Noname manuscript No.
(will be inserted by the editor)

Recursive utility with unbounded aggregators*

Juan Pablo Rincón–Zapatero[1], **Carlos Rodríguez–Palmero**[2]

[1] Departamento de Economía, Universidad Carlos III de Madrid, 28903 Getafe, Madrid, SPAIN, e-mail: jrincon@eco.uc3m.es
[2] Departamento de Economía Aplicada, Universidad de Valladolid, 47011 Valladolid, SPAIN, e-mail: cpalmero@eco.uva.es

The date of receipt and acceptance will be inserted by the editor

* Juan Pablo Rincón-Zapatero and Carlos Rodríguez-Palmero gratefully acknowledge financial support by the *Ministerio de Educación y Ciencia* and FEDER funds under Research Projects MTM2005-06534 and SEJ2005-08709/ECON, respectively, and by *Consejería de Educación* de la *Junta de Castilla y León* under Research Projects VA99/04 and VA017B05, respectively. This paper has substantially benefited from the comments of an anonymous referee. Particular and special thanks are due to Robert Becker for his very helpful comments and valuable suggestions.
Correspondence to: J.P. Rincón–Zapatero

Abstract A new framework is presented for the study of the existence and uniqueness of solutions to the Koopmans' equation in the unbounded case, that is based on the contraction mapping approach. In the bounded below case with bounded consumption streams, uniqueness of the solution in the whole class of weak–star continuous utility functions is obtained. When the aggregator is unbounded below and/or consumption streams are unbounded, existence of a weak–star continuous solution is shown, and a simple criterium to check the sufficient conditions for existence is provided.

Key words Koopmans' equation – recursive utility – contraction mapping.

JEL Classification Numbers: C61; D90.

1 Introduction

Traditionally, the functional relation between current utility and future utility has been considered additively separable over time. It is well–known that this assumption is restrictive since in risk environment models the consideration of expected additively separable utility makes the isolation of both effects, risk aversion and marginal substitution rate, impossible. Recursive utility remedies this drawback assuming that agents' current utility of a consumption stream is expressed as a function (the aggregator) of current consumption and the utility of the future consumption stream. In this fashion, recursive utility is adequate in models where the marginal substitution rates are different across agents, and depend on consumption.

An extensive economic literature starting with Koopmans [4], who stated a set of axioms guaranteeing the representation of the utility function by means of an aggregator, has generalized the additively separable models to the case of recursive utility models in which the marginal substitution rate is allowed to vary with the agents' consumption streams.

The approach initiated by Lucas and Stokey [6] is different, as these authors suppose that the aggregator is given, and then they impose conditions assuring the existence of an utility function associated to the aggregator. The method is based on contraction mapping techniques and is limited to bounded aggregators. Along this line, Boyd [2] and Becker and Boyd [1] considers unbounded aggregators by means of the introduction of a weighted norm on a certain space of continuous functions obeying an adequate growth condition. Such a condition on the aggregator, together with some assumptions linking this growth condition and the discounting rate, assure the existence of a unique recursive utility in the prescribed class of functions, via contraction mapping arguments as well. The approach we follow in this paper is related with contraction mapping techniques, but it differs in that we consider the whole space of continuous functions. Since this space is not normable, we need to introduce a suitable metric. This apparently slight generalization allows us to improve the results obtained in [1, 2] in the following directions:

(i) To obtain existence and uniqueness of the recursive utility function with respect to the whole class of continuous utility functions when the aggregator is bounded below and the consumptions streams are bounded sequences;

(ii) To strength the continuity properties of the utility function in regions where it is not identically $-\infty$, from weighted–norm continuity to weak–star continuity. Weak–star continuity of the recursive utility function is a property with major implications in optimal growth theory, in cases where agents have recursive preferences. This result immediately implies that the recursive utility function achieves its maximum value on any weak–star compact subset of the given commodity space;

(iii) To provide a simple way to check the fulfillment of the sufficient conditions assuring the existence of the utility function. The condition is based in the application of the root test to the appropriate power series.

A different framework has been developed in Streufert [8,9] by introducing the notions of lower and upper convergence, leading to the concept of biconvergence. In the more recent paper of Le Van and Vailakis [5] the convergence properties of the sequence of utility functions obtained from successive iteration of the Koopmans' operator is analyzed.

2 Definitions and preliminary results

In this section we establish some definitions and results that will be used in the study of the Koopmans' equation. We follow the approach introduced by the authors in Rincón–Zapatero and Rodríguez–Palmero [7] in the context of dynamic programming.

