Mean–variance portfolio and contribution selection in stochastic pension funding

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Abstract

In this paper we study the problem of simultaneous minimization of risks, and maximization of the terminal value of expected funds assets in a stochastic defined benefit aggregated pension plan. The risks considered are the solvency risk, measured as the variance of the terminal fund's level, and the contribution risk, in the form of a running cost associated to deviations from the evolution of the stochastic normal cost. To solve this bi–objective stochastic control problem the concept of efficient solution is used.

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1 Introduction

The optimal management of dynamic pension plans is an interesting problem due to the importance that pension funds have currently in financial markets, as well as their fundamental role in assuring the future wealth of participants in their retirement period.

Pension funds can be classified into the following two main categories: defined benefit (DB) pension plans and defined contribution (DC) pension plans. In a DB plan benefits are fixed in advance by the sponsor and contributions are designed to amortizes the fund according to a previously chosen actuarial scheme. Future benefits due to participants are thus a liability for the sponsor, who bears the financial risk. Of course, this risk is increased with the formation of a risky portfolio that, however, offers higher expected returns, with the possibility then of reducing the amortization quote. It is the concern of the sponsor to drive the dynamic evolution of the fund having into account the tradeoff between risk and contribution. In a DC plan contributions are fixed but benefits depend on the returns of the fund portfolio, so that the participants bear the risk.

It has been in recent years an increasing interest of researches in the study of the optimal management of DB pension plans. See e.g. Haberman and Sung (1994), Chang (1999), Cairns (2000), Haberman *et al* (2000), Taylor (2002), Chang *et al* (2002) and Josa–Fombellida and Rincón–Zapatero (2001, 2004, 2006a,b).

Beginning with the paper of Haberman and Sung, DB plans have been usually modelled as linear–quadratic optimal control problems. This is due to the fact that the dynamics of the fund is postulated linear, as in Merton's model, and that it is generally accepted that managers' objectives should be related with the minimization of solvency risk and contribution risk. These risk concepts are defined as quadratic deviations of fund wealth and amortization rates with respect to liabilities and normal cost, respectively. In an environment where liabilities are random, the risks so formulated do not correspond to the variance, which is by far the most common measure of risk. The aim of this paper is to study the optimal management of DB plans when the solvency risk is identified with the variance of the unfunded actuarial liability.

To this end, the problem is settled in the familiar mean–variance framework, translating the static model of Markowitz to the continuous–time setting of a DB plan that evolves with time.

Markowitz (1952) designed the mean–variance model to compare securities and portfolios based in a trade–off between their expected return and risk, measured as the variance of the return. From the point of view of optimization, the problem of portfolio selection is a multiobjective programming problem where it is desired to attain the highest possible expected return with the lowest possible variance. Since these objectives are in general mutually incompatible, the best can be done is to select portfolios where it is not possible to increase return without increasing risk, and reciprocally, where it is not possible to decrease risk without decreasing return. The set of pairs (return, variance) enjoying theses properties are called the Pareto frontier or efficient points set, and the associated portfolios are called efficient.

It has been several attempts in the literature to translate the mean–variance methodology from the static case to the dynamic setting. The most successful and fundamental is of course the one initiated by Merton. It is worth noting, however, that Merton's model does not exactly fit the structure of the mean variance approach. It has been recently, in the papers by Zhou and Li (2000) and Li and Ng (2000) that the methodology has been more faithfully carried out to the dynamic setting, in continuous and in discrete time, respectively. In our paper we follow the formulation of Zhou and Li (2000) but with some modifications due to the specificities of a DB plan, as the inclusion of the supplementary cost as a control variable in addition to the quantities invested in the risky assets. This point is explained in Remark 3.1 below. Moreover we use the Hamilton–Jacobi–Bellmam approach instead of the maximum principle.

Problems of mean–variance type have been recently considered in pension plans from a static point of view in Colombo and Haberman (2005) and in Huang and Cairns (2005). A dynamic model for asset and liability management under the mean–variance criteria has been studied in Chiu and Li (2006). The framework provided by these authors, although general, cannot be applied directly to a DB plan since several of the constitutive elements of the pension plan, as the amortization rate, normal cost, benefits and the technical actuarial rate, are not contemplated in the model. More fundamentally, a DB plan is identified by two different elements of control: investment decisions in the portfolio and amortization rate. The latter is absent in the framework provided by Chiu and Li (2006). The existence of an additional control variable requires a modification in the objective functional, introducing a running cost associated to the size of the amortization rate, more concretely, associated to quadratic deviations with respect to the stochastic normal cost. Thus our problem combines terminal payoffs due to the final levels of expected surplus/debt and of the variance of fund wealth (the *stock* variable) as well as an integral term or running cost that takes care of the contribution risk (the amortization rate is a *flow* variable).

Our paper follows Josa–Fombellida and Rincón–Zapatero (2004), where the benefits of the DB plan are stochastic, modelled by a geometric Brownian motion. Note that benefits is a non–tradable asset, hence the market is incomplete and, furthermore, we also consider the existence of correlation between the sources of uncertainty in the benefits and in the asset returns.

The paper is organized as follows. Section 2 defines the elements of the pension scheme and describes the financial market where the fund operates. Section 3 is devoted to formulate the management of the DB plan in a mean–variance framework, with the simultaneous objectives of minimizing the expected unfunded actuarial liability, as well as its variance at the final time, and to minimize the contribution rate risk over the planning interval. The problem is solved in Section 4 using first the well known scalarization method and then the device provided in Zhou and Li (2000). Once the Pareto frontier is obtained, we compute the total expected supplementary cost and the total expected contribution rate. Section 5 serves as a numerical illustration of previous results. Finally, Section 6 is dedicated to establishing some conclusions. All proofs are developed in Appendix A.

