ is a scalar. The vector ω is an order unit in A_ω. The principal ideal $(A_\omega, \|\bullet\|_\infty)$ together with its lattice norm is an AM-space with unit ω since the lattice norm satisfies the relation $\|C \bigvee C'\|_\infty = \max\{\|C\|_\infty \bigvee \|C'\|_\infty\}$ for each $C, C' \in A_\omega$.

Following ideas drawn from Boyd [12], and further developed in Becker and Boyd [8], weighted norms are introduced on this principal ideal. These norms are defined by strictly positive real-valued weight functions defined on A_ω. For example, the principal ideal's lattice norm defines a weighted norm on that space turning it into both a Banach space and a Banach lattice. This norm must agree with the lattice norm $\|\bullet\|_\infty$ since a Banach lattice may have only one norm.

There is an equivalent norm that offers an economically useful interpretation of the lattice norm. The $\alpha - norm$, $\|\bullet\|_\alpha$, is defined for elements of A_ω by the formula:

$$\|C\|_\alpha = \sup_{t\geq 1} \left| \frac{c_t}{\alpha^t} \right|. \tag{3}$$

The weighted normed vector space $\ell_\infty(\alpha)$ is defined by the pair $(A_\omega, \|\bullet\|_\alpha)$ where $\alpha \geq 1$. We note that the sequences in this space are $\alpha - norm - bounded$ since $(|c_t|/\alpha^t) \leq \lambda < +\infty$. This is so as $C \in A_\omega$ means there is some scalar $\lambda > 0$ such that $|c_t| \leq \lambda\alpha^t$ for each t. Hence, $\|C\|_\alpha \leq \lambda < +\infty$ whenever $C \in A_\omega$. This normed space is a vector lattice with the usual pointwise operations for join and meet of two vectors. The positive cone of this space is denoted by $\ell_\infty^+(\alpha)$, which is just A_ω^+ with the relative $\alpha - norm$ topology. The space $\ell_\infty(\alpha)$ is also a Banach lattice, so its positive cone is also $\alpha - norm$ closed. The lattice norm is equivalent to the $\alpha - norm$. This positive cone is also convex and has a nonempty $\alpha - norm$ interior. The latter fact follows from the observation that $\ell_\infty(\alpha)$ is an AM-space with unit ω.

2.1.2 The Space of Possible Utility Functions

The Koopmans equation and its companion operator act on an underlying domain of possible or trial real-valued utility functions with common domain $\ell_\infty^+(\alpha)$. These trial utility functions must also be bounded in an appropriately defined weighted norm. Marinacci and Montrucchio's [31] weight function is chosen to obtain an order interval of possible utility functions on which the solution to Koopmans' equation is sought.

Their **weight function**, φ_γ, is defined for each $C \in \ell_\infty^+(\alpha)$ by the formula:

$$\varphi_\gamma(C) = (1 + \|C\|_\alpha)^{1/\gamma}. \tag{4}$$

This weight function is uniformly continuous convex function on $\ell_\infty^+(\alpha)$ with respect to the $\alpha - norm$ topology.[4] Here, the parameter $0 < \gamma \leq 1$ is taken from $(M1)$. This weight function as well the $\alpha - norm$ entangle preference and technology parameters — the growth rate α is derived from a model's technology side while the parameter γ comes from the model's aggregator side.

Definition 1 *A function $U : \ell_\infty^+(\alpha) \to \mathbb{R}$ is $\varphi_\gamma-$ bounded provided*

$$\|U\|_\gamma := \sup_{C\in\ell_\infty^+(\alpha)} \frac{|U(C)|}{(1 + \|C\|_\alpha)^{1/\gamma}} < +\infty.$$

[4] The norm $\|\bullet\|_\alpha$ is a uniformly continuous convex real-valued function defined on the set A_ω. See Aliprantis and Burkinshaw ([3], p. 218). Hence, the function $\varphi_\gamma(C)$ is $\alpha-$ norm continuous as the composition of the continuous functions $1 + \|C\|_\alpha$ and $\phi(x) = x^{1/\gamma}$ for $x > 0$. This also shows that for $\gamma \leq 1$ that φ_γ is a convex function.

The set of all $\varphi_\gamma-$ bounded real-valued functions with domain $\ell_\infty^+(\alpha)$ is denoted by F_γ^α.

The zero function, θ, is the origin in F_γ^α. It is a Dedekind complete Riesz space with the usual pointwise partial order. Clearly the weight function φ_γ satisfies $\varphi_\gamma(\theta) = 1$ and $\varphi_\gamma(C) \geq 1$ for each C. Moreover, $\left\|\varphi_\gamma\right\|_\gamma = 1$ and φ_γ is an order unit in F_γ^α. The corresponding positive cone is denoted by $\left(F_\gamma^\alpha\right)^+$. The normed vector space $\left(F_\gamma^\alpha, \|\bullet\|_\gamma\right)$ is an AM-space with unit φ_γ. Moreover, it is a Banach lattice. Consequently, the positive cone is norm-closed and has a nonempty norm interior.

The space

$$C_\gamma^\alpha := \left\{U \in F_\gamma^\alpha : U \text{ is } \|\bullet\|_\alpha - \text{ continuous on } \ell_\infty^+(\alpha)\right\}$$

is a closed subspace of F_γ^α. However, it is *not* a Dedekind complete Riesz space: an increasing (isotone) sequence of continuous functions has a supremum and that sup is lower semicontinuous, need not be continuous. The corresponding positive cone is $\left(C_\gamma^\alpha\right)^+$ has a nonempty sup norm interior since $\varphi_\gamma \in \left(C_\gamma^\alpha\right)^+$ is an order unit.

2.2 The Koopmans Equation and Koopmans Operator

The aggregator approach to recovering recursive utility representations from an underlying concave Thompson aggregator is expressed in terms of a functional equation on the domain F_γ^α. The **Koopmans equation for recursive utility** is

$$U(C) = W(c_1, U(SC)). \tag{5}$$

Define the **shift operator** $S : \ell_\infty^+(\alpha) \to \ell_\infty^+(\alpha)$ according to the rule $C = \{c_1 c_2, c_3, \ldots\} \mapsto SC = \{c_2, c_{3,} \ldots\}$. A solution of this equation is a recursive utility function representation of the preference relation. Of course, it all depends on what is meant by a solution. Proving this functional equation has a solution turns on recasting the problem as demonstrating a corresponding nonlinear operator, the **Koopmans operator** (denoted by T_W), has a fixed point in the desired function space of possible solutions. The Koopmans operator is formally defined given a function $U \in \left(F_\gamma^\alpha\right)^+$ by the following equation for each $C \in \ell_\infty^+(\alpha)$:

$$(T_W U)(C) = W(c_1, U(SC)).$$

If $T_W U = U$, then U is a solution to (5) and is a recursive utility representation for some underlying preference ordering on the commodity space. The Koopmans operator for concave Thompson aggregators is easily shown to be a monotone operator: $U, U' \in \left(F_\gamma^\alpha\right)^+$ and $U \geq U'$ implies $T_W U \geq T_W U'$.

2.3 The Order Interval $\langle \theta, U^T \rangle$ and the Self-Mapping Koopmans Operator

The next step is to limit the search for a fixed point to an order interval contained in $\left(F_\gamma^\alpha\right)^+$. The order interval's least element is the zero function. The greatest element is defined so that application of either the TK FPT or the related Kleene FPT yields the LFP solution of the Koopmans equation. The Koopmans operator must be a self-map on the order interval to carry out this plan. Define Marinacci and Montrucchio's [31] greatest element as the function $U^T(C) = W(1, y_\alpha)\,\varphi_\gamma(C)$ for each $C \in \ell_\infty^+(\alpha)$. Here, the element $y_\alpha > 0$ is the solution to $W(1, y_\alpha) = \alpha^{-1/\gamma} y_\alpha$ (shown to exist in [31] using $(M1) - (M2)$). U^T is a $\|\bullet\|_\alpha -$ continuous function and belongs to $\left(C_\gamma^\alpha\right)^+$. Evidently $U^T \in \left(F_\gamma^\alpha\right)^+$ as well. Define the order interval $\langle \theta, U^T \rangle \subset \left(F_\gamma^\alpha\right)^+$, where we now say that θ is the **bottom** element and U^T is the **top** element. Observe that $\langle \theta, U^T \rangle$ is a complete lattice with its induced order taken as an order bounded subset of F_γ^α. Clearly $U^T \geq \theta$ (indeed, $U^T(C) > 0$ for each $C \geq 0$) and $\left\|U^T\right\|_\gamma = W(1, y_\alpha) < +\infty$ follows.

The order interval must have the properties $T_W\theta \geq \theta$ and $T_W U^T \leq U^T$ in order to verify T_W is a self-map on $\langle \theta, U^T \rangle$ given it is a monotone operator. Evidently $T_W\theta \geq \theta$ since for each $C \in \ell_\infty^+(\alpha)$ we have $T_W\theta(C) = W(c_1, 0) \geq 0$ as $\theta(SC) = 0$. Based on the additional properties $(M1) - (M2)$, Marinacci and Montrucchio [31] prove $T_W U^T \leq U^T$, which implies T_W *is a self-map on* $\langle \theta, U^T \rangle$.

