Minimization of risks in pension funding by means of contributions and portfolio selection

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Abstract

We consider a dynamic model of pension funding in a defined benefit plan of an employment system. The prior objective of the sponsor of the pension plan is the determination of the contribution rate amortizing the unfunded actuarial liability, in order to minimize the contribution rate risk and the solvency risk. To this end, the promoter invest in a portfolio with n risky assets and a risk–free security. The aim of this paper is to determine the optimal funding behavior in this dynamic, stochastic framework.

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1 Introduction

In a defined benefit plan of an employment system, it is common to apply the so known *Actuarial Cost Methods*, which allow us to determine an ideal contribution rate or normal cost and an ideal fund level (that could be related with the actuarial liability) in such a way that the benefits promised to a collective of members be guaranteed along the time. Actually, the existence of unexpected disturbances can make the evolution of the plan will not be in accordance with the valuation designed at the beginning. Therefore, the contribution rate must be the normal cost plus a positive or negative increment, called the supplementary cost. Of course, the promoter of the fund must planning how to drive the unfunded actuarial liability to zero. We suppose that the plan is built with the contributions and investment earnings. The sponsoring employer controls the contribution rate and the amount invested in a portfolio composed of n risky assets and a risk–free security. No shortselling is allowed but the manager have the possibility of borrowing.

In Haberman (1993), Haberman and Sung (1994) and Haberman (1997), are considered two main types of risks which the pension plan is confronted, the *contribution rate risk* and the *solvency risk*. The first is measured with the size of the deviations of the contributions from the normal cost and it is related with the stability of the plan. The second risk is measured with the size of the unfunding actuarial liability and it can serve as an indicator of the stability of the plan.

The promoter wishes to control the stability and security of the pension plan, by minimizing some convex combination of both types of risk. In this way, the weight in the convex combination measures the relative importance of the risk for the promoter. The attained solution is a Pareto optimum of this multiobjective problem. This means that there is no possible reduction of one of the risks without augmenting the other one.

Another important feature of the model is the presence of an unbounded horizon. We suppose that there is not a finish time for the plan but it extends forever and that the preferences of the sponsor are more concerned with the short run that with the long run; this is modeled by introducing a positive discount factor in the objective functional.

The assets of the fund can be invested in $n+1$ securities, n of them following a geometrical Brownian motion with independent Wiener processes. The risk–free security is constant in time. It then follows that the assets of the fund obeys a stochastic differential equation closely related with those proposed in Merton (1971) for portfolio and consumption selection.

Haberman and Sung (1994) considers the minimization of a linear combination of the aforementioned risks on a finite horizon, without discounting, both in deterministic and stochastic frameworks. The authors do not contemplate the investment as an instrumental variable but all the assets of the fund are inverted at a random rate of return. As a consequence the optimal control problem studied in Haberman and Sung (1994) turns out to be one of the linear–quadratic type, that can be explicitly solved. Although in the model we propose the objective functional is quadratic, the dynamics is nonlinear, so the control problem is not linear–quadratic. However, we are able to find a closed form solution for the problem of the optimal management of pension funding.

O'Brien (1987) analyze a stochastic optimal control problem which shows two sources of uncertainty, the investment returns and the benefit outcome. The author makes a linear approximation of the exponential fund model, see Bowers, Hickman and Nesbitt (1976, 1979), to retain analytical tractability of the problem. However, no investment decisions are available for the manager, who wishes to maintain a constant fund ratio (with respect to the actuarial liability), and penalizes fluctuations of the contribution rate from zero.

Our objective is to extend this prior work, allowing investment decisions in the model. The paper is structured as follows. In Section 2 we set up the model as a controlled diffusion problem. In section 3 we outline the dynamic programming approach and we find the optimal controls in feedback form, obtaining, roughly speaking, that the portfolio choice and contribution rate are (piecewise) proportional to the unfunded actuarial liability. We also study some properties of the solution. The last section is devoted to stablish some conclusions.

2 Mathematical model

All the variables listed is this section are related to all the participants in the aggregate pension fund we will consider. We assume that the actuarial valuation to estimate the main components of the plan are done at each instant of time.