2.1 Local Contractions

Let $\mathcal{U}$ be an arbitrary set in which a countable family of semidistances $\{d_j\}_{j=1}^{\infty}$ is defined so that $d_j(U,V)=0$ for all $j\in\mathbb{N}$ implies $U=V$ for all $U,V\in\mathcal{U}$. It is then straightforward to check that

$$d(U,V)=\sum_{j=1}^{\infty}2^{-j}\frac{d_j(U,V)}{1+d_j(U,V)} \tag{1}$$

defines a metric on $\mathcal{U}$. We are also interested in defining other metric on some subsets of $\mathcal{U}$, as follows: given a fixed element $U_0\in\mathcal{U}$, and strictly positive constants c and M, let the metric be

$$d_c(U,U_0)=\sum_{j=1}^{\infty}c^j\ d_j(U,U_0), \tag{2}$$

and the set be

$$\mathcal{U}_{c,U_0}=\{U\in\mathcal{U}\ :\ d_c(U,U_0)\le M\}. \tag{3}$$

We shall now define two local contraction concepts for operators defined on $\mathcal{U}$.

Definition 1 *Let $\mathscr{T}:\mathcal{U}\longrightarrow\mathcal{U}$ be an operator.*

(i) $\mathscr{T}$ is a 0–local contraction if and only if $d_j\big(\mathscr{T}U,\mathscr{T}V\big)\le\beta_j\ d_j(U,V)$ for all $j\in\mathbb{N}$ and for all $U,V\in\mathcal{U}$, where $0\le\beta_j<1$.
(ii) $\mathscr{T}$ is a 1–local contraction if and only if $d_j\big(\mathscr{T}U,\mathscr{T}V\big)\le\beta\ d_{j+1}(U,V)$ for all $j\in\mathbb{N}$ and for all $U,V\in\mathcal{U}$, where $\beta\ge 0$.

The two following theorems are the main tools for existence and uniqueness of the recursive utility in a fairly general context.

Theorem 1 *Suppose that* $(\mathcal{U}, d)$ *is a complete metric space. Let* $\mathscr{T} : \mathcal{U} \longrightarrow \mathcal{U}$ *be a 0–local contraction. Then, for every* $U_0 \in \mathcal{U}$*, the operator* $\mathscr{T}$ *maps the closed and bounded subset*[1]

$$\mathcal{V} = \left\{ U \in \mathcal{U} \, : \, d_j(U, U_0) \leq \frac{d_j(\mathscr{T}U_0, U_0)}{1 - \beta_j} \; \forall j \in \mathbb{N} \right\}$$

into itself. Furthermore,

(a) $\mathscr{T}$ *is a contraction on* $\mathcal{V}$ *and admits a fixed point* U^* *on* $\mathcal{V}$*, that is unique on* $\mathcal{U}$*.*
(b) *For any* $U \in \mathcal{U}$*,* $\mathscr{T}^n U$ *converges to* U^* *in the metric* d *as* $n \to \infty$*.*

Theorem 2 *Let* $\mathscr{T} : \mathcal{U} \longrightarrow \mathcal{U}$ *be a 1–local contraction. Then, for every* $U_0 \in \mathcal{U}$ *satisfying* $d_c(\mathscr{T}U_0, U_0) < \infty$ *for some* $c > \beta$*, the operator* $\mathscr{T}$ *maps the closed and bounded subset*

$$\mathcal{U}_{c,U_0} = \left\{ U \in \mathcal{U} \, : \, d_c(U, U_0) \leq \frac{d_c(\mathscr{T}U_0, U_0)}{1 - \frac{\beta}{c}} \right\}$$

into itself. Furthermore, if $(\mathcal{U}_{c,U_0}, d_c)$ *is complete, then*

(a) $\mathscr{T}$ *is a contraction on* $\mathcal{U}_{c,U_0}$ *and admits a unique fixed point* U^* *on* $\mathcal{U}_{c,U_0}$*.*
(b) *For any* $U \in \mathcal{U}_{c,U_0}$*,* $\mathscr{T}^n U$ *converges to* U^* *in the metric* d_c *as* $n \to \infty$*.*

2.2 Recursive Utility

Consumption streams are elements of $\mathbb{R}_+^{\mathbb{N}}$, that is, sequences $\mathbf{c} = (c_j)_{j=1}^{\infty}$ of non-negative real numbers. On this space we consider two linear operators: the projection, $\pi\mathbf{c} = c_1$, and the shift operator, $\sigma\mathbf{c} = (c_2, c_3, \ldots)$. In some economic models it is usual to consider a context in which agents' preferences on a given consumption path $\mathbf{c}$ depend both on current consumption, $\pi\mathbf{c}$, and on the utility of future consumption, $\sigma\mathbf{c}$. This idea of aggregation can be represented by means of a function $W : X \times Y \longrightarrow Y \cup \{-\infty\}$, called aggregator, where $X \subseteq \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $Y \subseteq \mathbb{R}$.

The following hypotheses on the aggregator W are standard in the literature (see [1,6]), and will be also used throughout this paper.

(W1) W is continuous on $X \times Y$.

(W2) W obeys a Lipschitz condition with respect to y

$$|W(x, y) - W(x, y')| \leq \; \beta(x) \; |y - y'|, \quad \forall x \in X, \; \forall y, y' \in Y,$$

where $\beta : X \longrightarrow [0, \infty)$ is a continuous function.