2 The pension model

Consider a DB pension plan of aggregated type where at every instant of time active participants as well as retired participants coexist. Benefits payed to the participants at the age of retirement are fixed in advance by the sponsor and are governed by an exogenous process which source of randomness is correlated with the financial market. To cover the liabilities in an efficient way, the manager creates a portfolio and design an amortization scheme varying with time.

The main elements intervening in a DB plan are the following.

T	:	Planning horizon or date of the end of the pension plan, with $0 < T < \infty$.
$F(t)$	:	Value of fund assets at time t.
$P(t)$	:	Benefits promised to the participants at time t. They are related with the salary at the moment of retirement.
$C(t)$	:	Contribution rate made by the sponsor at time t to the funding process.
$AL\,(t)$	:	Actuarial liability at time t, that is, total liabilities of the sponsor.
$NC\,(t)$	:	Normal cost at time t; if the fund assets match the actuarial liability, and if there are no uncertain elements in the plan, the normal cost is the value of the contributions allowing equality between asset funds and liabilities.
$UAL(t)$	:	Unfunded actuarial liability at time t, equal to $AL\,(t) - F(t)$.
$SC\,(t)$	:	Supplementary cost at time t, equal to $C(t) - NC\,(t)$.
δ	:	Constant rate of valuation of the liabilities, which can be specified by the regulatory authorities.

Following Josa–Fombellida and Rincón–Zapatero (2004) we suppose that disturbances there exist affecting the evolution of benefits and hence the evolution of the normal cost and the actuarial liability. To model this randomness, we consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathscr{F} = \{\mathscr{F}_t\}_{t\geq 0}$ is a complete and right continuous filtration generated by the one–dimensional Brownian motion $\{B(t)\}_{t\geq 0}$ and $\mathbb{P}$ is a probability measure on Ω. The more general case is to suppose benefits P is a diffusion process built from B, that is, P satisfies the stochastic differential equation (SDE)

$$dP(t) = \kappa(t, P(t))dt + \eta(t, P(t))dB(t), \quad 0 \leq t \leq T, \quad P(0) = P_0,$$

where P_0 is the value of the initial liabilities, and where κ and η are functions such that the SDE has a unique solution. For analytical tractability, we will need a more concrete specification for benefits, P. We will suppose that benefits follows a geometric Brownian motion. It is the natural extension of the deterministic case where P is an exponential function (see Bowers *et al* (1986)). This assumption is natural since in general benefits depends on salary and population plan, which show in the average exponential growth subject to random disturbances that may supposed to be proportional to the variables' size.

Assumption 1 *The benefits P satisfies*

$$dP(t) = \kappa P(t)\, dt + \eta P(t)\, dB(t), \quad t \geq 0,$$

where $\kappa \in \mathbb{R}$ and $\eta \in \mathbb{R}_+$. The initial condition $P(0) = P_0$ is a random variable that represents the initial liabilities.

To compute the actuarial functions AL and NC, we suppose that all information accumulated up to time t is used, under the real probability measure $\mathbb{P}$. The definitions of actuarial liability and normal cost given in Josa–Fombellida and Rincón–Zapatero (2004), extend to the stochastic case these concepts from Bowers *et al* (1986) as follows

$$AL\,(t) = \int_a^d e^{-\delta(d-x)}\, M(x)\mathbb{E}\left(P(t+d-x)|\mathscr{F}_t\right)\, dx,$$
$$NC\,(t) = \int_a^d e^{-\delta(d-x)}\, m(x)\mathbb{E}\left(P(t+d-x)|\mathscr{F}_t\right)\, dx,$$

for every $t \geq 0$, where $\mathbb{E}(\cdot|\mathscr{F}_t)$ denotes conditional expectation with respect to $\mathscr{F}_t$, and where $M(x)$ is a distribution function representing the percentage of actuarial value of future benefits accumulated until age x, and where $m(x)$ is its associated density function. Without lost of generality we are supposing that all members enter into plan at age a and retire at the common age d. Thus, to compute the actuarial functions at time t, the manager makes use of the information available up to that time, in terms of the conditional expectation. This procedure, instead of looking for a risk neutral probability measure and then to compute the conditional expectation under this measure, is justified since the liabilities are non tradeable in the financial market, thus the inherent risk cannot be hedged. The behavior of the actuarial functions AL and NC are then given in the following result, see Proposition 2.1 in Josa–Fombellida and Rincón–Zapatero (2004).

Proposition 2.1 *Under Assumption 1 there are constants* ψ_{AL} *and* ψ_{NC} *such that* $AL = \psi_{AL}P$ *and* $NC = \psi_{NC}P$. *Furthermore,* $\psi_{NC} = 1 + (\kappa - \delta)\psi_{AL}$ *and the identity* $NC(t) - P(t) = (\kappa - \delta)AL(t)$ *holds for every* $t \geq 0$.