We denote $F(t)$ as the value of the assets forming the fund at time t; $C(t)$ is the contribution rate made by the sponsor in order to accrue the amount of the defined benefit at the moment of retirement; the defined benefits are denoted by P; the normal cost for all participants, by *NC*; the actuarial liability, by *AL*; the unfunded actuarial liability, by *UAL* (this is simply the difference between *AL* and $F(t)$) and the supplementary contribution rate amortizing *UAL* at time t, by SC (this is simply the difference between $C(t)$ and NC).

Let us observe that we have considered constant values of P, NC and AL. This is justified, as in Haberman and Sung (1994), if the population in the pension plan is stationary from the start and there is no salary increase (or there is a fixed rate of salary inflation; in this case the rate of return is net the salary inflation).

If we assume that the valuation of the plan is done with a constant rate δ, then the main components of the plan are linked by the equation

$$\delta AL + NC - P = 0, \tag{1}$$

as is shown in Bowers, Hickman and Nesbitt (1976).

The sponsoring employer manages the funding process by making a portfolio choice of n risky assets $S^1(t), \dots, S^n(t)$ and a bond $S^0(t)$, $0 \le t < \infty$, with dynamics given by the equations

$$dS^0(t) = rS^0(t)\,dt, \quad S^0(0) = 1 \tag{2}$$

$$dS^i(t) = S^i(t)\Big(b_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t)\Big), \quad S^i(0) = s_i, \quad 1 \le i \le n. \tag{3}$$

Here b_i and σ_{ij}, $1 \le i, j \le n$ are positive constants. The vector $\mathbf{W}(t) = (W_1(t), \dots W_n(t))^T$ is an n–dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\{\mathcal{F}_t\}$ denotes the completion of the filtration $\sigma\{\mathbf{W}(s)\,|\,0 \le s \le t\}$.

We suppose that the interest rate r is strictly smaller than the mean rates of return b_i, $1 \le i \le n$. Next we introduce the matrix $\sigma = (\sigma_{ij})$ and the vectors $\mathbf{b} = (b_1, \dots, b_n)^T$, $\mathbf{\Lambda}(t) = (\lambda_1(t), \dots, \lambda_n(t))^T$.

A portfolio process or trading strategy $\mathbf{\Lambda}(t)$ is a $\mathbb{R}^n$–measurable process adapted to $\{\mathcal{F}_t\}$ such that

$$\int_0^\infty \|\mathbf{\Lambda}(s)\|^2 \, dx < \infty \quad \text{a.s.}$$

Here $\lambda_i(t) \geq 0$ denotes the proportion of the fund assets inverted by the promoter in asset i, $0 \leq i \leq n$. As explained in Section 1, the non negativity constraint on λ_i avoids shortselling.

The contribution rate process $C(t)$ is a measurable adapted process with respect to $\{\mathcal{F}_t\}$ verifying

$$\int_0^\infty |C(s)|^2 \, dx < \infty \quad \text{a.s.}$$

and

$$E_{F_0} \int_0^\infty \exp\left(-\rho t\right)(\beta SC^2(t) + (1-\beta)\, UAL^2(t)) \, dt < \infty.$$

Here, E_{F_0} denotes conditional expectation by F_0. The class of admissible controls is denoted by $\mathcal{A}_{F_0}$.

The quantity $\lambda_i(t)F(t)S^i(t)$ represents the amount invested in asset i, $0 \leq i \leq n$, and $\int_0^t C(s)\, ds$ is the total contribution made up to time t. Along the lines of Merton (1971), we suppose that changes in the fund level derive only from changes in the asset prices, interest of the bond and from the contribution. In consequence:

$$dF(t) = F(t)\Big(\sum_{i=1}^n \lambda_i(t)\frac{dS^i(t)}{S^i(t)}\Big) + F(t)\Big(1 - \sum_{i=1}^n \lambda_i(t)\Big)\frac{dS^0(t)}{S^0(t)} + (C(t) - P)\, dt. \qquad (4)$$

Taking into account (2), (3) and (4) we obtain that the fund amount satisfies the following stochastic differential equation:

$$dF(t) = \Big(rF(t) + \sum_{i=1}^n \lambda_i(t)(b_i - r)F(t) + C(t) - P\Big)dt + \sum_{i=1}^n\sum_{j=1}^n \lambda_i(t)\sigma_{ij}F(t)\, dW_j(t), \qquad (5)$$

with initial condition $F(0) = F_0 > 0$.