[1] The set $\mathcal{V}$ is closed and bounded, with respect to the topology generated by the metric d.

In the seminal paper by Lucas and Stokey [6], it is supposed that the aggregator is bounded. We eliminate this restriction from the assumptions.

Consider a subset $\mathbb{X} \subseteq \mathbb{R}_+^{\mathbb{N}}$ satisfying

$$\sigma\mathbb{X} \subseteq \mathbb{X} \quad \text{and} \quad \pi\left(\bigcup_{j=0}^{\infty} \sigma^j \mathbb{X}\right) \subseteq X, \tag{4}$$

where $\sigma^j \mathbf{c} = (c_{j+1}, c_{j+2}, \ldots)$ for all $j \in \mathbb{N}$. Then, the aggregator W defines a recursive operator $\mathscr{K}$ acting over the space of utility functions defined on $\mathbb{X}$ via $\mathscr{K}U(\mathbf{c}) = W(\pi\mathbf{c}, U(\sigma\mathbf{c}))$. A real function U defined on $\mathbb{X} \subseteq \mathbb{R}_+^{\mathbb{N}}$ is called recursive if it verifies the Koopmans' equation

$$U(\mathbf{c}) = \mathscr{K}U(\mathbf{c}) = W(\pi\mathbf{c}, U(\sigma\mathbf{c})),$$

that is to say, if U is a fixed point for the Koopmans's operator $\mathscr{K}$.

Given $\mathbb{X} \subseteq \mathbb{R}_+^{\mathbb{N}}$ with a topology τ, let $\mathcal{C}(\mathbb{X})$ be the space of τ–continuous real functions and let $\mathcal{C}_{c,U_0}(\mathbb{X})$ be the subset of $\mathcal{C}(\mathbb{X})$ satisfying (3) for some $U_0 \in \mathcal{C}(\mathbb{X})$ and $c > 0$. In order to find a countable family of semimetrics $\{d_j\}_{j=1}^{\infty}$ on both $\mathcal{U} = \mathcal{C}(\mathbb{X})$ and $\mathcal{U}_{c,U_0} = \mathcal{C}_{c,U_0}(\mathbb{X})$ in such a way that the metrics d and d_c (defined in (1) and (2) respectively) are complete, the set $\mathbb{X}$ must satisfy in addition to (4) some other properties, and the topology τ must be carefully selected. We suppose in the following that $\mathbb{X} \subseteq \mathbb{R}_+^{\mathbb{N}}$ is a subset of the norm dual of a Banach space and that τ is the weak–star topology[2]. The justification for this selection will be shown in Remark 1 below.

For general unbounded aggregators the existence of a recursive utility function on the whole $\mathbb{R}_+^{\mathbb{N}}$ cannot be expected, thus we consider commodity spaces $\mathbb{X}$ where consumption streams are bounded above and below in their growth rate, as [1,2].

Given a consumption stream $\mathbf{w} \geq 1$, consider the Riesz ideal

$$A(\mathbf{w}) = \{\mathbf{c} \in \mathbb{R}^{\mathbb{N}} : |c_t| \leq \lambda w_t \text{ for some } \lambda > 0, \text{ for all } t \geq 1\}.$$

Consider the norm $\|\mathbf{c}\|_{\mathbf{w}} = \sup_{j\in\mathbb{N}} |c_j/w_j|$. Notice that $(A(\mathbf{w}), \|\cdot\|_{\mathbf{w}})$ is the norm dual of the separable Banach space of sequences $\mathbf{q} \in \mathbb{R}_+^{\mathbb{N}}$ satisfying $\|\mathbf{q}\|_{\mathbf{w}}^1 = \sum_{j=1}^{\infty} |q_j| w_j < \infty$. We take as natural candidates for commodity spaces the non–negative cones

$$\mathbb{X} = A_+(\mathbf{w}) = \{\mathbf{c} \in \mathbb{R}_+^{\mathbb{N}} : \mathbf{c} \leq \lambda\mathbf{w} \ \text{ for some } \lambda > 0\}, \tag{5}$$

[2] For a Banach space E, the weak–star topology on the norm dual E^* is the weakest topology so that E is the topological dual of E^*. Notable properties of the weak–star topology are that every weak–star closed and norm–bounded set of E^* (in the dual norm defined on E^*) is compact by Alaoglu's Theorem, and that every weak–star compact subset is norm–bounded. All these facts can be found in, e.g., Willard [11].

with $\mathbf{w}$ increasing, or the sets

$$\mathbb{X} = A_+(\mathbf{v},\mathbf{w}) = \{\mathbf{c} \in \mathbb{R}_+^{\mathbb{N}} \,:\, \lambda_1 \mathbf{v} \le \mathbf{c} \le \lambda_2 \mathbf{w} \ \text{ for some } \ 0 < \lambda_1 < \lambda_2\}, \quad (6)$$

where $\mathbf{v}$ and $\mathbf{w}$ are consumption streams in $\mathbb{R}_+^{\mathbb{N}}$ such that $\mathbf{v}$ is decreasing, $\mathbf{w}$ is increasing, and $\mathbf{0} \le \mathbf{v} < \mathbf{1} \le \mathbf{w}$.