From this proposition we deduce:

$$dAL(t) = \kappa AL(t)\,dt + \eta AL(t)\,dB(t), \quad AL(0) = \psi_{AL}P_0 \tag{1}$$

and also

$$dNC(t) = \kappa NC(t)\,dt + \eta NC(t)\,dB(t), \quad NC(0) = \psi_{NC}P_0.$$

We will denote by AL_0 and NC_0 the initial values of the actuarial liability and the normal cost, respectively, that is $AL_0 = \psi_{AL}P_0$ and $NC_0 = \psi_{NC}P_0$.

In the rest of this section we describe the financial market where the fund operates. Given an $(n+1)$–dimensional standard Brownian motion $(w_0, w_1, \ldots, w_n)^\top$, we consider the complete probability space $(\Omega, \mathscr{G}, \mathbb{P})$ generated by it, that is to say, $\mathscr{G}$ is the filtration $\{\mathscr{G}_t\}_{t\geq 0}$, with $\mathscr{G}_t = \sigma\{w_0(s), w_1(s), \ldots, w_n(s); 0 \leq s \leq t\}$.

The plan sponsor manages the fund in the planning interval $[0, T]$ by means of a portfolio formed by n risky assets $\{S^i\}_{i=1}^n$, which are correlated geometric Brownian motions, generated by $w = (w_1, \ldots, w_n)^\top$, and a riskless asset S^0, as proposed in Merton (1971), that is, whose evolutions are given by the equations:

$$dS^0(t) = rS^0(t)dt, \quad S^0(0) = 1, \tag{2}$$

$$dS^i(t) = S^i(t)\Big(b_i dt + \sum_{j=1}^n \sigma_{ij} dw_j(t)\Big), \quad S^i(0) = s_i > 0, \quad i = 1, 2, ..., n. \tag{3}$$

Here $r > 0$ denote the short risk–free rate of interest, $b_i > 0$ the mean rate of return of the ith risky asset and $\sigma_{ij} \geq 0$ the covariance between asset i and j, for all $i, j = 1, \ldots, n$. It is assumed $b_i > r$ for all i, so the sponsor has incentives to invest with risk. We suppose that there exists correlation $q_i \in [-1, 1]$ between B and w_i, for $i = 1, \ldots, n$. As a consequence, B is expressed in terms of $\{w_i\}_{i=0}^n$ as $B(t) = \sqrt{1 - q^\top q}\, w_0(t) + q^\top w(t)$, where $q^\top q \leq 1$ for $q = (q_1, q_2, \ldots, q_n)^\top$. In this way the influence of salary and inflation in the evolution of liabilities P is taken into account, as well as the effect of inflation on the prices of the assets.

The amount of fund invested in time t in the risky asset S^i is denoted by $\lambda_i(t)$, $i = 1, 2, \ldots, n$. The remainder, $F(t) - \sum_{i=1}^{n} \lambda_i(t)$, is invested in the bond. Borrowing and shortselling is allowed. A negative value of λ_i means that the sponsor sells a part of his risky asset S^i short while, if $\sum_{i=1}^{n} \lambda_i$ is larger than F, then he or she gets into debt to purchase the stocks, borrowing at the riskless interest rate r. We suppose the investment strategy $\{\mathbf{\Lambda}(t) : t \geq 0\}$, with $\mathbf{\Lambda}(t) = (\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))^\top$, is a control process adapted to filtration $\{\mathscr{G}_t\}_{t\geq 0}$, $\mathscr{G}_t$–measurable, markovian and stationary, satisfying

$$\mathbb{E} \int_0^T \mathbf{\Lambda}(t)^\top \mathbf{\Lambda}(t) dt < \infty, \tag{4}$$

where $\mathbb{E}$ is the expectation operator. The contribution rate process $C(t)$ is also an adapted process with respect to $\{\mathscr{G}_t\}_{t\geq 0}$ verifying

$$\mathbb{E} \int_0^T SC^2(t) dt < \infty. \tag{5}$$

Therefore, the fund dynamic evolution under the investment policy $\mathbf{\Lambda}$ is[1]:

$$dF(t) = \sum_{i=1}^{n} \lambda_i(t) \frac{dS^i(t)}{S^i(t)} + \left(F(t) - \sum_{i=1}^{n} \lambda_i(t) \right) \frac{dS^0(t)}{S^0(t)} + (C(t) - P(t))\, dt. \tag{6}$$

By substituting (2) and (3) in (6), we obtain:

$$dF(t) = \left(rF(t) + \sum_{i=1}^{n} \lambda_i(t)(b_i - r) + C(t) - P(t) \right) dt + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i(t) \sigma_{ij}\, dw_j(t), \tag{7}$$

with initial condition $F(0) = F_0 > 0$.

Next we will assume the matrix notation: $\sigma = (\sigma_{ij})$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)^\top$, $\mathbf{1} = (1, 1, \ldots, 1)^\top$ and $\Sigma = \sigma\sigma^\top$. We take as given the existence of Σ^{-1}, that is to say, σ^{-1}. Finally the vector of standardized risk premia or Sharpe ratio of the portfolio is denoted by $\theta = \sigma^{-1} (\mathbf{b} - r\mathbf{1})$. So, we can write (7) as:

$$dF(t) = \left(rF(t) + \mathbf{\Lambda}^\top(t)(\mathbf{b} - r\mathbf{1}) + C(t) - P(t) \right) dt + \mathbf{\Lambda}^\top(t) \sigma dw(t), \tag{8}$$

that, with the initial condition $F(0) = F_0$, determines the fund evolution.

[1]This is the familiar equation obtained and justified in e.g. Merton (1990, p. 124). The only difference is that consumption is replaced here by $P - C$.