The symmetric matrix $\Sigma = \sigma\sigma^T$ is positive definite. We denote $\theta = \sigma^{-1}(\mathbf{b} - r\mathbf{1})$ the market price of risk, where $\mathbf{1}$ denotes a (column) vector of 1's. Obviously, $\theta^T\theta$ is a positive scalar.

Now, we turn with the preferences of the controller. We assume that he or she wish to minimize a convex combination of the contribution rate risk and the solvency risk. In consequence, the objective functional to be minimized over $\mathcal{A}_{F_0}$ is

$$J(F, C, \mathbf{\Lambda}) = E_{F_0} \int_0^\infty \exp(-\rho t) \left(\beta SC^2(t) + (1-\beta)\, UAL^2(t)\right) dt.$$

The parameter β verifies $0 < \beta \leq 1$ and is a weighting factor reflecting the relative importance of one type of risk with respect the other one. The restriction in the range of possible values of β means that the sponsor choose a compromise solution or Pareto optimal solution in the multiobjective problem arising in this decision model. There is also a positive actualization rate, ρ. A high actualization rate implies that the promoter is more concerned with the present than with the distant future. Throughout the paper we make the assumption that δ equals r.

3 Optimal feedback pension funding control

In this section we stablish some properties of the value function of the control problem introduced in Section 2 and we prove that it is a generalized solution to the Hamilton–Jacobi–Bellman equation (HJB henceforth). The value function is defined as

$$\widehat{V}(F) = \inf_{(C,\mathbf{\Lambda}) \in \mathcal{A}_{F_0}} \left\{ J(F, C, \mathbf{\Lambda}) \,|\, \text{s.t. } (5) \right\}.$$

The function $\widehat{V}(F)$ is the minimum value of the deviations from the objectives when the initial wealth in the fund is F. Given that the problem is autonomous and the horizon is unbounded, we may suppose that $\widehat{V}$ is time independent. It is clear that the value function so defined is nonnegative, strictly convex and that $\widehat{V}(AL) = 0$.

As is well known since Bellman (1957), the knowledge of the value function implies the knowledge of the optimal controls (at least in an implicit way). The connection between value functions in optimal control theory (deterministic or stochastic) and optimal feedback controls is accomplished by the HJB equation, see Fleming and Soner (1993). For our problem of optimal pension funding, the HJB equation becomes:

$$\begin{aligned}\rho V(F) &= \min_{C,\mathbf{\Lambda}\geq 0}\Big\{\beta(C-NC)^2+(1-\beta)(AL-F)^2+(rF+\mathbf{\Lambda}^T(\mathbf{b}-r\mathbf{1})F+C-P)V'(F)\\ &\quad +\frac{1}{2}\mathbf{\Lambda}^T\Sigma\mathbf{\Lambda}F^2\,V''(F)\Big\} \qquad (6)\end{aligned}$$

Whenever a solution V of the above equation is smooth enough, the minimizing arguments are given by

$$\widetilde{C}(V'(F)) = NC - \frac{1}{2\beta}V'(F), \qquad (7)$$

$$\widetilde{\mathbf{\Lambda}}(V'(F),V''(F)) = \left(-\frac{V'(F)}{FV''(F)}\Sigma^{-1}(\mathbf{b}-r\mathbf{1})\right)_+. \qquad (8)$$

Here, given $\mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{R}^n$, $\mathbf{a}_+$ denotes the vector which i–coordinate is $\max\{a_i,0\}$. The equalities (7) and (8) are the link between the HJB equation and the optimal controls of the problem. Observe that the scalar magnitude in (8) is the inverse of the Arrow–Pratt measure of risk aversion of the value function.