3 Order Continuity, the Koopmans Operator, the Information Ordering and the Principal Solution

Order continuity of the Koopmans operator is understood in terms of order convergent sequences. Order convergence depends on the Riesz space's defining partial order and lattice operations. The formal definition of order continuity is technically a more restrictive property than the monotonic sup/inf-preservation property. Only the latter is required for application of the TK FPT and deduction that the LFP and the GFP exist. Order convergence for sequences, suitably abstracted, leads to us to Scott's topology and a possible application of Kleene's FPT to the Koopmans operator's LFP existence problem. The formal proof that monotonic sup-preservation holds for the principal limit of monotonic Scott convergent nets follows from showing T_W is Scott continuous. The monotonic sup-preservation for sequences, required for a TK FPT application, constitutes special cases of monotonic sup-preservation for Scott convergent nets when the sup of a monotonic sequence is treated as the principal limit of that sequence in Scott's topology (see Section 4).

3.1 Order Convergent Sequences

We follow Kantorovich's [21] order convergence and continuity ideas. They depend only on each underlying Riesz space's order structure. An order convergent sequence is formally defined via the sequence's upper and lower limits. They exist as elements of (F_α^γ) by Dedekind completeness provided the given sequence $\{U^N\} \in (F_\alpha^\gamma)$ is **order bounded**. A sharper claim is available for sequences in $\langle \theta, U^T \rangle$. Upper and lower limits are well-defined as the various sups and infs exist in $\langle \theta, U^T \rangle$ since it is a $\sigma-$ **closed order interval** (here N and K are natural numbers), that is, $\langle \theta, U^T \rangle$ contains the sups and infs of countable subsets.[5]

$$\begin{aligned} \limsup_N U^N &= \inf_N \left(\sup_{K \geq N} U^K \right); \\ \liminf_N U^N &= \sup_N \left(\inf_{K \geq N} U^K \right). \end{aligned} \tag{6}$$

The sequence $\{U^N\}$ is said to $\sigma-$*order converge* to U provided $\limsup_N U^N = U = \liminf_N U^N$ in $\langle \theta, U^T \rangle$. If $\{U^N\}$ is **isotonic (increasing)**: $U^N \leq U^{N+1}$, then $\{U^N\}$ is a countable chain in $\langle \theta, U^T \rangle$. Moreover, $\liminf_N U^N = \sup_N \{U^N\} = \bigvee_N U^N \in \langle \theta, U^T \rangle$, using lattice notation for supremum in the last expression. We also use the notation $U^N \nearrow \bigvee_N U^N$ (with some abuse of notation). Similarly, if $\{U^N\}$ is **antitone (decreasing):** then $\{U^N\}$ is a countable chain in $\langle \theta, U^T \rangle$ and $\limsup_N U^N = \inf_N U^N = \bigwedge_N U^N \in \langle \theta, U^T \rangle$. Denote this by $U^N \searrow \bigwedge_N U^N$. The reader is reminded that these convergence relations hold pointwise, that is, for each $C \in \ell_\infty^+(\alpha)$. As $\langle \theta, U^T \rangle$ is a complete lattice with its induced order, it is also a countably chain complete poset — a requirement for applying the TK FPT.

3.2 Monotonically Sup/Inf-Preserved Sequences and the CRT

Two particular sequences, $\{T_W^N \theta\}$ and $\{T_W^N U^T\}$, are found by iterating the Koopmans operator starting with the initial seeds θ and U^T, respectively. Here, $T_W(T_W \theta) = T_W^2 \theta \geq T_W \theta \geq \theta$ (by T_W monotone) and so on for each N. That is, $\{T_W^N \theta\}$ is isotone. Likewise, $\{T_W^N U^T\}$ is antitone. Application of the TK FPT only requires the Koopmans operator be a monotonically sup/inf-preserving self-map on $\langle \theta, U^T \rangle$. This implies these two iterations converge to their respective fixed points: the LFP for $\{T_W^N \theta\}$ and the GFP for $\{T_W^N U^T\}$. These fixed points may differ.

One advantage in checking the Koopmans operator is monotonically sup/inf-preserving lies in checking the condition for $\{T_W^N \theta\}$ and $\{T_W^N U^T\}$ only. Otherwise, we must verify T_W is $\sigma-$*order continuous*: for *each* $\sigma-$ order convergent

[5] Order convergence for sequences and nets is covered by Aliprantis and Border [2]. Our constructions occur in Dedekind complete Riesz spaces. This avoids some subtle definitional issues raised by Abramovich and Sirotkin [1] about the exact meaning of order convergence.

sequence $\left\{U^N\right\} \subset \left\langle \theta, U^T \right\rangle$ with limit U, $\{T_W U^N\}$ σ – order converges to $T_W U$. The sequences appearing in this condition need not be monotone.

Kantorovich's ([21], pp. 66 - 68) order continuity condition for the LFP is basically monotonic sup preservation as used here. The Koopmans operator is a **monotonically sup-preserving self-map on** $\left\langle \theta, U^T \right\rangle$ provided for any isotone sequence $\left\{U^N\right\}$:

$$T_W \left(\liminf_N U^N \right) = T_W \left(\bigvee\nolimits_N U^N \right) = \bigvee\nolimits_N T_W U^N.$$

Note that T_W monotone implies $\left\{T_W U^N\right\}$ is an isotone sequence. Setting $U = \bigvee_N T_W U^N$, $T_W(U) = U$ follows and it is a fixed point of T_W. In particular, set $T_W U^N = T_W^N \theta$ and note $T_W^N \theta \nearrow \left(\bigvee_N T_W^N \theta\right) = U_\infty = T_W U_\infty$ is a fixed point of T_W. Likewise, the Koopmans operator is a **monotonically inf-preserving self-map on** $\left\langle \theta, U^T \right\rangle$ provided for an antitone sequence $\left\{U^N\right\}$:

$$T_W \left(\limsup_N U^N \right) = T_W \left(\bigwedge\nolimits_N U^N \right) = \bigwedge\nolimits_N T_W U^N.$$

Monotonicity of T_W implies $\left\{T_W^N U^T\right\}$ is antitone. Let $U^\infty = \left(\bigwedge_N T_W^N U^T\right)$ and conclude $T_W(U^\infty) = U^\infty$ is also a fixed point of T_W.

The Koopmans operator is a **monotonically sup/inf-preserving self-map on** $\left\langle \theta, U^T \right\rangle$ if it is both monotonically sup and monotonically inf preserving on $\left\langle \theta, U^T \right\rangle$. That the Koopmans operator is monotonically sup/inf-preserving *rests on the joint continuity assumption* imposed on each concave Thompson aggregator. This joint continuity property also shows up in our LFP Constructive Existence Theorem (Section 4). The main observation about sup/inf preservation is demonstrated next. This result builds on the fact T_W is a monotone self-map on $\left\langle \theta, U^T \right\rangle$.

Proposition 2 *Suppose W is a concave Thompson aggregator. Then $\left\langle \theta, U^T \right\rangle$ is a countably chain complete poset set and the associated Koopmans operator is a monotonically sup/inf-preserving self-map on $\left\langle \theta, U^T \right\rangle$.*

Proof. Suppose $\left\{U^N\right\} \equiv \left\{U^N\right\}_{N=1}^\infty$ is a sequence of φ_γ — bounded functions in the order interval $\left\langle \theta, U^T \right\rangle \subset (F_\alpha^\gamma)^+$. Clearly both the sup and inf of this sequence exist as elements of $\left\langle \theta, U^T \right\rangle$. This implies $\left\{U^N\right\}$ is a a countably chain complete set in $\left\langle \theta, U^T \right\rangle$ provided it is a chain. Therefore, $\left\langle \theta, U^T \right\rangle$ is a countably chain complete poset follows immediately as $\left\{U^N\right\}$ may be an arbitrarily chosen countable chain in $\left\langle \theta, U^T \right\rangle$.

The order interval $\left\langle \theta, U^T \right\rangle$ evidently contains a smallest and largest element. Now suppose $\left\{U^N\right\}$ is any monotone increasing sequence of functions in $\left\langle \theta, U^T \right\rangle$. By countable chain completeness, we find $\bigvee U^N$ exists since each $U^N \leq U^T$. Hence, there is a function $U = \bigvee U^N \in \left\langle \theta, U^T \right\rangle$. In fact, $U^N \nearrow U$ pointwise on $\ell_\infty^+(\alpha)$. That is $\lim_{N\to\infty} U^N(C) = U(C)$ for each $C \in \ell_\infty^+(\alpha)$. Since W is increasing in its second argument and *continuous in its second argument*, $(T1)$ implies for each $C \in \ell_{\infty}^{+}(\alpha)$ the following equalities:

$$\begin{aligned}
\bigvee [T_W U^N](C) &= \bigvee W\left(c_1, U^N(SC)\right) \text{ (by definition of } T_W) \\
&= \lim_N W\left(c_1, U^N(SC)\right) \text{ (by the monotone property of } W) \\
&= W\left(c_1, \lim_N U^N(SC)\right) \text{ (by continuity of } W) \\
&= W\left(c_1, U(SC)\right) \\
&= T_W\left(\bigvee U^N\right)(C).
\end{aligned}$$

Hence, the Koopmans operator is monotonically sup-preserving. Apply the analogous argument for monotone decreasing sequences $\{U^N\}$, bounded below by the zero function. This shows that T_W is also monotonically inf-preserving. Hence, the Koopmans operator is monotonically sup/inf-preserving. ∎

This Proposition's proof seemingly depends only on assumption that the aggregator is jointly continuous. However, the other properties come into play when verifying T_W is a monotone self-map on the order interval $\langle \theta, U^T \rangle \subset (F_\alpha^\gamma)^+$. The related Scott continuity property for monotonic nets is demonstrated in Section 4. Application of the Proposition yields the Constructive Recovery Theorem, which is our version of Marinacci and Montrucchio's [31] Recovery Theorem. The order continuity property shown above is the place where our approach differs from theirs.