The sets given in (6) work well in cases of aggregators for which it is necessary to keep $\mathbf{c}$ away from the null consumption stream $\mathbf{0}$ (typically, when $W(0,y) = -\infty$). Note that $A_+(\mathbf{0},\mathbf{w}) = A_+(\mathbf{w})$, so we refer in the following analysis to $A_+(\mathbf{w})$ or $A_+(\mathbf{0},\mathbf{w})$, indistinctly.[3]

Commodity spaces $\mathbb{X}$ of the sort given in (5) and (6) are subsets of the dual of a Banach space and satisfy (4): If $\mathbf{c} \in \mathbb{X} = A_+(\mathbf{v},\mathbf{w})$, then $\lambda_1 v_j \le c_j \le \lambda_2 w_j$ for all $j \in \mathbb{N}$, for some $0 < \lambda_1 < \lambda_2$. Define $\overline{\lambda}_1 = \lambda_1 \,{}_{\mathbf{v}}\|\sigma\mathbf{v}\|$ and $\overline{\lambda}_2 = \lambda_2 \|\sigma\mathbf{w}\|_{\mathbf{w}}$, where ${}_{\mathbf{v}}\|\mathbf{v}\|$ denotes $\inf_{j\in\mathbb{N}} v_{j+1}/v_j$. Then $\overline{\lambda}_1 v_j \le c_{j+1} \le \overline{\lambda}_2 w_j$ for every $j \in \mathbb{N}$, so that $\sigma\mathbf{c} \in A_+(\mathbf{v},\mathbf{w})$. Thus, $\sigma\mathbb{X} \subseteq \mathbb{X}$. On the other hand, $\pi(\bigcup_{j=0}^{\infty} \sigma^j \mathbb{X}) \subseteq X$ is clear.

In the following lemma it is supposed that if $\mathbf{v}$ is not the null sequence, then ${}_{\mathbf{v}}\|\sigma\mathbf{v}\| < 1$ and also that $\|\sigma\mathbf{w}\|_{\mathbf{w}} > 1$. Obviously, these inequalities always can be obtained.

Lemma 1 *Let* $\mathbb{X}$ *given as in* (5) *or* (6) *with the weak–star topology. Then,* $(\mathcal{C}(\mathbb{X}), d)$ *and* $(\mathcal{C}_{c,U_0}(\mathbb{X}), d_c)$ *are complete metric spaces.*

Proof Consider the sets K_j defined as

$$K_j = \left\{\mathbf{c} \in A_+(\mathbf{v},\mathbf{w}) \,:\, ({}_{\mathbf{v}}\|\sigma\mathbf{v}\|)^j \le {}_{\mathbf{v}}\|\mathbf{c}\|, \quad \|\mathbf{c}\|_{\mathbf{w}} \le (\|\sigma\mathbf{w}\|_{\mathbf{w}})^j\right\}, \quad \forall j \in \mathbb{N}. \quad (7)$$

It is readily seen that each K_j is weak–star closed and norm bounded, thus weak–star compact by Alaoglu's Theorem. The family $\{K_j\}_{j=1}^{\infty}$ satisfies $K_j \subset K_{j+1}$ and $A_+(\mathbf{v},\mathbf{w}) = \cup_{j=1}^{\infty} K_j$. The first property is obvious. To show the latter, let $\mathbf{c} \in A_+(\mathbf{v},\mathbf{w})$. Then, $\lambda_1 v_j \le c_j \le \lambda_2 w_j$ for all $j \in \mathbb{N}$, for some $0 < \lambda_1 < \lambda_2$. By the hypotheses imposed, it is then possible to choose $j \in \mathbb{N}$ satisfying $({}_{\mathbf{v}}\|\sigma\mathbf{v}\|)^j \le \lambda_1$ and $(\|\sigma\mathbf{w}\|_{\mathbf{w}})^j \ge \lambda_2$. Thus, $\mathbf{c} \in K_j$.

Consider now the countable family of seminorms $\{p_j\}_{j=1}^{\infty}$ given by

$$p_j(U) = \sup\{|U(\mathbf{c})| : \mathbf{c} \in K_j\} = \|U\|_{K_j}, \quad \forall j \in \mathbb{N},$$

and the corresponding countable family of semidistances, $\{d_j\}_{j=1}^{\infty}$, defined, as usual, by $d_j(U,V) = p_j(U-V)$ for all $j \in \mathbb{N}$. The metrics d and d_c appearing in (1) and (2) transform into

$$d(U,V) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|U-V\|_{K_j}}{1 + \|U-V\|_{K_j}} \quad \text{and} \quad d_c(U,V) = \sum_{j=1}^{\infty} c^j \, \|U-V\|_{K_j},$$