We assume throughout the paper, as in Josa–Fombellida and Rincón–Zapatero (2004), that the technical interest rate coincides with the rate of return of the bond plus an additional term related with the market risk of the liabilities. In fact, this definition of δ adjusts the risk of the discounted future value of the liabilities, as if the preferences of the sponsor were risk–neutral. We are using here the equilibrium approach of Constantinides (1978), as it is detailed in the Appendix. On the other hand, this value of δ allows us to obtain the optimal contribution and portfolio in explicit form.

Assumption 2 *The technical rate of actualization is $\delta = r + \eta q^\top \theta$.*

Notice that if either benefits are deterministic of there is no correlation between benefits and the financial market, then δ is the risk–free rate of interest. With positive (*resp.* negative) correlation, the valuation of liabilities is r plus a positive (*resp.* negative) term, weighted by the product of the instantaneous variance of P and the Sharpe ratio of the assets. This is the right way to price liabilities, since with positive (*resp.* negative) correlation it is expected that liabilities and assets move in the same (*resp.* opposite) direction.

By (1), equation (8) in terms of $X = -UAL = F - AL$ and of $SC = C - NC$ is

$$dX(t) = \Big(rF(t) + \mathbf{\Lambda}^\top(t)(\mathbf{b} - r\mathbf{1}) + SC(t) + NC(t) - P(t) - \kappa AL(t)\Big)dt + \mathbf{\Lambda}^\top(t)\sigma dw(t) - \eta AL(t)dB(t).$$

By Proposition 2.1 and Assumption 2, the above can be written

$$dX(t) = \Big(rX(t) + \mathbf{\Lambda}^\top(t)(\mathbf{b} - r\mathbf{1}) + SC(t) - \eta q^\top \theta AL(t)\Big)dt + \mathbf{\Lambda}^\top(t)\sigma dw(t) - \eta AL(t)dB(t),$$

and using the independent Brownian motions $\{w_i\}_{i=0}^n$

$$\begin{aligned} dX(t) = {} & \Big(rX(t) + \mathbf{\Lambda}^\top(t)(\mathbf{b} - r\mathbf{1}) + SC(t) - \eta q^\top \theta AL(t)\Big)dt \\ & - \eta AL(t)\sqrt{1 - q^\top q}\, dw_0(t) + (\mathbf{\Lambda}^\top(t)\sigma - \eta AL(t) q^\top) dw(t), \end{aligned} \tag{9}$$

with the initial condition $X(0) = X_0 = F_0 - AL_0$.

To fix the nomenclature, we will suppose along the paper that the fund is underfunded at time 0, $X_0 < 0$, so that X has the meaning of debt. The same interpretation of the results are valid when the fund is overfunded, but then X is surplus.

3 The problem formulation

The objective of the manager is double. On the one hand, it is to minimize the expected unfunded actuarial liability $\mathbb{E}\,UAL(T) = -\mathbb{E}X(T) = -(\mathbb{E}F(T) - \mathbb{E}AL\,(T))$, or equivalently to maximize the expected value of fund's assets. Note that as we are supposing $X < 0$ most often we refer to X as debt. On the other hand, the aim is to minimize the variance of the terminal debt, $\mathbb{V}\mathrm{ar}X(T)$, and the contribution risk SC^2 on the interval $[0,T]$. This bi–objective problem reflects the concern of the promoter of increase fund assets to pay due benefits, but at the same time not subject the pension fund to large variations to provide stability to the plan. Minimization of the contribution risk (related with the security of plan) has been considered in other works as Haberman and Sung (1994), Haberman *et al* (2000) and Josa–Fombellida and Rincón–Zapatero (2001, 2004).

Thus we are considering a multi–objective optimization problem with two criteria[2]:

$$\min_{(SC,\mathbf{\Lambda})\in\mathcal{A}_{X_0,AL_0}} (J_1(SC,\mathbf{\Lambda}), J_2(SC,\mathbf{\Lambda})) \doteq \min_{(SC,\mathbf{\Lambda})\in\mathcal{A}_{X_0,AL_0}} \left(-\mathbb{E}X(T), \mathbb{E}\int_0^T SC^2(t)\,dt + \mathbb{V}\mathrm{ar}X(T)\right), \tag{10}$$

subject to (9), (1). Here $\mathcal{A}_{X_0,AL_0}$ is the set of measurable processes $(SC,\mathbf{\Lambda})$, where SC satisfies (5), $\mathbf{\Lambda}$ satisfies (4) and such that (1) and (9) admit a unique solution $\mathscr{G}_t$–measurable adapted to the filter $\{\mathscr{G}_t\}_{t\geq 0}$.