Theorem 3 *(Constructive Recovery Theorem (**CRT**)). Suppose W is a concave Thompson aggregator satisfying (M1) and (M2).*

1. *There is a $\|\bullet\|_\alpha -$ upper semicontinuous function $U^\infty \in \langle \theta, U^T \rangle$ such that $T_W U^\infty = U^\infty$.*
2. *There is a $\|\bullet\|_\alpha -$ lower semicontinuous function $U_\infty \in \langle \theta, U^T \rangle$ such that $T_W U_\infty = U_\infty$.*
3. *U_∞ is the least fixed point in $\langle \theta, U^T \rangle$, U^∞ is the greatest fixed point in $\langle \theta, U^T \rangle$, and fix$(T_W)$ is a countably chain complete subset of $\langle \theta, U^T \rangle$.*

Proof. (1): Iterate T_W using U^T as the initial seed. That is, for each natural number, N, let

$$U^N = T_W U^{N-1} \text{ and } U^0 \equiv U^T.$$

Clearly for each $N \geq 1$,

$$\theta \leq U^N \leq U^{N-1} \leq \cdots \leq U^1 \leq U^T.$$

Hence, there is a function U^∞ such that

$$U^\infty = \bigwedge_N U^N \in \langle \theta, U^T \rangle$$

since $\langle \theta, U^T \rangle$ is a countably chain complete subset of $(F^\gamma_\alpha)^+$.

The function U^T is $\|\bullet\|_\alpha -$ continuous on $\ell^+_\infty(\alpha)$. Hence, since, by $(T1)$, W is a continuous function on $\mathbb{R}^2_+$, the function $U^1 = T_W U^T$ is also a $\|\bullet\|_\alpha -$ continuous function on $\ell^+_\infty(\alpha)$, and so on for each U^N. Hence, U^∞ is a $\|\bullet\|_\alpha -$ upper semicontinuous real-valued function on $\ell^+_\infty(\alpha)$ as it is the pointwise infimum of the collection of continuous functions, $\{T^N_W U^T\}$. Proposition 2 shows that T_W is monotonically sup/inf-preserving. Therefore, T_W satisfies the hypotheses of the TK FPT. Hence, we may conclude U^∞ is a fixed point of the Koopmans operator. That is,

$$T_W U^\infty = U^\infty = \bigwedge_N T^N_W U^T.$$

Moreover U^∞ is the Koopmans operator's GFP in $\langle \theta, U^T \rangle$. Suppose $U \in$ fix(T_W). Then $U \leq U^T$ implies upon iteration that $T^N_W U = U \leq T^N_W U^T$ for each N. Passage to the limit as $N \to \infty$ implies $U \leq U^\infty$.

(2): A parallel argument establishes that there is also a LFP in $\langle \theta, U^T \rangle$, denoted $U_\infty = \bigvee_N T^N_W \theta$. It is found by iterating T_W with the initial seed, θ. Moreover, U_∞ is a $\|\bullet\|_\alpha -$ lower semicontinuous real-valued function on $\ell^+_\infty(\alpha)$ as the pointwise supremum of the family of continuous functions $\{T^N_W \theta\}$.

(3): Let fix(T_W) denote the nonempty set of fixed points belonging to our Koopmans operator in the order interval $\langle \theta, U^T \rangle$. Balbus, et al (see [5], Theorem 7, p. 109) implies fix(T_W) is a countably chain complete poset in $\langle \theta, U^T \rangle$ with its induced order. ■

Recall $T^N_W \theta \nearrow U_\infty$ says that each $T^N_W \theta$ approximates U_∞ from below. This construction of U_∞ receives additional attention in Sections 3.3 and 4. Concavity of U_∞ follows since W is a concave Thompson aggregator and $T^N_W \theta$ is concave for each N (apply Boyd's Lemma [12], p. 331).

The CRT implies that **IF** $U^\infty = U_\infty \equiv U^*$, then U^* is the unique $\|\bullet\|_\alpha -$ continuous φ_γ-bounded real-valued function in $\langle \theta, U^T \rangle$ satisfying the Koopmans equation when W is a concave Thompson aggregator. That is, in this situation $U^* \in (C^\gamma_\alpha)^+$ as well! The interesting problem at this point is to provide conditions under which there is a unique $\|\bullet\|_\alpha -$ continuous and φ_γ-bounded solution to this aggregator's Koopmans equation. The uniqueness question is addressed in Marinacci and Montrucchio's papers ([31], [32]). We also address this problem using concave operator theory in our working papers ([9], [10]). The main difficulty, as seen in our aforementioned working papers, is extremal fixed points may differ at $C = 0$ when the commodity space is ℓ^+_∞ $(\alpha = 1)$ and $W(0,0) = 0$ holds (as with our examples). Uniqueness theorems obtained to date apply on the nonempty norm-interior of ℓ^+_∞, so it is the case that LFP and GFP agree on that subset of the commodity space and the recursive utility function so obtained is norm-continuous on that set as well.

The CRT has implications for dynamic optimization models (e.g. optimal growth theory) with a recursive utility objective defined on a sequence space. The CRT's LFP (GFT) is norm lower (upper) semicontinuous. Suppose that the commodity space is ℓ_∞^+ and the Koopmans equation has a unique solution in $\langle \theta, U^T \rangle$ and is an element of the norm interior of this commodity space. This recursive utility function is norm continuous on that subset of ℓ_∞^+. The question of existence of an optimum and its continuity in underlying parameters, taken as an application of the Maximum Theorem, is problematic. The required norm compactness condition is not satisfied in many applications. Marinacchi and Montrucchio ([31], p. 1790; p. 1801 (iii)) show that *given the uniqueness of the Koopmans operator's fixed point on the norm interior of* ℓ_∞^+, the LFP construction's LFP is continuous in the relative product topology on that domain. Product compact feasible sets are common in optimal growth models. The Maximum Theorem's hypotheses may be satisfied (provided the feasible set is a subset of the norm interior of ℓ_∞^+ (this is a restriction). The order convergence based CRT is consistent with this viewpoint since it is a pointwise convergence concept. Marinacchi and Montrucchio's proof that the recursive utility is continuous in the product topology, at least on the norm interior of ℓ_∞^+, applies. This is one advantage of the CRT as it yields both the LFP and GFP with their corresponding norm semicontinuity properties. The LFP is product continuous. Given the uniqueness hypothesis, the LFP also inherits the GFP's upper semicontinuity property in the product topology. Hence, the LFP is product continuous. The LFP in the Scott topological theory cannot be shown to have an upper semicontinuity property on the norm interior of ℓ_∞^+, as occurs in the CRT framework. A GFP might not exist within the Scott topology framework. By comparison, the Scott topological continuity property offers information about the quality of the successive approximations to the LFP that is unavailable using order continuity alone, as will be argued in more detail below.

3.3 Monotonic Sup-Preservation and LFP as the Principal Solution: An Information Ordering Interpretation

An intuitive understanding of the monotonic sup-preserving property of the Koopmans operator is available for the special case where $\{U^N\} = \{T_W^N \theta\}$, which is an isotone sequence. Let $U_\infty \equiv \liminf_N T_W^N \theta$. with $T_W^N \theta \nearrow U_\infty$, the LFP.

The corresponding pointwise expression, given $C \in \ell_\infty^+(\alpha)$ is, $\left(T_W^N \theta\right)(C) \nearrow U_\infty(C)$. Set $(T_W \theta)(C) = U^1(C) = W(c_1, 0)$ and so for each $N \geq 1$. Then:

$$\begin{aligned} \left(T_W^N \theta\right)(C) &= \left(T_W U^{N-1}\right)(C) = W\left(c_1, W\left(c_2, W\left(c_3, \ldots, W\left(c_N, 0\right) \cdots\right)\right)\right), \text{ or} \\ 0 &= \theta(C) \leq U^1(C) \leq U^2(C) \leq \cdots \leq U_\infty(C). \end{aligned}$$

Rewriting this in aggregator terms, the isotonic sequence of "finite horizon" approximations of the infinite horizon value $U_\infty(C)$, yields:

$$0 \leq W(c_1, 0) \leq W(c_1 W(c_2, 0)) \leq \cdots \leq U_\infty(C).$$

Successive approximations starting from θ provide approximations, from below, for the value $U_\infty(C)$. Each approximation incorporates the consumption of a finite number of consecutive periods further interpreted as consumption over a finite horizon of length N. This truncation assigns zero consumption to all periods beyond N. More consumption periods are incorporated in the N^{th} approximation than its predecessors. In this sense, there is **more information** in $W(c_1, W(c_2, 0))$ about $U_\infty(C)$ than provided by $W(c_1, 0)$, and so on. This theoretical computation of $U_\infty(C)$ starts with **no information** about $U_\infty(C)$ as $\theta(C) = 0$ for each C. This interpretation is consistent with the *information ordering* notion in the computer science literature on theoretical computation and successive approximation.[6] Formally interpret the inequality $T_W^N \theta \leq T_W^{N+1}\theta$ to also mean iterate $T_W^{N+1}\theta$ **has more information than iterate** $T_W^N\theta$. Clearly, $\sup_N T_W^N \theta = U_\infty$ contains more information than any iterate: $T_W^N \theta \leq U_\infty$. The bottom element, θ, is interpreted as **totally uninformative** about U_∞ as $\theta \leq T_W^N\theta$ for each N. Successive approximation improves on the information known about U_∞ at each step.