In order to find V satisfying (6) we make the following observations: (i) deviations of the wealth of the fund from the actuarial liability are penalized, so the investment in the risky assets must be zero if $F > AL$, because their mean of return is higher than that of the bond, and (ii) the value function $\widehat{V}$ is a solution of (6) if it is smooth enough. Since the running cost is quadratic we will postulate a piecewise quadratic value function of class $\mathcal{C}^1$ (although the dynamics is non linear)[1]. The necessity for consider a piecewise smooth function comes from the impossibility of shortselling. Based on the above considerations, we make the guessing

$$\widetilde{\mathbf{\Lambda}}(F) = \begin{cases} -\dfrac{V'(F)}{FV''(F)}\Sigma^{-1}(\mathbf{b} - r\mathbf{1}), & \text{if } F < AL, \\ \\ \mathbf{0}, & \text{if } F > AL, \end{cases} \tag{9}$$

where $\mathbf{0}$ is a (column) vector of 0's, and

$$V(F) = \begin{cases} a(AL - F)^2, & \text{if } F < AL, \\ \\ \alpha(AL - F)^2, & \text{if } F > AL, \end{cases} \tag{10}$$

as a smooth solution of (6) on the regions $F < AL$ and $F > AL$ with the exception of the point $F = AL$. By substituting (10) in (6) we find that the positive constants a, α, must satisfy the two equations

$$a^2 + \beta\left(\rho - 2r + \theta^T\theta\right)a - \beta(1-\beta) = 0, \tag{11}$$

$$\alpha^2 + \beta(\rho - 2r)\alpha - \beta(1-\beta) = 0. \tag{12}$$

It is easy to see that (11) and (12) admits each one a unique positive solution verifying $a < \alpha$. It turns out that V satisfies the conditions $V'(AL^-) = V'(AL^+) = 0$ and $V''(AL^+) = 2\alpha >$

[1] A function is of class $\mathcal{C}^k(\mathbb{R})$ if its first k–derivatives exists and are continuous functions on $\mathbb{R}$.

$2a = V''(AL^-)$.

We now proceed to confirm the validity of our conjecture, that is, V as defined in (10) coincides with the value function $\widehat{V}$. To this end we will use a Verification Theorem based on a generalized Itô's rule.

Theorem 1 *The optimum contribution rate in feedback form is given by*

$$C^*(F) = \begin{cases} NC + \dfrac{a}{\beta}(AL - F), & \text{if } F < AL, \\ \\ NC + \dfrac{\alpha}{\beta}(AL - F), & \text{if } F > AL. \end{cases} \tag{13}$$

The optimal investment policy is

$$\mathbf{\Lambda}^*(F) = \begin{cases} \dfrac{AL - F}{F}\Sigma^{-1}(\mathbf{b} - r\mathbf{1}), & \text{if } 0 < F < AL, \\ \\ \mathbf{0}, & \text{if } F > AL. \end{cases} \tag{14}$$

Here, a is the unique positive solution of (11) verifying

$$a > \beta\left(r - \theta^T\theta\right), \tag{15}$$

and α is the unique positive solution of (12) such that

$$\alpha > \beta r. \tag{16}$$

Proof. Denote F^* the wealth of the fund associated with C^* and $\mathbf{\Lambda}^*$, when the initial condition is $F^*(0) = F$. The pair $(C^*, \mathbf{\Lambda}^*)$ belongs to the admissible class of controls $\mathcal{A}_F$ because of Proposition 1 below, where it is proved that the expected value of F^* converges to AL.

Now we consider an arbitrary pair $(C, \mathbf{\Lambda}) \in \mathcal{A}_F$. Applying the generalized Itô's rule (Karatzas and Shreve (1991), p. 219) to $e^{-\rho t}V(t)$ we obtain, after taking expectations

$$\begin{aligned} e^{-\rho t}E_F V(F(t)) &= V(F) + E_F \int_0^t \Big[\frac{1}{2}\mathbf{\Lambda}^T(s)\Sigma\mathbf{\Lambda}(s)F(s)^2 V''(F(s)) \\ &\quad +(rF(s) + (\mathbf{b} - r\mathbf{1})^T\mathbf{\Lambda}(s)F(s) + C(s) - P)V'(F(s)) - \rho V(F(s))\Big]\, ds. \end{aligned}$$