[3] When $\mathbf{v}$ and $\mathbf{w}$ are of the type $v_j = \delta^j$ and $w_j = \alpha^j$ for $0 < \delta < 1 < \alpha$, for all $j \in \mathbb{N}$, then the sets $A_+(\mathbf{w})$ and $A_+(\mathbf{v},\mathbf{w})$ are the weighted spaces $\ell_+^{\infty}(\alpha) = \{\mathbf{c} \in \mathbb{R}_+^{\mathbb{N}} \,:\, \sup_{j\in\mathbb{N}} c_j/\alpha^j < \infty\}$ and $\ell_+^{\infty}(\delta,\alpha) = \{\mathbf{c} \in \mathbb{R}_+^{\mathbb{N}} \,:\, \inf_{j\in\mathbb{N}} c_j/\delta^j > 0,\ \sup_{j\in\mathbb{N}} c_j/\alpha^j < \infty\}$, respectively.

respectively. We prove the lemma only for $(\mathcal{C}(\mathbb{X}), d)$, since for $(\mathcal{C}_{c,U_0}(\mathbb{X}), d_c)$ the proof is similar.

First note that, for each $j \in \mathbb{N}$, d_j satisfies all the axioms in order to be a distance except that $d_j(U, V) = 0$ does not imply $U = V$. However, whenever $d_j(U, V) = 0$ for every $j \in \mathbb{N}$, then clearly $U = V$ because the family of compacts subsets $\{K_j\}_{j=1}^{\infty}$ is increasing and their union fill $\mathcal{C}(\mathbb{X})$. With this property at hand, and the fact that the map $x \mapsto x/(1+x)$ is increasing, it is obvious that d is a distance. To prove that it is complete, let $\{U_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the metric d. Then $\|U_n - U_m\|_{K_j} \to 0$ for every $j \in \mathbb{N}$ as $n, m \to \infty$, so that $\{U_n\}_{n=1}^{\infty}$ converges uniformly on every compact K_j to a continuous function U_j. Then define the function $U = U_j$ on every K_j. To show that U is continuous, let $\{\mathbf{c}_a\}_{a \in I}$ be a net of elements of $\mathbb{X}$ converging in the weak–star topology to a limit $\mathbf{c} \in \mathbb{X}$, which is unique since this topology is Hausdorff. Weak–star convergent nets are norm bounded, thus the set $K = \{\mathbf{c}_a\}_{a \in I} \cup \{\mathbf{c}\}$ is norm bounded and in consequence there exists $j \in \mathbb{N}$ such that $K \subseteq K_j$. Now, since U is continuous on K_j, $U(\mathbf{c}_a) \to U(\mathbf{c})$. In consequence, U is weak–star continuous and d is complete. □

Notice that convergence in distance d means uniform convergence on the compact subsets of $\mathbb{X}$, and the same property is true for the metric d_c.

Remark 1 The commodity space $A_+(\mathbf{w})$ endowed with the topology derived from the norm $\|\cdot\|_{\mathbf{w}}$ cannot be covered with a countable family of compact sets, since the unit ball is not compact in the topology generated by the norm. In consequence, is not possible to define on the space of norm–continuous functions a countable family of seminorms with the properties described in the proof of Lemma 1 and $\mathcal{C}(\mathbb{X})$ is not metrizable. On the other hand, a weaker topology such as the product topology is not suitable either: consider for instance that $A_+(\mathbf{w}) = \ell_+^{\mathbb{N}}$ is the space of all bounded consumption streams, and take an arbitrary increasing family of compact subsets $\{K_j\}_{j=1}^{\infty}$ covering $\ell_+^{\mathbb{N}}$. Then, $\pi^t(K_j) \subseteq \mathbb{R}_+$ is compact for all $t \in \mathbb{N}$, since the projection π is continuous in the product topology, so $\alpha_{tj} = \max \pi^t(K_j)$ is well-defined. It is obvious, however, that the compact set $K = \{\mathbf{c}\}_{j=1}^{\infty} \cup \{\mathbf{0}\}$, where $\mathbf{c}_j = (0, \ldots, 0, 1 + \alpha_{jj}, 0, \ldots)$, for all $j \in \mathbb{N}$, is not contained in any element of the countable family $\{K_j\}_{j=1}^{\infty}$. Thus the space of continuous functions with respect to the product topology is metrizable, but $\mathcal{C}(\mathbb{X}, d)$ is not complete.