An admissible control process $(SC^*,\mathbf{\Lambda}^*)$ is *Pareto efficient* (or simply *efficient*) if there exists no admissible $(SC,\mathbf{\Lambda})$ such that

$$J_1(SC,\mathbf{\Lambda}) \leq J_1(SC^*,\mathbf{\Lambda}^*), \qquad J_2(SC,\mathbf{\Lambda}) \leq J_2(SC^*,\mathbf{\Lambda}^*),$$

with at least one of the inequalities being strict. The pairs $(J_1(SC^*,\mathbf{\Lambda}^*), J_2(SC^*,\mathbf{\Lambda}^*)) \in \mathbb{R}^2$ form the *Pareto frontier*. We will call to SC^* an efficient supplementary cost (or efficient contribution rate), and $\mathbf{\Lambda}^*$ an efficient portfolio. Throughout the text the term *optimal* must be understood in the sense of efficiency. Actually, we are not interested in the representation and properties

[2]The complete notation for the objective functionals would be $J_1((t,x,y);(SC,\mathbf{\Lambda})) = -\mathbb{E}_{txy}X(T) = -\mathbb{E}(X(T)|X(t)=x, AL\,(t)=y)$ and $J_2((t,x,y);(SC,\mathbf{\Lambda})) = \mathbb{E}_{txy}\int_t^T SC^2(s)ds + \mathbb{V}\mathrm{ar}_{txy}X(T)$.

of the Pareto frontier, but in the pairs $(-\mathbb{E}X(T), \mathbb{V}\mathrm{ar}X(T))$ for optimal $X(T)$, that we call the *mean–variance efficient frontier.*

Remark 3.1 Problem (1), (9), (10) is a mean–variance problem similar to the one studied in Zhou and Li (2000), but with the additional control variable SC in the state equation (9), and an additional running cost. The model could be also formulated as a family of problems depending on $z \in \mathbb{R}$

$$\min_{(SC,\mathbf{\Lambda})\in\mathcal{A}_{X_0,AL_0}} \left\{ J_2(SC,\mathbf{\Lambda}) = \mathbb{E}\int_0^T SC^2(t)\,dt + \mathbb{E}(X(T)-z)^2 \right\},$$

subject to $\mathbb{E}X(T) = z$ and (1), (9).

According to Da Cunha and Polak (1967) when the objective functionals defining the multi-objective program are convex, the Pareto optimal points can be found solving a scalar optimal control problem where the dynamics remain the same and where the objective functional is a convex combination of the original cost functionals. In our case the equations (1), (9) are linear, so both J_1 and J_2 are obviously convex. Therefore, the original problem (1), (9), (10) is equivalent to the scalar problem

$$\min_{(SC,\mathbf{\Lambda})\in\mathcal{A}_{X_0,AL_0}} J_1(SC,\mathbf{\Lambda})+\mu J_2(SC,\mathbf{\Lambda}) = \min_{(SC,\mathbf{\Lambda})\in\mathcal{A}_{X_0,AL_0}} -\mathbb{E}X(T)+\mu\left(\mathbb{E}\int_0^T SC^2(t)\,dt + \mathbb{V}\mathrm{ar}X(T)\right), \tag{11}$$

subject to (1), (9), with $\mu > 0$ a weight parameter. As μ varies in the interval $(0,\infty)$, the solutions of (11) describe the Pareto frontier. Notice that μ serves the manager to transfer linearly units of risk to units of expected return, and reciprocally. The size of μ indicates which one of the objectives is of more concern for the manager, to reduce risk or to reduce debt.

Problem (1), (9), (11) is not a standard stochastic optimal problem due to the term $(\mathbb{E}X(T))^2$ in the variance, and the dynamic programming approach can not be applied here. Following Zhou and Li (2000) or Li and Ng (2000) we propose an auxiliar problem that turns out be a stochastic problem of linear quadratic type:

$$\min_{(SC,\mathbf{\Lambda})\in\mathcal{A}_{X_0,AL_0}} J(SC,\mathbf{\Lambda}) \doteq \min_{(SC,\mathbf{\Lambda})\in\mathcal{A}_{X_0,AL_0}} \mathbb{E}\int_0^T SC^2(t)\,dt + \mathbb{E}\left(X^2(T) - 2\gamma X(T)\right), \tag{12}$$

subject to (1), (9), where $\gamma \in \mathbb{R}$.

The relationship between problems (1), (9), (11) and (1), (9), (12) is shown in the following result.

Proposition 3.1 *For any $\mu > 0$, if $(SC^*, \mathbf{\Lambda}^*)$ is an optimal control of* (1), (9), (11) *with associated optimal debt X^*, then it is an optimal control of* (1), (9), (12) *for $\gamma = (2\mu)^{-1} + \mathbb{E}X^*(T)$.*

The main consequence of Proposition 3.1 is that any optimal solution of problem (1), (9), (11) can be found solving problem (1), (9), (12). This will be done in the following section.

4 Optimal contributions and portfolio of the auxiliar problem and efficient frontier

In this section we find the efficient frontier for the original problem (1), (9), (10). Previously we solve the problem (1), (9), (12), depending on the parameter γ.

Theorem 4.1 *The optimal rate of supplementary cost and the optimal investment in the risky assets are given by*

$$SC^*(t, X, AL\,) = f(t)\left(\gamma e^{-r(T-t)} - X\right), \tag{13}$$

$$\mathbf{\Lambda}^*(t, X, AL\,) = \Sigma^{-1}(\mathbf{b} - r\mathbf{1})\left(\gamma e^{-r(T-t)} - X\right) + \eta\sigma^{-\top} qAL\,, \tag{14}$$

where

$$f(t) = \frac{(1-c_1)e^{(2r-\theta^\top\theta)(T-t)}}{1-c_1 e^{(2r-\theta^\top\theta)(T-t)}}, \quad \forall t \in [0,T], \tag{15}$$

with $c_1 = 1/(-2r + \theta^\top\theta + 1)$.