The Koopmans operator is monotonically sup-preserving means:

$$T_W\left(\liminf_N T_W^N \theta\right) = \liminf_N T_W^N \theta. \tag{7}$$

This equality is the same as requiring:

$$T_W\left(\liminf_N T_W^N \theta\right) \leq \liminf_N T_W^N \theta, \tag{8}$$

and

$$T_W\left(\liminf_N T_W^N \theta\right) \geq \liminf_N T_W^N \theta. \tag{9}$$

The right-hand side of inequality (9) is just the $\vee_N T_W^N \theta$ follows since T_W is monotone and $\vee_N T_W^N \theta \geq T_W^K \theta$ for each natural number K. Apply the Koopmans operator to obtain $T_W(\vee_N T_W^N \theta) \geq T\left(T_W^K \theta\right) \geq T_W^K \theta$. Take the supremum of the right-hand side (after changing the indices from K back to N). This verifies (9).

Inequality (8) is more interesting as it has a theoretical computation interpretation suggesting why monotonic sup-preservation for the sequence $\left\{T_W^N \theta\right\}$ is important. Recast this inequality as

$$T_W\left(\liminf_N T_W^N \theta\right) \leq \liminf_N T_W^N \theta = U_\infty.$$

It stands to reason the Koopmans operator applied to $U_\infty = \liminf_N T_W^N \theta$ again CANNOT yield more information about U_∞. This is the interpretation of inequality (8). Suppose $T_W\left(\liminf_N T_W^N \theta\right)$ added "more information" about U_∞. That is, $T_W\left(\liminf_N T_W^N \theta\right) > \liminf_N T_W^N \theta$. Vickers ([39], p. 96) reminds us that this new information can only be *"available to us by waiting until the Crack of Doom (the time when all infinite computations are completed), and*

[6] See Gierz et al ([16], pp. xxviii-xxix; p. 135) Goubault-Larrecq ([17], pp. 56-64), Stoletenberg-Hansen, et al ([36], p.23), and Vickers [39]. We formalize an abstract definition of the information ordering in Section 4.

that is too late." Hence, it is reasonable to conjecture the Koopmans operator satisfies (8), and the monotonic sup-preservation condition follows.

This intuitive importance of monotonic sup-preservation with the information ordering interpretation is our first argument in favor of U_∞ as the Koopmans equation's **principal solution**, or **principal fixed point** (after Kantorovich [21]). Each step in the iterative procedure, initiated at θ, requires knowledge of a finite number of coordinates of the given consumption sequence and the form of the aggregator function only. Iteration from the top element fails on this issue as it requires the exact value $\|C\|_\alpha$ and knowing the entire infinite horizon consumption stream. It would seem more information must be secured to carry through the iteration initiated at U^T than at θ. Calculating the value $U_\infty(C)$, or a "good approximation" of it, requires information on a finite string of consecutive consumption dates. Calculating $U^\infty(C)$, or a "good approximation" of it, demands inputting the complete sequence, C. From a theoretical computational perspective the approximation of the LFP value $U_\infty(C)$ by successive approximation (from below) offers informational advantages over the succession of approximations (from above) to $U^\infty(C)$. Furthermore, $U_\infty \leq U^\infty$ suggests U^∞ encapsulates more information (computationally) than U_∞. In this respect, the LFP has a *minimal information ordering interpretation among all fixed points of the Koopmans operator in* $\langle \theta, U^T \rangle$. For each $U \in \mathsf{fix}(T_W) \subset \langle U_\infty, U^\infty \rangle$, $U \neq U_\infty$, the LFP is always approximated using less information than would be necessary to approximate U and in particular, U^∞. However, the GFP is the only other fixed point known to be approximated by an iterative construction.

One drawback of the order theoretic approach is there is no way to deduce the speed at which the successive approximations converge since order convergence is not formalized in a metric space. More modestly, there is no way to infer if a particular iterate, $T_W^N\theta$ is "close" to U_∞, a general point emphasized by Gierz et al ([16], p. xxvii). Since there is no topology in this context, it is impossible to know if an iterate is eventually in a neighborhood of U_∞ and, in that intuitive sense, a "good approximation of U_∞." The Scott topology and its related continuity idea allows us to formally show "$T_W^N\theta$ is eventually in a Scott neighborhood of the point U_∞." It is a valid statement about "good approximations" in Scott's topology, as applied to Scott convergent nets, and where $\vee_n T_W^N\theta$ is the principal Scott limit of the net $\{T_W^N\theta\}$. Formalizing this is the subject of the next section.

4 Scott Continuity of T_W and Construction of Its Least Fixed Point

The existence of the Koopmans operator's LFP, U_∞, utilizes monotonic sup-preservation of the lower limit (i.e., the sup) of $\{T_W^N\theta\}$ and the underlying Riesz space's order structures. No topological meaning is associated with an order convergent sequence (or, more generally, net). The Koopmans operator's monotonic sup-preservation property for sequences, abstracted to nets describes a topological continuity idea. The possibility for unifying order and topological properties for the Koopmans operator falls into place.

Scott [35] proposes a topology for a complete lattice by abstracting the notions of a lower limit for sequences and lower semicontinuity for real-valued functions on a metric space. His motivation arose from foundational questions in computational theory. His **induced topology** permits consideration of continuous self-maps on a complete lattice. The literature following Scott's fundamental paper refers to the induced topology as the **Scott topology**. Assigning the Scott topology to $\langle \theta, U^T \rangle$ turns that set into a $T_0 - space$: given the points U and V in $\langle \theta, U^T \rangle$, there is a Scott open set containing one and not the other point. The space $\langle \theta, U^T \rangle$ endowed with its Scott topology is neither a T_1 space nor a T_2 space. Convergent nets may have more than one limit!

There are two ways to define the Scott topology. One specifies the open sets directly. The other, which we implement, defines the class of convergent nets and their limits. Sequences hardly suffice in this setup. Both approaches are found in the literature. Kelly ([22], [23]) shows how to specify a topology by describing convergent nets on the given space. It is an analytical approach with a direct link to our proof the Koopmans operator is Scott continuous. Both descriptions of Scott's topology are presented in Gierz et al [16].[7] Scott's [35] original paper also develops both approaches.

We prove the Koopmans operator is a Scott continuous self-map on the order interval $\langle \theta, U^T \rangle$ with its Scott topology. This order interval's complete lattice structure plays an integral role in this demonstration. The monotonic sequence $\left\{ T_W^N \theta \right\}$ once again constructs the LFP, U_∞, by successive approximations, but Kleene's FPT rather than the TK FPT is the foundation. Kleene's FPT (adapted to our setup) requires the monotonicity of the Koopmans operator and the generalization of monotonic sup-preservation for Scott convergent nets. That the sequence $\left\{ T_W^N \theta \right\}$ yields, in its Scott limit, the LFP, is a surprising conclusion given that we must use nets to describe the topology. However, monotonic sequences are particular monotonic nets where the natural numbers form the net's directed index set. A similar construction of the largest fixed point, U^∞, is **not** available using the Scott topological structure! Scott's topological setup abstracts properties enjoyed by real-valued lower semicontinuous functions defined on a metric space and may differ from related properties characteristic of upper semicontinuous functions. For this reason, we argue that the Scott continuity property of the Koopmans operator, and the subsequent fixed point theory (via Kleene's FPT), form another rationale for calling the LFP, U_∞, the principal solution to the operator equation, $T_W U = U \in \langle \theta, U^T \rangle$.

A **net** $u : \Lambda \to \langle \theta, U^T \rangle$ is a mapping from a directed set, Λ, to the complete lattice $\langle \theta, U^T \rangle$. Denote the net by setting $u(\lambda) = U^\lambda \in \langle \theta, U^T \rangle$. The set Λ (with generic elements λ, μ, and ν) is the net's **index set**. This set is **directed** by a binary relation $\geq$ which is reflexive and transitive. Moreover, if λ and μ

[7] See Gierz, et al ([16], pp. 131-138) for detailed motivation, formal definitions of Scott open sets, and the formal development of his topology via net convergence.

are elements of Λ, then there is a $\nu \in \Lambda$ such that $\nu \geq \lambda$ and $\nu \geq \mu$. Write this net as $\left(U^{\lambda}\right)_{\lambda \in \Lambda}$ or, when the meaning is clear, as $\left(U^{\lambda}\right)$. We say that $\left(U^{\lambda}\right)$ is a net in $\left\langle \theta, U^{T}\right\rangle$. This net is **monotonic (isotonic)** when $\mu \geq \lambda$ implies $U^{\mu} \geq U^{\lambda}$. Monotone nets play an important role in Scott's topological theory.