Remark 2 The compact subsets K_j just defined in (7) satisfy $\sigma K_j \subseteq K_{j+1}$, for all $j \in \mathbb{N}$. To show this, notice the following properties: $\|\sigma\mathbf{c}\|_{\mathbf{w}} \leq \|\mathbf{c}\|_{\mathbf{w}} \|\sigma\mathbf{w}\|_{\mathbf{w}}$, and ${}_{\mathbf{v}}\|\sigma\mathbf{c}\| \geq {}_{\mathbf{v}}\|\mathbf{c}\| \; {}_{\mathbf{v}}\|\sigma\mathbf{v}\|$. Then, for $\mathbf{c} \in K_j$, it follows $\|\sigma\mathbf{c}\|_{\mathbf{w}} \leq (\|\sigma\mathbf{w}\|_{\mathbf{w}})^j \|\sigma\mathbf{w}\|_{\mathbf{w}} = (\|\sigma\mathbf{w}\|_{\mathbf{w}})^{j+1}$, and a similar argument shows ${}_{\mathbf{v}}\|\sigma\mathbf{c}\| \geq ({}_{\mathbf{v}}\|\sigma\mathbf{v}\|)^{j+1}$, which proves the statement. On the other hand, in cases where $\mathbf{w}$ is a bounded consumption stream, and $\mathbf{v} = \mathbf{0}$, we can consider the compact subsets defined as

$$K_j = \prod_{j=1}^{\infty} [0, j]^{\mathbb{N}}, \quad \forall j \in \mathbb{N}, \tag{8}$$

selection which permits to trivially obtain $A_+(\mathbf{v}) = \cup_{j=1}^{\infty} K_j$, and $\sigma K_j \subseteq K_j$. This last property is important for proving that the Koopmans' operator $\mathscr{K}$ is a 0–local contraction, a case for which global uniqueness of the recursive utility function can be obtained.

3 Main results

Our first result in this section shows that $\mathscr{K}$ is a 0–local contraction on $\mathcal{C}(\mathbb{X})$, whenever $\mathbb{X} = A_+(\mathbf{w})$ and $\mathbf{w}$ is a bounded consumption stream. Therefore, the conclusions of Theorem 1 are applicable to the operator $\mathscr{K}$. In consequence, the utility function whose existence is asserted is the *unique* weak–star continuous function satisfying the Koopmans' equation.

Theorem 3 *Let W be an aggregator satisfying* (W1) *and* (W2). *Let $\mathbf{w} \in \mathbb{R}_+^{\mathbb{N}}$ be a bounded consumption stream and consider $\beta_j = \max_{\mathbf{c} \in K_j} \beta(\pi(\mathbf{c})) < 1$, for all $j \in \mathbb{N}$, where $K_j = \prod_{j=1}^{\infty} [0, j]^{\mathbb{N}}$. Then,*

(a) *The Koopmans' equation has a unique solution U^* on $\mathcal{C}(A_+(\mathbf{w}))$. Furthermore, U^* satisfies*

$$\|U^*\|_{K_j} \leq \frac{\|\mathscr{K}0\|_{K_j}}{1 - \beta_j}, \quad \forall j \in \mathbb{N}.$$

(b) *For any $U \in \mathcal{C}(A_+(\mathbf{w}))$, $\mathscr{K}^n U$ converges to U^* in the metric d as $n \to \infty$.*

Proof According to Lemma 1 and Theorem 1, we only need to prove that $\mathscr{K}$ is a 0–local contraction on $\mathcal{C}(A_+(\mathbf{w}))$. Given a function $U \in \mathcal{C}(A_+(\mathbf{w}))$, $\mathscr{K}U$ is weak–star continuous from (W1), and because π and σ are weak–star continuous linear operators. Let $U, V \in \mathcal{C}(A_+(\mathbf{w}))$ and let $\mathbf{c} \in K_j$, where K_j is defined as in (8). Then,

$$\begin{aligned}
|\mathscr{K}U(\mathbf{c}) - \mathscr{K}V(\mathbf{c})| &= |W(\pi\mathbf{c}, U(\sigma\mathbf{c})) - W(\pi\mathbf{c}, V(\sigma\mathbf{c}))| \\
&\leq \beta(\pi\mathbf{c})\ |U(\sigma\mathbf{c}) - V(\sigma\mathbf{c})| \qquad \text{(from (W2))} \\
&\leq \max_{\mathbf{c} \in K_j} \beta(\pi\mathbf{c})\ |U(\sigma\mathbf{c}) - V(\sigma\mathbf{c})| \\
&\leq \beta_j \max_{\mathbf{c} \in K_j} |U(\mathbf{c}) - V(\mathbf{c})| \qquad \text{(since } \sigma K_j \subseteq K_j\text{)} \\
&= \beta_j\ \|U - V\|_{K_j}.
\end{aligned}$$

Thus, we have $\|\mathscr{K}U - \mathscr{K}V\|_{K_j} \leq \beta_j\ \|U - V\|_{K_j}$ and $\mathscr{K}$ is a 0–local contraction. □

Now, we establish the corresponding result for the commodity spaces appearing in Lemma $\mathbb{X} = A_+(\mathbf{v}, \mathbf{w})$. The result, which permits to cover aggregators unbounded below, $W(0, y) = -\infty$, and undiscounted or even upcounted models, states some joint restrictions on the preferences and the rate of growth of the consumption streams. Such joint restrictions are unavoidable whether another techniques are used, as those derived in [2], [8], or [5].