The efficient strategies depend on the term $\gamma e^{-r(T-t)} - X(t)$ that, by the definition of γ in Proposition 3.1, decomposes in three terms that we collect into two summands

$$\frac{1}{2\mu}e^{-r(T-t)} + \Big[\mathbb{E}\big(X(T)e^{-r(T-t)}\big) - X(t)\Big].$$

The first summand is always positive, increasing with time, and depends inversely on μ, the parameter weighing the relative importance of the objective of variance minimization with respect to the objective of debt reduction. The summand in brackets is the expected value of debt reduction planned, valued at time t. Notice from the expression of SC^* that if this reduction is positive, then amortization rate is higher than the normal cost. In the same way, the first summand in $\mathbf{\Lambda}^*$ is also positive. Of course, this behavior is also observed for small values of μ, even if there is no reduction of the expected debt. As the control of variance becomes less important for the sponsor, that is, μ decreases, the investment strategies are riskier.

In contradistinction to the supplementary cost, optimal investment depends also on AL and on the elements giving the randomness of assets and benefits. If the actuarial liability AL is positively correlated with the financial market (an extreme case being uncorrelated, where $q = 0$), then the investment in the risky assets is greater than if the correlation is negative. It is remarkable that it does not depend on the rate of growth of benefits, κ.

A technical assumption to obtain some properties of the optimal solutions is necessary. We suppose that twice the risk–free rate of interest is lesser than the norm square of the Sharpe ratio.

Assumption 3 *The Sharpe ratio vector satisfies* $2r < \theta^\top \theta$.

This hypothesis implies for the constant c_1 and the function f defined in Theorem 4.1 that $0 < c_1 < 1$ and $0 < f(t) < 1$, for all $0 \leq t < T$.

Theorem 4.1 gives also a linear relationship between the supplementary cost and investment strategies, which vector coefficient is $1/f(t)$ times the optimal growth portfolio, $\Sigma^{-1}(\mathbf{b} - r\mathbf{1})$:

$$\mathbf{\Lambda}^* = \frac{1}{f(t)} \Sigma^{-1}(\mathbf{b} - r\mathbf{1}) SC^* + \eta \sigma^{-\top} q AL \,. \tag{16}$$

This can be considered as a “rule of thumb” for the sponsor: at time t, each monetary unit of additional amortization with respect to the computed normal cost, must be accompanied by an investment of $\frac{1}{f(t)}\Sigma^{-1}(\mathbf{b} - r\mathbf{1})$ monetary units in risky assets, plus $\eta\sigma^{-\top} q AL$ units due to the stochastic elements defining the pension plan.

The following result characterizes the efficient frontier in terms of the expected returns and variance (disregarding the influence of the contribution risk).

Theorem 4.2 *The mean–variance efficient frontier of the problem* (1)–(9)–(10) *is given by*

$$\mathbb{V}\mathrm{ar}X(T) = \left(\frac{1-\beta}{\beta}\right)^2 \left(e^{\theta^\top \theta T} - 1\right) \left(\mathbb{E}X(T) - e^{rT}X_0\right)^2 + \nu, \tag{17}$$

where

$$\begin{aligned} \beta &= 1 - e^{-\theta^\top \theta T} \frac{1-c_1}{1-c_1 e^{(2r-\theta^\top\theta)T}} = 1 - e^{2rT} f(0), \\ \nu &= \eta^2 (1 - q^\top q) A L_0^2 e^{(2\kappa+\eta^2)T} \int_0^T \frac{e^{(2r-\theta^\top\theta-2\kappa-\eta^2)t}}{(1-c_1 e^{(2r-\theta^\top\theta)t})^2} dt. \end{aligned}$$

Expression (17) shows the familiar quadratic relation between debt and its variance. The minimum possible variance, $\mathbb{V}\mathrm{ar}X(T) = \nu \geq 0$, is attained when the sponsor borrows money for the total amount of debt at date $t = 0$ for T years, so that $\mathbb{E}X(T) = e^{rT}X_0$.

From (17), the expected debt and the standard deviation, $\sigma_X^2(T)$, at time T are related by

$$\mathbb{E}X(T) = e^{rT}X_0 + \frac{\beta}{(1-\beta)\sqrt{e^{\theta^\top\theta T} - 1}} \sqrt{\sigma_X^2(T) - \nu}.$$

There are two cases where it is a straight line: when the benefits are exponential and deterministic, $\eta = 0$, and when the market is complete (Brownian B only depends on w), $q^\top q = 1$. In both cases the *capital market line* is

$$\mathbb{E}X(T) = e^{rT}X_0 + \frac{\beta}{(1-\beta)\sqrt{e^{\theta^\top\theta T} - 1}} \sigma_X(T),$$

The slope, $\beta/\left((1-\beta)\sqrt{e^{\theta^\top\theta T} - 1}\right)$, is the *price of risk*. This is positive because $0 < \beta < 1$ by Assumption 3. It shows how much the expected optimal debt decreases if its standard deviation increases by one unit.

Observe that parameter ν and in consequence the terminal variance in (17) does not depend on the sign of correlations q_i.

Remark 4.1 The optimal investment decisions, contribution rate and fund's wealth evolution can be expressed in terms of the optimal expected debt at time T, $\mathbb{E}X^*(T)$, instead of using the parameters γ or μ. This provides a more clever interpretation of the results. The substitution of γ may be done from the equality $e^{rT}(1-\beta)X_0 + \beta\gamma = \mathbb{E}X^*(T)$, which is obtained in (29) in the Appendix. Taking into account (14) and (29), the investment at instant t is

$$\Lambda^*(t,X,AL) = \Sigma^{-1}(\mathbf{b}-r\mathbf{1})\left(\frac{e^{-r(T-t)}}{\beta}\left(z-e^{rT}X_0\right)-\left(X-e^{rt}X_0\right)\right)+\eta\sigma^{-\top}qAL\,,$$

where $z = \mathbb{E}X^*(T)$. This shows the existing relation between the desired expected levels of debt at time T and the optimal composition of the portfolio at every instant of time t.