For any net $\left(U^{\lambda}\right)$ in $\left\langle \theta, U^{T}\right\rangle$ define the net's **lower limit**, or lim inf, by

$$\liminf_{\lambda} \left(U^{\lambda}\right) = \sup_{\lambda} \left[\inf_{\mu \geq \lambda} U^{\mu}\right]. \tag{10}$$

Scott [35] refers to the net's lower limit as its **principal limit**. We adopt this terminology as well and justify it below. Note that if $\left(U^{\lambda}\right)$ is an isotonic net in $\left\langle \theta, U^{T}\right\rangle$, then $\liminf_{\lambda} \left(U^{\lambda}\right) = \sup_{\lambda} \left(U^{\lambda}\right)$. This follows as the sup exists in $\left\langle \theta, U^{T}\right\rangle$ as it is a complete lattice in the induced order inherited from F_{γ}^{α}. Clearly $\left(U^{\lambda}\right)$ order bounded (from below by θ) implies $\inf_{\mu \geq \lambda} U^{\mu} = U^{\lambda}$ exists as well for each $\lambda \in \Lambda$.

Scott's topology is defined by describing the class of Scott convergent nets. Let $\mathcal{S}$ denote the class of those pairs $\left(\left(U^{\lambda}\right), U\right)$ such that

$$U \leq \liminf_{\lambda} \left(U^{\lambda}\right). \tag{11}$$

For such a pair we say that U is an $\mathcal{S}-$ *limit* of the net $\left(U^{\lambda}\right)$ and we denote this limit

$$\left(U^{\lambda}\right) \xrightarrow{\mathcal{S}} U. \tag{12}$$

The convergence conditions and inequality (11) hold pointwise for each $C \in \ell_{\infty}^{+}(\alpha)$. That is, (11) is equivalent to the pointwise condition:

$$U(C) \leq \liminf_{\lambda} \left(U^{\lambda}(C)\right) = \sup_{\lambda} \left[\inf_{\mu \geq \lambda} U^{\mu}(C)\right]. \tag{13}$$

The monotonic net $\left(U^{\lambda}\right)$ has the property $\left(U^{\lambda}\right) \xrightarrow{\mathcal{S}} U^{\mu}$ for each $\mu \in \Lambda$. That is, each U^{μ} is an $\mathcal{S}-$ *limit* of the net $\left(U^{\lambda}\right)$! The reason is simple: each $U^{\mu} \leq \sup_{\lambda} \left(U^{\lambda}\right)$; hence $U^{\mu} \leq \liminf_{\lambda} \left(U^{\lambda}\right)$. This shows a net's $\mathcal{S}-$ *limit* may not be unique. For an arbitrary net in $\left\langle \theta, U^{T}\right\rangle$ we refer to the particular limit function, $\liminf_{\lambda} \left(U^{\lambda}\right)$, as the net's **principal limit** to distinguish it from other points in $\left\langle \theta, U^{T}\right\rangle$ which are also limits for this net. The Scott topology on $\left\langle \theta, U^{T}\right\rangle$ is definitely NOT Hausdorff! This description of net convergence defines the Scott topology on the complete lattice $\left\langle \theta, U^{T}\right\rangle$.

Note that a **Scott open set** $\mathcal{O} \subseteq \left\langle \theta, U^{T}\right\rangle$, as defined through the convergent net description, has two properties: First, if $V \in \mathcal{O}$ and $U \geq V$, then $U \in \mathcal{O}$. Second, if the net $\left\{U^{\lambda}\right\}$ is also a *directed set* in $\mathcal{O}$ and $\vee_{\lambda} U^{\lambda} \in \mathcal{O}$, then there exists an index $\mu \in \Lambda$ such that $U^{\mu} \in \mathcal{O}$. An isotonic net is a directed set. Hence, the latter condition implies each Scott convergent isotonic net is eventually in a neighborhood of its principal limit. This is precisely the concept needed to infer a successive approximation sequence such as $\left\{T_{W}^{N} \theta\right\}$, viewed as an isotonic net with principal limit $\vee_{N} T_{W}^{N} \theta = U_{\infty}$, is eventually in each Scott neighborhood of U_∞. Since the net (sequence) is isotonic, for a given neighborhood of U_∞, once $T_W^N\theta$ belongs to that neighborhood, so will each successive iteration by the Koopmans operator's monotonicity property. Our intuitive understanding of the meaning of $\left\{T_W^N\theta\right\}$ produces "good approximations" of U_∞ is that successive approximations produces a net (sequence) that is eventually in any given neighborhood of U_∞.

The Koopmans operator is **Scott continuous** if and only if for each $\left(U^\lambda\right) \xrightarrow{\mathcal{S}} U$, the corresponding values $\left(T_W U^\lambda\right) \xrightarrow{\mathcal{S}} T_W U$. That is, the **abstract lower semicontinuity property** holds (pointwise):

$$(T_W U) \leq \liminf_\lambda \left(T_W U^\lambda\right) \tag{14}$$

whenever $\left(U^\lambda\right) \xrightarrow{\mathcal{S}} U$. Writing out the pointwise version of the above inequality in terms of the underlying Thompson aggregator yields the condition

$$W\left(c_1, U\left(SC\right)\right) \leq \liminf_\lambda \left[W\left(c_1, U^\lambda\left(SC\right)\right)\right]. \tag{15}$$

The next result specializes a general theorem in Gierz et al ([16], Proposition II-2.1, p. 157) for our setup. We verify the Koopmans operator satisfies its necessary and sufficient conditions for the Scott continuity in the upcoming Scott Continuity Proposition. Their general theorem links the formal notion of continuity in the Scott topology with preservation of suprema for directed sets and the preservation of suprema for isotonic nets. The implication is we can limit our analysis to checking the preservation of suprema for isotonic nets. This implies we need verify the monotonic sup-preservation property for isotonic nets in order to apply the Kleene FPT to our LFP construction.

Proposition 4 *The Koopmans operator is a Scott continuous self-map on $\left\langle\theta, U^T\right\rangle$ if and only if it is an order-preserving (monotone) operator and for any net $\left(U^\lambda\right)$ in $\left\langle\theta, U^T\right\rangle$ such that $\liminf_\lambda \left(U^\lambda\right)$ and $\liminf_\lambda \left(T_W U^\lambda\right)$ both exist,*

$$T_W\left(\liminf_\lambda U^\lambda\right) \leq \liminf_\lambda \left(T_W\left(U^\lambda\right)\right) \tag{16}$$

Inequality (16) expresses the abstract lower semicontinuity inequality (14) for the case where $U = \liminf_\lambda U^\lambda$ *is the net's principal limit.* Note that $\liminf_\lambda \left(U^\lambda\right)$ and $\liminf_\lambda \left(T_W U^\lambda\right)$ both exist since $\left\langle\theta, U^T\right\rangle$ is a complete lattice in its induced order. The nets appearing in this proposition may, or may not, be monotonic. The pointwise analog of (16) expressed in terms of the Thompson aggregator is

$$W\left(c_1, \liminf_\lambda U^\lambda\left(SC\right)\right) \leq \liminf_\lambda W\left(c_1, U^\lambda\left(SC\right)\right), \tag{17}$$

where

$$T_W\left(\liminf_\lambda U^\lambda(C)\right) = W\left(c_1, \liminf_\lambda U^\lambda(SC)\right), \text{ and}$$
$$\liminf_\lambda \left(T_W\left(U^\lambda(C)\right)\right) = \liminf_\lambda W\left(c_1, U^\lambda(SC)\right).$$

Since the Koopmans operator is already known to be a monotone operator it suffices to verify (16) obtains for an arbitrary convergent net of functions in $\langle\theta, U^T\rangle$ in order to conclude the Koopmans operator is Scott continuous.

Observe that if $(U^\lambda) \xrightarrow{S} U$, then $U \leq \liminf_\lambda U^\lambda$, so T_W monotone implies

$$T_W U \leq T_W\left(\liminf_\lambda U^\lambda\right).$$

Hence, if (16) also holds, then the previous inequality yields

$$T_W U \leq T_W\left(\liminf_\lambda U^\lambda\right) \leq \liminf_\lambda \left(T_W\left(U^\lambda\right)\right),$$

which is the abstract lower semicontinuity inequality (14) and T_W is Scott continuous. The following Continuity Proposition sets up the proof that fix(T_W) is nonempty and has a LFP.