We have made use of the fact that V' is continuous, so there is no *local time* term in the stochastic integral. On the other hand, (6) implies

$$\rho V(F) \leq \beta(C-NC)^2+(1-\beta)(AL-F)^2+(rF+(\mathbf{b}-r\mathbf{1})^T\mathbf{\Lambda}F+C-P)V'(F)+\frac{1}{2}\mathbf{\Lambda}^T\Sigma\mathbf{\Lambda}F^2\, V''(F) \tag{17}$$

for all $F \neq AL$, with equality whenever we replace C by C^* and $\mathbf{\Lambda}$ by $\mathbf{\Lambda}^*$. Integrating and taking expectations in (17) and making use of the existence of side limits of V'' in AL, we have

$$e^{-\rho t}E_F V(F(t)) + E_F \int_0^t e^{-\rho s}[\beta(C(s) - NC)^2 + (1 - \beta)(AL - F(s))^2]ds \geq V(F), \tag{18}$$

with equality when $C = C^*$ and $\mathbf{\Lambda} = \mathbf{\Lambda}^*$.

The transversality condition

$$\lim_{t\to\infty} e^{-\rho t}E_F V(F(t)) = 0$$

holds, because of Proposition 1 below. Hence inequality (18) implies $J(F; C, \mathbf{\Lambda}) \geq V(F)$ and $J(F; C^*, \mathbf{\Lambda}^*) = V(F)$. It then follows that $(C^*, \mathbf{\Lambda}^*)$ is optimum on $\mathcal{A}_F$. □

It is interesting to note that, as mentioned above, the value function is not of class $\mathcal{C}^2(\mathbb{R})$. This means that the *smooth pasting* conditions (Krylov (1980, p. 32)), cannot be fulfilled by the value function of our problem, that is

$$V''(AL^-) = a \neq \alpha = V''(AL^+).$$

Note that (6) is not uniformly elliptic because the second order term $\frac{1}{2}\mathbf{\Lambda}^T \Sigma \mathbf{\Lambda} F^2 V''(F)$ becomes zero when $\mathbf{\Lambda} = 0$, so we can not be sure of the existence of a smooth solution of the equation, see Krylov (1980).[2] In fact, the optimal $\mathbf{\Lambda}^*$ goes to zero as F goes to AL. At this point it is worth comparing with the model of optimal consumption and portfolio choice with no shortselling, where the value function is proved to be of class $\mathcal{C}^2$, see Vila and Zariphopoulou (1997).

The expressions for the optimal rate of contribution and the optimal vector of investments given in the above theorem can be rephrased as

$$SC = \begin{cases} \dfrac{a}{\beta} UAL, & \text{if } UAL > 0, \\ \\ \dfrac{\alpha}{\beta} UAL, & \text{if } UAL < 0, \end{cases} \qquad \mathbf{\Lambda}^*(F)F = \begin{cases} UAL\, \Sigma^{-1}(\mathbf{b} - r\mathbf{1}), & \text{if } UAL > 0, \\ \\ \mathbf{0}, & \text{if } UAL < 0. \end{cases}$$

The total investment vector $\mathbf{\Lambda}^* F$ is a constant proportion policy on the region $UAL > 0$, because regardless the gap between the wealth of the fund and the goal, the proportion of wealth invested in the risky stocks is fixed. Thus the manager of the plan gets increasingly more cautious as the wealth of the fund takes closer values to the actuarial liability. This implies investing less at each time, eventually reaching zero investment in the limit. Similar comments apply to the optimal contribution rate on each of the regions $UAL > 0$ and $UAL < 0$. Let us observe that the supplementary cost depends on β but the investment strategy does not.

[2]However, it is an easy exercise to prove that the function appearing in (10) is a viscosity solution of (6). See Fleming and Soner (1993) for the appropriated definitions.

The optimal funding process can be summarized in the two following rules: (i) keep the supplementary cost proportional to the unfunded actuarial liability, with different constants of proportionality $\frac{a}{\beta}$ or $\frac{\alpha}{\beta}$, as soon as the wealth of the fund is above or below the actuarial liability, respectively and (ii) make an investment in the risky assets proportional to the unfunded actuarial liability whenever the wealth of the fund is below the objective. Do not invest anything in other case.