Theorem 4 *Let W be an aggregator satisfying* (W1) *and* (W2). *Let $U_0 \in \mathcal{C}(A_+(\mathbf{v}, \mathbf{w}))$ be such that the series*

$$d_c(\mathscr{K} U_0, U_0) = \sum_{j=1}^{\infty} c^j \; \| \mathscr{K} U_0 - U_0 \|_{K_j}$$

converges for some $c > \beta$ with $\beta = \sup_{x \in X} \beta(x)$. Then,

(a) *The Koopmans operator has a unique solution U^* on the set*

$$\mathcal{C}_{c,U_0}(A_+(\mathbf{v}, \mathbf{w})) = \left\{ U \in \mathcal{C}(A_+(\mathbf{v}, \mathbf{w})) \; : \; d_c(U, U_0) \leq \frac{d_c(\mathscr{K} U_0, U_0)}{1 - \beta/c} \right\}$$

(b) *For any $U \in \mathcal{C}_{c,U_0}(A_+(\mathbf{v}, \mathbf{w}))$, $\mathscr{K}^n U$ converges to U^* in the metric d_c as $n \to \infty$.*

Proof According to Lemma 1 and Theorem 2 we only need to prove that $\mathscr{K}$ is a 1–local contraction on $\mathcal{C}_{c,U_0}(A_+(\mathbf{v}, \mathbf{w}))$. Given a function $U \in \mathcal{C}_{c,U_0}(A_+(\mathbf{v}, \mathbf{w}))$, $\mathscr{K} U$ is weak–star continuous from (W1), and because π and σ are weak–star continuous linear operators. Let $U, V \in \mathcal{C}_{c,U_0}(A_+(\mathbf{v}, \mathbf{w}))$ and let $\mathbf{c} \in K_j$, where K_j is defined in (7). Then,

$$\begin{aligned}
|\mathscr{K} U(\mathbf{c}) - \mathscr{K} V(\mathbf{c})| &= |W(\Pi \mathbf{c}, U(\sigma \mathbf{c})) - W(\Pi \mathbf{c} \,, V(\sigma \mathbf{c}))| \\
&\leq \beta \; |U(\sigma \mathbf{c}) - V(\sigma \mathbf{c})| \qquad \text{(from (W2))} \\
&\leq \max_{\mathbf{c} \in K_j} \beta \; |U(\sigma \mathbf{c}) - V(\sigma \mathbf{c})| \\
&\leq \beta \; \|U - V\|_{K_{j+1}} \qquad \text{(since } \sigma K_j \subseteq K_{j+1}\text{)}.
\end{aligned}$$

Thus, we have $\|\mathscr{K} U - \mathscr{K} V\|_{K_j} \leq \beta \; \|U - V\|_{K_{j+1}}$ and $\mathscr{K}$ is a 1–local contraction. □

Remark 3 In applications, it is important to determine the range of values of c for which the series $\sum_{j=1}^{\infty} c^j \; \| \mathscr{K} U_0 - U_0 \|_{K_j}$ converges. Here is a well–known result: let $\sum_{j=1}^{\infty} c^j a_j$ be a power series of real (or complex) numbers and let $\lambda = \limsup_{j \to \infty} \; |a_j|^{\frac{1}{j}}$. If $c\lambda < 1$, then the series converges uniformly. If $c\lambda > 1$, then the series diverges.

Notice that in this context, where $a_j = \| \mathscr{K} U_0 - U_0 \|_{K_j}$, the limit always exists (it can be $+\infty$), since a_j is an increasing sequence of real numbers.

When $U_0 \equiv 0$ we can summarise the conditions established in Theorem 4 as follows:

$$\limsup_{j\to\infty} \|W(\pi\mathbf{c},0)\|_{K_j}^{1/j} = \lambda, \quad \text{and} \quad \beta\lambda < 1.$$

This is a joint restriction in the growth of $W(x,0)$, the rate of growth of consumption, and the size of the implicit discount factor.

4 Examples

The following examples illustrates the applicability of theorems 3 and 4. In the first model the case where $\mathscr{K}$ is a 0–local contraction is studied, obtaining uniqueness of the utility function. It also shows how our techniques can be applied to undiscounted models with $\beta = 1$. The second example illustrate the case of an unbounded below aggreagator, $W(0,y) = -\infty$. We improve the analysis with respect to previous results on the statement of the weak–star continuity of the utility function.

Example 1 (Uzawa–Epstein–Hynes). Let $W(x,y) = (u(x)+y)e^{-v(x)}$, introduced in [10] and also studied in [3], where it is supposed that u, v are continuous and strictly increasing on $X = [0,\infty)$, with $u < 0$ and $v(0) \geq 0$. Assumption (W1) holds as well as (W2) for $\beta = e^{-v(0)}$. Suppose first that $v(0) > 0$. Let $\mathbf{w}$ be a bounded consumption stream, and consider the space $A_+(\mathbf{w}) = \ell_+^{\mathbb{N}}$. Since $\beta = e^{-v(0)} < 1$, Theorem 3 applies assuring the existence of a unique fixed point U^* on $\mathcal{C}(A_+(\mathbf{w}))$. On the other hand, let $\mathbf{w}$ be a consumption stream satisfying $1 < \|\sigma\mathbf{w}\|_{\mathbf{w}} < \infty$. Now, let U_0 be the null function. Since u is strictly increasing, it follows $\limsup_{j\to\infty} \|\mathscr{K}0\|_{K_j}^{1/j} \leq \limsup_{j\to\infty}(\beta|u(0)|)^{1/j} = \lambda = 1$. Thus, Theorem 4 and Remark 3 assure the existence of a fixed point U^* on $\mathcal{C}(A_+(\mathbf{w}))$. Furthermore, uniqueness of the fixed point is verified on the sets of the form $\mathcal{C}_{c,0}(A_+(\mathbf{w}))$, for all $c \in (\beta, 1)$.