Proposition 5 *(Scott Continuity Proposition) T_W is a Scott continuous self-map on $\langle\theta, U^T\rangle$.*

Proof. We prove the pointwise inequality (17) obtains. Fix a consumption sequence $C \in \ell_\infty^+(\alpha)$. Note that a concave Thompson aggregator function, $W(x,y)$, is jointly continuous on $\mathbb{R}_+^2$. In particular, given $c_1 \geq 0$, the function $W(c_1, \bullet)$ is a lower semicontinuous function on $\mathbb{R}_+^* = [0, +\infty]$, the nonnegative extended real numbers endowed with its usual topology. Now consider $\left(U^\lambda(C)\right)$ and $\left(U^\lambda(SC)\right)$ as defining nets in $\mathbb{R}_+^*$. In fact, the values taken by the nets for each index are nonnegative real numbers as each U^λ is φ_γ *− bounded*. Indeed, each $U^\lambda \leq U^T$ implies $\left\|U^\lambda\right\|_\gamma \leq \left\|U^T\right\|_\gamma$. This lower semicontinuity property for the aggregator implies:

$$W\left(c_1, \liminf_\lambda U^\lambda(SC)\right) \leq \liminf_\lambda W\left(c_1, U^\lambda(SC)\right),$$

which is (17). Therefore (16) holds and T_W is Scott continuous by the previous Proposition. ■

Extend the information ordering to monotonic nets by interpreting $U^\mu \leq U^\lambda$ as U^λ is **more informative** than U^μ when $\lambda, \mu \in \Lambda$ and μ precedes λ in the directed index set Λ. With this interpretation of the monotone nets (U^λ) and $\left(T_W\left(U^\lambda\right)\right)$ we observe by the abstract lower semicontinuity property that $T_W(U) \leq \liminf_\lambda T_W\left(U^\lambda\right)$ when the monotonic net $(U^\lambda) \xrightarrow{S} U$.

Now, $\left(T_W\left(U^\lambda\right)\right)$ is also a monotonic net since T_W is a monotone operator. Then $\liminf_\lambda T_W\left(U^\lambda\right) = \sup_\lambda T_W\left(U^\lambda\right)$ contains more information than $T_W\left(U\right)$ by the information ordering. If T_W is applied to the principal limit of $\left(U^\lambda\right)$, $\liminf_\lambda U^\lambda = \sup_\lambda\left(U^\lambda\right)$, then, Vickers ([39], p.96) "Crack of Doom" metaphor applied to these nets tells us the information ordering inequality:

$$T_W\left(\sup_\lambda\left(U^\lambda\right)\right) = T_W\left(\liminf_\lambda U^\lambda\right) \leq \liminf_\lambda\left(T_W\left(U^\lambda\right)\right) = \sup_\lambda T_W\left(U^\lambda\right)$$

must hold, at least for monotonic nets. Thus, combining a monotone operator with Scott continuity applied to "successive approximations" should be a consistent way of adding information at each computational stage no matter how many iterations have been undertaken and whether the index set Λ is an arbitrary directed set or $\Lambda = \mathbb{N}$. That is, the use of nets to model a series of abstract computations generalizes the idea of computations undertaken by iteration over the natural numbers. In particular, inequality (16) is verified for the Koopmans operator.

Definition 6 *The Koopmans operator T_W is said to* ***preserve the supremum of the monotonic net*** $\left(U^\lambda\right)$ *in* $\left\langle\theta, U^T\right\rangle$ *whenever*

$$T_W\left(\liminf_\lambda\left(U^\lambda\right)\right) = \liminf_\lambda\left(T_W U^\lambda\right). \tag{18}$$

Put differently, T_W preserves the supremum of monotonic nets provided that

$$T_W\left(\sup\left(U^\lambda\right)\right) = \sup\left(T_W U^\lambda\right).$$

These suprema correspond to the principal limits of the monotonic nets $\left(U^\lambda\right)$ *and* $\left(T_W U^\lambda\right)$, where the latter net is also monotonic as T_W is a monotone operator. This is a subtle distinction with respect to monotonic sup-preservation for order convergent sequences. Notice that if this property holds for arbitrary monotone nets, then it holds in particular for monotonic (nondecreasing) sequences, such as $\left\{T_W^N\theta\right\}$ PROVIDED only the principal limits are to be preserved. This observation is the key to reducing the existence of a fixed point for the Koopmans operator to the application of the Kleene FPT (for *monotonically* $\sup -$*preserving sequences*). The smallest fixed point, U_∞, is constructed as before by iteration of T_W indexed on the natural numbers with initial seed θ. The existence of the smallest fixed point by successive approximations is available even though sequences do not suffice to describe the Scott topology. Hence, the key step in showing this construction applies is the following Corollary to the Scott Continuity Proposition. It applies to the principal limit in the relevant Scott convergent nets.

Corollary 7 *T_W preserves the supremum of each monotonic net* $\left(U^\lambda\right)$ *in* $\left\langle\theta, U^T\right\rangle$.

Proof. Let $\left(U^\lambda\right)$ be a monotonic net in $\left\langle\theta, U^T\right\rangle$ with its principal Scott limit $\vee_\lambda U^\lambda$. The net $\left(T_W U^\lambda\right)$ is also a monotonic net in $\left\langle\theta, U^T\right\rangle$ since T_W

is monotone. Its principal Scott limit is $\vee_\lambda T_W U^\lambda$. Since T_W is a Scott continuous self-map on $\langle \theta, U^T \rangle$, inequality (16) holds in the following form:

$$T_W\left(\bigvee_\lambda U^\lambda\right) \leq \bigvee_\lambda \left(T_W U^\lambda\right).$$

The converse inequality follows since T_W is a monotone operator. To see this, note that for each index $\mu \in \Lambda$,

$$\bigvee_\lambda U^\lambda \geq U^\mu,$$

and by T_W monotone,

$$T_W\left(\bigvee_\lambda U^\lambda\right) \geq T_W U^\mu.$$

The supremum of the right-hand side, after changing back to the λ index notation, is just $\vee_\lambda T_W U^\lambda$. Hence,

$$T_W\left(\bigvee_\lambda U^\lambda\right) \geq \bigvee_\lambda \left(T_W U^\lambda\right).$$

Therefore,

$$T_W\left(\bigvee_\lambda U^\lambda\right) = \bigvee_\lambda \left(T_W U^\lambda\right),$$

and the Koopmans operator preserves the supremum of monotonic nets. ■

The main result in this section is the existence of a smallest or least fixed point for the Koopmans operator and its construction by successive approximations.

Theorem 8 *(Least Fixed Point Existence and Construction Theorem) The Scott continuous Koopmans operator has a least fixed point, U_∞. Moreover, $U_\infty = \vee_N T_W^N \theta$ and it is constructed by successive approximations indexed on the natural numbers.*

Proof. The existence and construction of U_∞ follows from the proof of Kleene's FPT in Goubault-Larrecq ([17], p.64) as T_W preserves the supremum of each monotonic net $\left(U^\lambda\right)$ in $\langle \theta, U^T \rangle$. In particular, this property holds for the principal Scott limit of the monotone sequence $\left\{T_W^N \theta\right\} \nearrow U_\infty = \vee_N T_W^N \theta = T_W U_\infty$ by the previous corollary. Thus, $U_\infty \in \text{fix}(T_W)$.

Suppose that $U \in \text{fix}(T_W)$. Then $\theta \leq U$ and T_W monotone implies $T_W \theta \leq T_W U = U$. Iterate this to yield the inequality $T_W^N \theta \leq U$. Hence, passing to the limit we find $U_\infty \leq U$ and U_∞ is the least fixed point of the Koopmans operator acting on $\langle \theta, U^T \rangle$. ■

The sequence $\left\{T_W^N \theta\right\}$ has many Scott limits besides its principal limit, U_∞. But NONE of the other Scott limits, such as $T_W^N \theta$, are also fixed points. That is, the LFP is the *unique* Scott limit of $\{T_W^N \theta\}$ that is also a fixed point of the Koopmans operator.

An important implication of the LFP construction within the Scott topology framework is that the sequence yields "good approximations" to the LFP. Given the LFP we find that the sequence of iterates is eventually in any neighborhood (open set) containing the LFP. The monotone properties of $\{T_W^N \theta\}$ imply that once an iterate belongs to a particular neighborhood of the LFP, then all succeeding iterates also belong to that neighborhood and improves the information about the LFP as one step follows another.

The CRT says that $U^\infty = \wedge_N T_W^N U^T$ is the GFP. This is demonstrated by showing the antitone sequence $\{T_W^N U^T\}$ is inf-preserving: $T_W \left(\wedge_N T_W^N U^T\right) = \wedge_N T_W^N U^T$. This inf-preservation property required by the TK FPT does NOT have an analog in the Scott topology approach.[8] The sequence $\{T_W^N U^T\} \searrow U^\infty$ fails to satisfy the monotonic net sup-preservation property simply because it is not isotone. Scott continuity, acting alone, yields the inequality $T_W U^\infty \leq \liminf_N \left(T_W^N U^T\right) = U^\infty$. Absent a form of inf-preservation, Scott's topological structure does not imply U^∞ is a fixed point for the Koopmans operator. Even though we know from the CRT construction that U^∞ is the GFP of T_W , this fact is not provable from Scott continuity and monotonicity of T_W alone.

Now suppose that we establish U^∞ as the GFP (say, invoke the TK FPT based CRT). Then we observe that each $U \in \langle \theta, U^\infty \rangle$ is also a Scott limit of the sequence $\{T_W^N U^T\}$! In particular, each $U \in$fix(T_W), including the LFP, is a Scott limit of $\{T_W^N U^T\}$. Therefore, the sequence $\{T_W^N U^T\}$ *does not have a* ***unique*** *Scott limit which is also a fixed point*, unlike the LFP theory's case. We cannot reasonably say that the GFP is *constructed* as the unique Scott limit of $\{T_W^N U^T\}$ which is also a fixed point in the manner we can say the LFP is constructed by successive approximations under the Scott continuity hypothesis.