Although the investment behavior seems to be paradoxical, we must remember that the prior objective of the employer sponsor is to reduce the inherent risks of the process funding, and not to maximize the wealth of the fund. It is very interesting to note that the optimal supplementary cost for this problem corresponds to a *spread method* of contribution, see Bowers, Hickman and Nesbitt (1976). This funding method is very used in the literature and has been proved to have good properties for the stabilization of the pension plan. Here we find another justification for this method, because it arise naturally as a consequence of an optimal or extremal aiming of the controller. Let us observe, however that the constant of proportionality is different in the regions $F < AL$ and $F > AL$, being higher, $\alpha > a$, in the last. This fact has an easy explanation; the no shortselling condition on the amount invested in the risky assets impose a higher rate of reduction on the contribution rate on the region $F > AL$ than that would be in the control problem without constraints.

Let us note that borrowing at rate r is optimal to be invested in asset i ($\lambda_i > 1$), whenever the fund level is below the critical value

$$\frac{\mathbf{e_i}\Sigma^{-1}(\mathbf{b} - r\mathbf{1})}{\mathbf{e_i}(\mathbf{1} + \Sigma^{-1}(\mathbf{b} - r\mathbf{1}))}AL,$$

where $\mathbf{e_i} = (0, \ldots, \overset{i)}{1}, 0, \ldots, 0)$.

The main concern of the promoter is to keep the contribution rate and the level of the fund as close as possible to the ideal values. As the following proposition shows, (15)–(16) implies the stabilization of the expected wealth of the fund on the desired target. In fact, an additional constraint on the weighing factor β assure convergence a.s. of the fund to the actuarial liability.

Proposition 1 *If (15)–(16) hold, then*

1. *The fund, the rate of contribution and the total investment converge a.s. to the actuarial liability, the normal cost and zero, respectively; that is to say,*

$$\lim_{t \longrightarrow \infty} E_{F_0} F^*(t) = AL, \quad \lim_{t \longrightarrow \infty} E_{F_0} C^*(t) = NC, \quad \textit{and} \quad \lim_{t \longrightarrow \infty} E_{F_0} \mathbf{\Lambda}^*(t) F^*(t) = 0.$$

2. *If the parameters of the problem verify*

$$a > \beta \left(r - \frac{1}{2} \theta^T \theta \right), \tag{19}$$

then the variance of the fund, the rate of contribution and the total investment amount converge to zero, that is to say,

$$\lim_{t \longrightarrow \infty} Var_{F_0} F^*(t) = 0, \quad \lim_{t \longrightarrow \infty} Var_{F_0} C^*(t) = 0 \quad \textit{and} \quad \lim_{t \longrightarrow \infty} Var_{F_0} \mathbf{\Lambda}^*(t) F^*(t) = 0$$

Proof. By substituting the optimal values of the control variables, the evolution of the fund is given by the stochastic differential equation

$$dF^*(t) = (rF^*(t) + \theta^T \theta \, UAL^*(t) + \frac{a}{\beta} UAL^*(t) + NC - P) \, dt + UAL^*(t) \theta^T \, d\mathbf{W}(t),$$

with $F^*(0) = F_0$, whenever $F^* < AL$ and by the ordinary differential equation

$$dF^*(t) = \left(\frac{\alpha}{\beta} - r\right)(F^*(t) - AL)dt$$

with $F^*(0) = F_0$, if $F^* \geq AL$.

The first expression can be rewritten in terms of the unfunded actuarial liability as follows:

$$d\, UAL^*(t) = \left(r - \theta^T\theta - \frac{a}{\beta}\right) UAL^*(t)dt + UAL^*(t)\theta^T\, d\mathbf{W}(t),$$

with $UAL^*(0) = AL - F_0$. The solution of this equation, a geometrical Brownian motion, is given by

$$UAL^*(t) = (AL - F_0)\exp\left\{(r - \frac{3}{2}\theta^T\theta - \frac{a}{\beta})t\right) + \theta^T W(t)\right\}.$$

Letting t to infinity and taking into account (15) and the properties of Brownian motion, we have that UAL^* converges to zero a.s. and its conditional expected value converges to AL. On the other hand (19) imply

$$\lim_{t\longrightarrow\infty} E_{F_0}(UAL^*(t))^2 = 0,$$

see Arnold (1974, p. 140). Hence

$$\mathrm{Var}_{F_0}\, UAL^*(t) = E_{F_0}(UAL^*(t))^2 - E^2_{F_0}(UAL^*(t))$$

tends to zero as t goes to infinity.