When $v(0) = 0$ Theorem 3 is not applicable on $\mathcal{C}(A_+(\mathbf{w}))$, as $\beta = 1$. However, Theorem 4 again applies on the commodity space $\mathcal{C}_{c,0}(A_+(\mathbf{v},\mathbf{w}))$, for all consumption streams $\mathbf{v},\mathbf{w}$ satisfying $0 < \epsilon < {}_{\mathbf{v}}\|\sigma\mathbf{v}\| < 1 < \|\sigma\mathbf{w}\|_{\mathbf{w}} < \infty$, where now $X = [\epsilon,\infty)$ (the reason is that, for such a selection, $\beta = e^{-v(\epsilon)} < 1$). Hence, Theorem 4 guaranties the existence and uniqueness of a fixed point U^* on $\mathcal{C}_{c,0}(A_+(\mathbf{v},\mathbf{w}))$, for all $c \in (\beta, 1)$.

Example 2 (Logarithmic case). Suppose that W satisfies (W1), (W2) and is such that $a + b\ln x \leq W(x,0) \leq A + B\ln x$ for all $x > 0$, for some non-negative constants a, b, and $A, B > 0$. Obviously, $W(0,y) = -\infty$, so that W is unbounded below. Take then consumption streams $\mathbf{v} = \{v_j\}_{j=1}^{\infty}$ and $\mathbf{w} = \{w_j\}_{j=1}^{\infty}$ satisfying the hypotheses of Theorem 2, and define $\lambda_1 = \limsup_{j\to\infty} |A + B\ln w_j|^{1/j}$, and $\lambda_2 = \limsup_{j\to\infty} |a + b\ln v_j|^{1/j}$. Then, for any $\mathbf{c} \in K_j$, it follows

$$\mathscr{K}0(\mathbf{c}) \geq a + b\ln \pi\mathbf{c} = a + b\ln c_1 \geq a + b\ln v_j,$$

and, in the same way,

$$\mathscr{K}0(\mathbf{c}) \leq A + B \ln \pi \mathbf{c} = A + B \ln c_1 \leq A + B \ln w_j .$$

Hence, $\limsup_{j\to\infty} \|\mathscr{K}0\|_{K_j}^{1/j} \leq \min\{\lambda_1, \lambda_2\}$. Therefore, Theorem 4 and Remark 3 show that, for all $\beta < \min\{\lambda_1, \lambda_2\}$, a unique recursive utility function U^* exists on $\mathcal{C}_{c,0}(A_+(\mathbf{v}, \mathbf{w}))$, for all $c \in (\beta, 1)$.

References

1. Becker, R.A., Boyd III, J.H.: Capital theory, equilibrium analysis and recursive utility. Oxford: Blackwell 1997
2. Boyd III, J.H.: Recursive utility and the Ramsey problem. Journal of Economic Theory **50**, 326–345 (1990)
3. Epstein, L., Hynes, J.A.: The rate of time preference and dynamic economic analysis. Journal of Political Economy **91**, 611–635 (1983)
4. Koopmans, T.J.: Stationary ordinal utility and impatience. Econometrica **28**, 287–309 (1960)
5. Le Van, C., Vailakis, Y.: Recursive utility and optimal growth with bounded or unbounded returns. Journal of Economic Theory **123**, 187-209 (2005)
6. Lucas Jr., R.E., Stokey, N.L.: Optimal growth with many consumers. Journal of Economic Theory **32**, 139–171 (1984)
7. Rincón–Zapatero, J.P., Rodríguez–Palmero, C.: Existence and uniqueness of solutions to the Bellman equation in the unbounded case. Econometrica **71**, 1519-1555 (2003)
8. Streufert, P.A.: Stationary recursive utility and dynamic programming under the assumption of biconvergence. Review of Economic Studies **57**, 79–97 (1990)
9. Streufert, P.A.: Abstract recursive utility. Journal of Mathematical Analysis and Applications **175**, 169–185 (1993)
10. Uzawa, H.: Time preference, the consumption function, and optimum asset holding. In: Wolfe, J.N. (ed.) Value, capital and growth: Papers in honor of sir John Hicks. Edinburgh: Edinburgh University Press 1968
11. Willard, S.: General topology. Reading (Mass.): Addison–Wesley 1970