The regions of no short–selling and no borrowing in a given asset S^i, i.e. $0 \le \lambda_i^*(t) \le F(t)$, for $i = 1,\dots,n$, are given in the following inequalities

$$\frac{\mathbf{e_i}\Sigma^{-1}(\mathbf{b}-r\mathbf{1})}{1+\mathbf{e_i}\Sigma^{-1}(\mathbf{b}-r\mathbf{1})}\varphi(t)+k_iAL\,(t) \le F(t) \le \varphi(t)+k_i'AL\,(t),$$

with $\mathbf{e_i} = (0,\dots,1,0\dots,0)$, $\varphi(t) = e^{-r(T-t)}\left(z-e^{rT}X_0\right)/\beta + e^{rt}X_0$,

$$k_i = \frac{\mathbf{e_i}\Sigma^{-1}(\mathbf{b}-r\mathbf{1})+\eta\mathbf{e_i}\sigma^{-\top}q}{1+\mathbf{e_i}\Sigma^{-1}(\mathbf{b}-r\mathbf{1})},\quad k_i' = \frac{\mathbf{e_i}\Sigma^{-1}(\mathbf{b}-r\mathbf{1})+\eta\mathbf{e_i}\sigma^{-\top}q}{\mathbf{e_i}\Sigma^{-1}(\mathbf{b}-r\mathbf{1})}.$$

Again from (29) in the Appendix and Proposition 3.1, we obtain $z = e^{rT}X_0+\beta/(2\mu(1-\beta)) > e^{rT}X_0$, by Assumption 3, that is, the expected terminal unfunded liability is lesser than the debt accrued at $t = T$ for borrowing money at date $t = 0$ at an interest rate r. Then, $\varphi > 0$ and in consequence there is a minimum floor for the amount invested in the risky portfolio, which is obtained as $\mu \to \infty$

$$\Lambda^*(t,X,AL) > \Sigma^{-1}(\mathbf{b}-r\mathbf{1})\left(e^{rt}X_0-X\right)+\eta\sigma^{-\top}qAL \doteq \Lambda^*_{\text{inf}}(t,X,AL).$$

Formally, Λ^*_{inf} is the optimal portfolio corresponding to the minimum variance, $\mathbb{V}\text{ar}X(T) = \nu$ and $\mathbb{E}X(T) = e^{rT}X_0$, when $\mu \to \infty$.

Analogously, (13) and (29), allows us to rewrite the optimal rate of contribution at instant t as

$$C^*(t,X) = NC(t) + f(t)\left(\frac{e^{-r(T-t)}}{\beta}\left(z-e^{rT}X_0\right)-\left(X-e^{rt}X_0\right)\right).$$

From this, it is easily obtained that $C^*(t,X) > NC(t)$ in the underfunded region, $X < 0$. As $f > 0$ by Assumption 3, the contribution is bounded below by

$$C^*(t,X) > NC(t) + f(t)\left(e^{rt}X_0 - X\right) \doteq C^*_{\text{inf}}(t,X),$$

with $C^*_{\inf}(t,X)$ the limit of the optimal contribution rate as $\mu \to \infty$.

The contribution rate C^* is not exactly a spread method of amortizing the liability, that is, the supplementary cost is not proportional to the unfunded actuarial liability. However, for a particular level of terminal debt, $\mathbb{E}X(T) = e^{rT}(1-\beta)X_0$, the supplementary cost reduces to

$$SC^*(t,X) = -f(t)X(t) = f(t)\,UAL(t),$$

with variance $\mathbb{V}\mathrm{ar}X(T) = (1-\beta)^2(e^{\theta^\top\theta T}-1)X_0^2 e^{2rT} + \nu$.

The following proposition gives the total optimal contribution.

Proposition 4.1 *The total expected discounted value of the optimal contribution and the optimal supplementary cost in the interval* $[0,T]$ *of problem* (1)*,* (9)*,* (10)*, denoted* $\overline{C}$ *and* $\overline{SC}$ *respectively, are given by*

$$\overline{SC} \doteq \mathbb{E}\int_0^T e^{-rt} SC^*(t,X(t))\,dt = \pi\left(\mathbb{E}X(T) - e^{rT}X_0\right),$$
$$\overline{C} \doteq \mathbb{E}\int_0^T e^{-rt} NC(t)\,dt + \overline{SC} = \frac{1-e^{-(r-\kappa)T}}{r-\kappa} NC_0 + \overline{SC},$$

where

$$\pi = \frac{1-\beta}{\beta}\frac{e^{2rT}-1}{2r}e^{-rT}.$$

The relation between $\overline{SC}$ and $\mathbb{E}X(T)$ is linear, with positive slope π, since $0<\beta<1$ by Assumption 3. Thus, a reduction of one monetary unit of expected debt at time T is attained with an extra expected amortization of π monetary units over the total expected discounted normal cost computed along the time horizon $[0,T]$.