In the region $F^* \geq AL$ the evolution of UAL^* is given by

$$d\,UAL^*(t) = (r - \frac{\alpha}{\beta})\,UAL^*(t))dt, \qquad UAL^*(0) = AL - F_0.$$

In this case it is clear that UAL^* converges to zero when (16) holds. Now, the statements of the proposition are immediate consequences of the behavior of UAL^*. $\square$

Consider the natural scale function φ of the Brownian motion UAL^*. It is defined as

$$\varphi(x) = \int^x \exp\left(- \int^y \frac{2\mu(z)}{a(z)}\, dz \right)\, dy,$$

where

$$\mu(x) = (r - \theta^T\theta - \frac{a}{\beta})x, \quad \text{and} \quad a(x) = \theta^T\theta x^2, \quad x \in \mathbb{R}$$

are the drift and difussion coefficients of the process UAL^*, respectively. We obtain

$$\varphi(x) = -(2\gamma+1)^{-1}|x|^{2\gamma+1}, \quad \text{where} \quad \gamma = 1 - \frac{\left(r - \frac{a}{\beta}\right)}{\theta^T\theta}$$

is a positive constant if (15) holds. The velocity density of the process is defined as

$$m(x) = \left(a(x)\frac{d\varphi}{dx}(x) \right)^{-1} = \theta^T\theta|x|^{-2(\gamma+1)}, \quad x \in \mathbb{R}.$$

Now, for any $z \in \mathbb{R}$ let us define τ_z^* as the first instant of time such that UAL^* hits z. The conditioned probability P_x of the event $\tau_z^* > \tau_0^*$ when $UAL(0) = x$ is

$$P_x(\tau_z^* > \tau_0^*) = \frac{\varphi(x) - \varphi(z)}{\varphi(0) - \varphi(z)} = \frac{|z|^{2\gamma} - |x|^{2\gamma}}{|z|^{2\gamma}}, \tag{20}$$

see Karlin and Taylor (1981). In the next proposition we prove that the funding process never achieve the point AL with positive probability in finite time although this value is an attracting barrier.

Proposition 2 *If (15)–(16) hold,* AL *is an unattainable and attracting barrier for* F^*.

Proof. The statement is equivalent to prove that 0 is an unattainable and attracting barrier for UAL^*. First we claim it is unattainable: if $x < 0$

$$\int_x^0 (\varphi(0) - \varphi(y))m(y)\, dy = \frac{\theta^T\theta}{2\gamma+1}\int_x^0 \frac{1}{y}\, dy = \infty.$$

Hence we can apply a result in Durret (1996, p. 241). If $x > 0$ the result follows from uniqueness of solutions to ordinary differential equations. Then the claim follows. Second, we claim that 0 is attracting for UAL^*, because $P_x(\tau_z^* > \tau_0^*)$ as defined in (20) converges to 1 as x go to 0. That is to say, the conditioned probability that UAL^* takes the value 0 before any other value $z \neq 0$ tends to 1 as the gap between AL and F^* shrinks. $\square$

4 Conclusion

This paper has analyzed the pension funding problem from the point of view of a manager that tries to minimize the solvency risk and the contribution rate risk on an unbounded horizon. To this end she or he makes a portfolio choice with n risky assets and a bond and also controls the size of the contribution rate. A *spread method* of funding arises, as a result of the postulated optimal behavior. We are able to give a closed form solution to the problem and we also prove strong stability properties of the solution. In further research we intend to eliminate the constancy assumption on the actuarial (deterministic) functions used in the model. Suitable hypothesis from a tractable point of view would include exponential growth for the population of the plan and the salary. The autonomous character of the optimal control problem could be preserved in this case with the introduction of additional state variables (the normal cost, the actuarial liability and the benefits). We shall also consider modeling uncertainty on some elements of the fund.

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