The fact that the fixed point U_∞ is shown to exist as a consequence of verifying the Koopmans operator is Scott continuous provides us with a *topological*, as well as *order-theoretic*, defense for considering this fixed point as the operator equation's principal solution. These two elements, acting in combination, have their roots in "logic and computer science," according to Goubault-Larrecq ([17], p.58).

The Least Fixed Point Existence and Construction Theorem does not yield either the existence of the GFP nor any statement about fix(T_W) other than it is nonempty and U_∞ is its smallest element. By contrast, the TK FPT constructions yield the extremal fixed points and fix(T_W) is a countably chain complete poset. The Least Fixed Point Existence and Construction Theorem's hypotheses are stronger than monotonicity of T_W assumed in Tarski's Theorem [37]. The formal argument is also more elementary (by reduction to the monotonic sup-preservation of sequences) in comparison to Tarski's Theorem.[9]

[8] The inf-preservation property is the analog of saying the Koopmans operator is Scott upper semicontinuous. However, convergence of antitone nets indexed on the natural numbers is well-defined in this concept, but there is no topological requirement that anitone inf-preservation obtain as a matter of Scott continuity. .

[9] See the comments in Gierz et al ([16], p. 160). Also see Goubault-Larrecq ([17], p. 64) in

In particular, the constructive TK FPT and Kleene FPT proofs based on successive approximations by iteration over the natural numbers are certainly more elementary than the recent "constructive" versions for Tarski's Theorem due to Cousot and Cousot [13] and Echenique [15] obtained by iterating over the ordinals. In the case of the Kleene FPT we exploit the natural numbers as an index set for defining a net that has nice limiting and approximation properties in Scott's topology.

5 Conclusion

Our development of the Scott continuity and fixed point machinery for the Koopmans equation has a broader methodological perspective. The same arguments favoring the LFP over the GFP can be made for those two solutions offered in Kantorovich's [21] original paper. Assign his operator's domain, an order interval in a Dedekind complete Riesz space, the Scott topology. The principal solution as defined by Kantorovich is found by iteration from the order intervals bottom element. It differs from his GFP in the Scott context in the same manner discussed by us: each Scott limit of the iterative process initiated at the top element has multiple Scott limits including the LFP. In fact, Scott continuity does not, by itself, show the GFP actually exists as the principal Scott limit of the successive approximation initiated at the top element. Scott continuity of the operator must be verified in applications. Our paper, using net convergence, demonstrates this verification is possible in at least one applied nonlinear operator fixed point problem on a Dedekind complete Riesz space.

Economic models employing monotone operator methods are widespread in the macrodynamics literature. Our computationally motivated methodology offers a selection principle for numerical solutions of recursive macrodynamic models where theory suggests multiple fixed points exist. Choose the Least Fixed Point.

reference to the Kleene Fixed Point Theorem.

6 Mathematical Appendix

6.1 Posets, Lattices, and the Tarski-Kantorovich Theorem

A set X is said to be **partially ordered**, or a **poset**, if it is nonempty and for certain pairs (x, y) in $X \times X$ there is a binary relation $x \leq y$ which is reflexive, transitive, and antisymmetric.

A poset X is a **lattice** provided each pair of elements has a **supremum (sup, meet)** and an **infimum (inf, join)**. Standard lattice notation for sups and infs is followed: $\sup\{x, y\} = x \vee y$ and $\inf\{x, y\} = x \wedge y$. A **complete lattice** is a lattice in which each nonempty subset Y has a supremum $\bigvee Y$ and an infimum $\bigwedge Y$. The element $x \in Y$ is called **greatest,** or **largest** (**smallest,** or **least)** in Y if and only if $y \leq x$ $(x \leq y)$ respectively, for all $y \in Y$. Note that a complete lattice has a greatest element (**top**) and and a bottom element (**bottom**). An **order interval** in X, denoted by $\langle \underline{x}, \bar{x} \rangle \subseteq X$, is defined by $\underline{x} \leq \bar{x}$, $\underline{x} \neq \bar{x}$, and $x \in \langle \underline{x}, \bar{x} \rangle$ if and only if $\underline{x} \leq x \leq \bar{x}$. Clearly $\underline{x}$ is the least element of the order interval while $\bar{x}$ is the corresponding largest element.

Suppose that $Y \subseteq X$ and let X be a poset. The set Y is called a **chain** (of X) if and only if Y is nonempty and for all $x, y \in Y$, one of the two conditions $x \leq y$ or $y \leq x$ holds. If the chain is countable, then it is called a **countable chain**. Let $\{x^n\}_{n=0}^{\infty} \subset X$ be a monotone sequence (either $x^n \leq x^{n+1}$, or $x^n \geq x^{n+1}$ for each n). The monotone sequence $\{x^n\}_{n=0}^{\infty}$ is **increasing (decreasing)** when $x^n \leq x^{n+1}$ $(x^n \geq x^{n+1})$ for each n. A monotone sequence is a countable chain. The supremum and infimum of a monotone sequence are denoted in lattice notation as follows:

$$\bigvee_n x^n = \sup_n x^n; \text{ and } \bigwedge_n x^n = \inf_n x^n.$$

The subscript n in the meet and join notation is omitted when the index set is clearly understood from the context. If, for every chain $Y \subseteq X$, we have $\inf Y \equiv \bigwedge Y \in X$ and $\sup Y \equiv \bigvee Y \in X$, then X is said to be a **chain complete poset**. If this condition obtains only for every countable chain $Y \subseteq X$, then X is said to be a **countably chain complete poset**. If Y has greatest and smallest elements, then monotone sequences $\{x^n\} \subseteq Y$ are countably chain complete posets in Y.

A function $F : X \to X$ is said to be a **self-map on** X. By $F^N(x)$, we are denoting the $N^{th}-$iteration of F with initial seed x. That is, $F^N(x) = F\left(F^{N-1}(x)\right)$ for each natural number N and $F^0(x) \equiv x$. This self-map is said to be **monotone** whenever $x, y \in X$ and $x \leq y$, then $F(x) \leq F(y)$. Some writers refer to a monotone self-map as an **isotone self-map or an increasing self-map**. A point $x^* \in X$ with $F(x^*) = x^*$ is a fixed point of the self-map, F. The set of all fixed points of this self-map is denoted $\text{fix}(F)$.

The classical Tarski Fixed Point Theorem [37] asserts that a monotone self-map on a complete lattice has a nonempty set of fixed points. Moreover, there is a smallest and a largest fixed point. These are the **extremal fixed points**. The set of all fixed points forms a complete lattice in the induced order (the partial order inherited from X). Successive approximations iterating the monotone self-map by **transfinite induction** yields the largest fixed point with initial seed the top element, and the smallest fixed point when the bottom element is the initial seed.[10] Iteration using transfinite induction is not a constructive procedure in any sense of that term. The Tarski-Kantorovich Theorem is similar to Tarski's result, but combines a weaker property for the self-map's domain with a stronger order continuity condition imposed on the operator. That property implies the operator is a monotone self-map.

We consider two distinct forms of order continuity. The first is defined entirely in terms of the underlying order properties of our domain's (and range's) function space. This approach, introduced below, implies the set of fixed points is a countably chain complete subset of the operator's domain. The successive approximation procedure used in this result is constructive in so far as the iterations are indexed on the natural numbers in contrast to the transfinite iterative procedure underlying Tarski's Theorem. The second order continuity idea is topological and its recursive utility application is new.[11] This is the notion of continuity when the order interval of possible utility functions is endowed with Scott's induced topology. This topology's definition and the development of its properties as applied to the Koopmans operator are deferred to Section 4. Scott's topological structure yields a constructive foundation for the operator's least fixed point. We argue in Section 4 that this result reenforces the arguments supporting the least fixed point as the operator equation's **principal solution**.

Definition 9 *A self-map F defined on a countably chain complete poset X with the greatest element $\bar{x}$ and smallest element $\underline{x}$ is* ***monotonically sup-preserving*** *if for any increasing $\{x^n\}$ we have*

$$F\left(\bigvee x^n\right) = \bigvee F(x^n),$$

and ***monotonically inf-preserving*** *if for any decreasing $\{x^n\}$, we have*

$$F\left(\bigwedge x^n\right) = \bigwedge F(x^n).$$

F is said to be ***monotonically sup/inf-preserving*** *if and only if it is both monotonically sup-preserving and monotonically inf-preserving.*

Evidently, a monotonically sup (respectively, inf)-preserving self-map on the ordered space X must be an increasing self-map. The sup/inf preservation

[10] Cousot and Cousot [13] provide a so-called constructive proof without monotonic sup-inf continuity. However, their argument employes transfinite induction. Echenique [15] simplifies their proof while maintaining a tranfinite induction argument. Gierz ([16], p.20) sketches an iterative least fixed point theorem that applies to a monotone self-map on complete lattice. However, that proof also employs transfinite induction indexed by the ordinals.

[11] See Vassilakis [38] for economic and game theoretic applications of Scott domains and Scott continuity (in terms of sequencs as opposed to nets).

property is a type of **order continuity** introduced in Kantorovich's [21] seminal article on monotone methods with successive approximations. In the case of a monotonically increasing sequence the sup is regarded as the sequence's limit and continuity is taken to mean $F(\sup\{x^n\}) = \sup[\{F(x^n)\}]$ where the countable chain is denoted $\{x^n\}$. Likewise for the inf of a decreasing sequence. Some authors (e.g. Granas and Dugundji [18]) refer to order continuity as used here by the term $\sigma -$ **order continuity** to stress the restriction to countable chains and also drop the monotonicity requirement for the sequences. *The conclusions of the Tarksi-Kantorovich Theorem based on iteration indexed on the natural numbers can fail without order continuity. Davey and Priestley ([14], p.93) offer an elementary counterexample.*

The **Tarski-Kantorovich Fixed Point Theorem** (**TK FPT**) as refined by Balbus, Reffett and Woźny ([5], Theorem 7), states the following:[12]

Theorem 10 *Suppose that X is a countably chain complete partially ordered set with the greatest element, $\bar{x}$, and the smallest element, $\underline{x}$. Let F be a monotone self-map on X.*

1. *If F is monotonically inf-preserving; then $\bigwedge_N F^N(\bar{x})$ is the **greatest fixed point** of F, denoted x^∞;*
2. *if F is monotonically sup-preserving; then $\bigvee_N F^N(\underline{x})$ is the **least fixed point** of F, denoted x_∞.*
3. *fix(F) is a nonempty countably chain complete poset in X.*

The result that fix(F) is a countably chain complete poset in X is due to Balbus, Reffett, and Woźny [5]. It is the analog of Tarski's result that fix(F) is a complete lattice in the induced order. The Tarski-Kantorovich theorem tells us that successive approximations (iteration of F indexed on the natural numbers) initiated at either the smallest or greatest element of the set X produces the smallest or largest fixed point in the limit, respectively. Moreover, it is clear that $x_\infty \leq x^\infty$. If x^* is any other fixed point for F, and $\underline{x} \leq x^*$, then $\underline{x} \leq F(\underline{x}) \leq F(x^*) = x^*$. Iteration produces the sequence $\{F^N(\underline{x})\}_{N=1}^{\infty}$ such that for each N, $F^N(\underline{x}) \leq x_\infty \leq x^*$ and $F^N(\underline{x}) \nearrow F(x_\infty) = x_\infty \leq x^*$. Hence, the fixed point x_∞ is the **least fixed point (LFP)**. Likewise, x^∞ is the **greatest fixed point (GFP)**. The notation $F^N(\underline{x}) \nearrow F(x_\infty)$ indicates that $F^N(\underline{x})$ **approximates the LFP from below for each** N. Likewise, $F^N(\bar{x}) \searrow F(x^\infty) = x^\infty$ says $F^N(\bar{x})$ **approximates the GFP from above.**

[12] Granas and Dugundji([18], p. 26) name this result. The earliest published version is in Kantorovich [21]. Baranga [6] presents it as the "Kleene Fixed Point Theorem." Jachymski et al ([19], p. 249) argue it is equivalent to the TK FPT. Also, see Stoltenberg-Hansen, et al ([36], p. 21) on Kleene's Fixed Point Theorem. Kamihigashi et al [20] apply the Kleene Fixed Point Theorem to dynamic programming. These authors assume the operator in question is σ-order continuous.

6.2 The Kleene Fixed Point Theorem

The TK FPT is order theoretic; it is not a topological fixed point theorem. The Kleene Fixed Point Theorem (Kleene FPT) combines order and the Scott topology for the poset X with some additional structure. The formal development of Scott's topology is developed in the main text (Section 4). Kleene's FPT is closely related to the TK FPT, but it stands formally as a distinct result. Both Giabault-Larrecq ([17], p. 64) and Gierz, et al ([16], p. 160) treat Kleene's FPT via the Scott topology. The Theorem's proof offers a surprise: the LFP is constructed by the same successive approximation argument used to construct the LFP in the TK FPT. However, Scott continuity is not necessarily monotonic inf-preserving, so there is a subtle difference in the conclusions of these two fixed point theorems for monotone mappings. See Section 4 for additional comments on the use of the Scott topology and its use in distinguishing the LFP from the GFP (when the latter exists). The Kleene FPT is stated below is a special case of the result given by Giabault-Larrecq [17]. These restrictions are sufficient for application of his stated Kleene FPT and directly connect the Koopmans operator equation and our proof that a LFP exists by successive approximations. The Scott continuity of the self-map F on X implies that F must be a monotone operator.

Theorem 11 *Suppose that X is a chain complete partially ordered set with the smallest element, $\underline{x}$. Let F be a Scott continuous self-map on X. Then F has a LFP in X, denoted $x_\infty = F(x_\infty)$. Moreover, the LFP may be obtained $x_\infty = \bigvee_N F^N(\underline{x})$.*

6.3 Positive Cones and Nonlinear Operators in Riesz Spaces

Let E denote a real vector space. The zero element in E is denoted by θ. A nonempty subset P of E is said to be a **cone** if $x \in P$, then $\lambda x \in P$ for each scalar $\lambda \geq 0$. In particular this definition of a cone implies $\theta \in P$. A cone induces a partial order on the vectors belonging to E. A vector x is said to be **positive**, written $x \geq \theta$, provided $x \in P$. The cone is then called the **positive cone** of E and is denoted by E^+ in the sequel. The standard partial relation expressing $x \geq y$ whenever $x, y \in E$ is defined by requiring $x - y \in E^+$. Write $x > \theta$ whenever $x \geq \theta$ and $x \neq \theta$. Likewise, $x > y$ provided $x \geq y$ and $x \neq y$.

Our application requires the vector spaces are Riesz spaces where E is equipped with the partial order derived from the cone E^+. A **Riesz space** is a partially ordered vector space that is also a lattice. For each element $x \in E$, we define its **positive part**, x^+, its **negative part** x^-, and its **absolute value**, $|x|$, by the formulas:

$$x^+ = x \vee \theta, x^- = x \wedge \theta\,, \text{ and } |x| = x \vee (-x)\,.$$

An **order interval** in the Riesz space E is a set of the form $\langle x, y\rangle = \{z \in E : x \leq z \leq y\}$. A subset G of a Riesz space is **order bounded from above** if there is a $y \in E$ such that $z \leq y$ for each $z \in G$. The dual notion that this subset is order bounded from below is defined similarly. A subset of a Riesz space is **order bounded** if it is contained in an order interval. E is **order complete**, or **Dedekind complete**, if every nonempty subset that is order bounded from above has a supremum (and dually, every nonempty subset that is order bounded from below has an infimum).

Suppose further that E is a real Banach space. The notation $x >> \theta$ means $x \in int(E^+)$, where $int\,(E^+)$ denotes the norm interior of the cone E^+. Of course, this latter inequality is only meaningful when $int\,(E^+) \neq \varnothing$ — a strong topological restriction on the underlying Banach space. An arbitrary cone P contained in E with nonempty interior in its norm topology is said to be a **solid cone**. The positive cones turns out to be solid in our applications.

We consider an abstract nonlinear operator, denoted by A, that is positive on E^+. That is, it is a self-map: $A : E^+ \to E^+$. We also write this as $AE^+ \subseteq E^+$. The operator A is said to be **monotone (isotone, increasing) on** E^+ if $x \leq y, (x, y \in E^+)$ implies $Ax \leq Ay$. It is **antitone** whenever $Ax \geq Ay$ instead. The Koopmans operator is shown in Section 4 to be **monotone** whenever the aggregator is also monotone in its arguments.

Given a nonlinear operator satisfying $AE^+ \subseteq E^+$ we are concerned with the *existence of fixed points* as well as whether or not there is a *unique solution in the cone* E^+. The **operator equation** is $Ax = x$ with $x \in E^+$; a solution is a fixed point of the operator, A. In some applications there may be a trivial fixed point, θ. We are only interested in **nontrivial fixed points** $x \in E^+$ with $x \neq \theta$. The Koopmans operator does not admit a trivial fixed point under our assumptions.

The present paper addresses the existence of a solution in the cone E^+. We do this by showing the operator is an order continuous self-map on a particular order interval in that cone. Application of the TK FPT yields extremal fixed points.

All spaces in this paper are complete normed Riesz spaces. They are also **Banach lattices**. That is, they are Riesz spaces which are Banach spaces whose norms are also lattice norms. A norm $\|\bullet\|$ on a Riesz space is a **lattice norm** provided for each point x and y, $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. Indeed, the spaces on which the Koopmans operator acts turn out to be *abstract* $M - spaces$**, or** $AM - spaces$ **with an order unit**. $AM - spaces$ are Banach lattices for which $\|x \vee y\| = \max\{\|x\|, \|y\| \text{ for each } x, y \in E^+\}$. An $AM - space$ E possesses an **order unit** whenever there exists an element $e \in E$, $e > \theta$, such that for each $x \in E$ there is a scalar $\lambda > 0$ satisfying $|x| \leq \lambda e$. If an $AM - space$ has a unit, then its lattice norm is defined for each $x \in E$ by $\|x\|_\infty = \inf\{\lambda > 0 : |x| \leq \lambda e\}$. This norm is equivalent to the given norm on E. One advantage to this setup is that the positive cone of an $AM - space$ with unit is norm-closed, convex and has a nonempty norm interior. A Banach lattice has an order unit if and only if that order unit is an interior point of the space's positive cone. In this case, the original sup norm and lattice norm topologies are equivalent.

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