In the case that the manager borrows money to cover the debt X_0 at $t=0$ and does not invest in risky assets, then $\mathbb{E}X(T) = e^{rT}X_0$, i.e. the efficient point with minimum variance, the total contribution is

$$\overline{C}_{\inf} = \mathbb{E}\int_0^T e^{-rt} C^*_{\inf}(t,X(t))dt = \frac{1-e^{-(r-\kappa)T}}{r-\kappa} NC_0.$$

So we have that as $\pi > 0$ then $\overline{C} > \overline{C}_{\inf}$. In conclusion the efficient strategy with $\mathbb{E}X(T) = e^{rT}X_0$ gives the smaller terminal variance, $\mathbb{V}\mathrm{ar}X(T) = \nu$, and the smaller total contribution, $\overline{C}_{\inf}$.

Other interesting fact is that total contribution does not depend on diffusion parameter η, so it coincides with total contribution in the case of deterministic benefits.

The following result confirms that the higher returns obtained from the mixed portfolio in comparison with the returns of the bond, lead to a lower expected contribution in the former case.

Corollary 4.1 *a) The total expected value of the supplementary cost is less when the manager invests in the mixed portfolio than when he or she invests only in the bond.*

b) If $q^\top \theta \geq 0$ then the total expected value of the optimal contribution rate is less when the manager invests in the mixed portfolio than when he or she invests only in the bond.

5 A numerical illustration

In this section we illustrate the results of section above in a specific example. In order to give a more sound illustration of the model's properties we consider two risky assets. The objective is to observe the behavior of the terminal standard deviation, the initial investment and the total expected optimal contribution, with respect to the terminal date, the expected terminal debt and the correlations between benefits and risky assets.

Assumptions. The values of parameters that we consider are the following.

- benefits are random with $\eta = 0.03$ and $\kappa = 0.2$;
- the risk free rate of interest is $r = 0.06$;
- risky investment is in two assets ($n = 2$) with $\mathbf{b} = (0.12,\, 0.10)^\top$ and $\sigma = \begin{pmatrix} 0.15 & 0.07 \\ 0.07 & 0.10 \end{pmatrix}$; this implies a Sharpe ratio $\theta = (0.317,\, 0.178)^\top$;
- the initial values are $AL_0 = 1$, $F_0 = 0.8$, so that the initial liability is $X_0 = -0.2$, that is, the fund is 20% underfunded; benefits at time $t = 0$ are supposed to be 1% of AL_0, that is, $P_0 = 0.01$.

We consider four values of the time horizon, $T = 1$, 2, 5 and 10 years. The goal of the manager is to reduce expected debt $\mathbb{E}X(T)$ to values -0.15, -0.10, -0.05 and 0, that is to say, to attain the 25%, 50%, 75% and 100%, of debt reduction, respectively. The last variable elements that we consider are the correlations $q = (q_1, q_2)^\top$. We suppose the norm square of correlation vector, $q^\top q = q_1^2 + q_2^2$, takes the values 0, 0.50 and 1. More values could be considered obtaining similar properties. The extreme values are uncorrelation and perfect correlation, respectively. In order to fix the correlations we suppose symmetric cases, i.e. $q_1 = \pm q_2$, so the vectors q considered are $(0, 0)^\top$, $(\pm 1/2, \pm 1/2)^\top$ and $(\pm\sqrt{2}/2, \pm\sqrt{2}/2)^\top$.

The remainder elements depending on correlations are easily found. For instance, the technical interest rate δ can be calculated from Assumption 2, and from this Proposition 2.1 for $t = 0$ allows to obtain NC_0 in each case. Observe that $\mathbb{E}X(T) \geq e^{rT}X_0$ is satisfied for all values of the parameters proposed.

Table 1 shows for several final dates of the pension plan what the terminal standard deviation must be in order to reduce debt. We observe that the standard deviation does not depend on the sign of the correlations and it grows with respect to the planning horizon when $q^\top q < 1$, but it decreases when $q^\top q = 1$. In fact, it is more sensible to changes in the horizon length than to changes in the reduction of debt.

Another interesting fact is that the standard deviation is reduced when the norm square of correlation vector is increased, attaining the minimum value when the market is complete.

[INSERT TABLE 1 HERE]

The total amount of initial investment proportion in the risky assets, $(\lambda_1 + \lambda_2)/F_0$, chosen to reduce debt to the prescribed levels, is shown in the Table 2. The investment in the bond is $1 - (\lambda_1 + \lambda_2)/F_0$, that can be obtained from the table. There are two cases of short–selling, which appear of course when both correlations are the more negative allowed values. In some cases borrowing to invest in the risky assets is needed. Specially, this happens when the target is to eliminate completely debt in a short period of time. As expected, the risky investment increases with higher debt reduction levels. Note that there is a “horizon effect” in the investment strategies, since that they do not follow a monotononic pattern with respect to final time T.

The investment is sensible with respect to the sign of the correlations and the norm square of the correlations vector. With negative signs, a more conservative strategy is implemented and the investment behavior is more aggressive for higher levels of correlations.

[INSERT TABLE 2 HERE]

Table 3 shows the total expected terminal optimal value of the contribution rate. The total contribution grows with the debt reduction and with the planning horizon. The correlation has little influence in the total contribution although it is smaller with positive correlation.

[INSERT TABLE 3 HERE]

Table 4 shows the total expected optimal value of the supplementary cost when the portfolio comprises the bond and two risky assets, whereas Table 5 shows the result when investment is only in the bond. In both cases the supplementary cost grows with debt reduction, and a horizon effect appears as in the other components of the plan previously analyzed.

We observe in the tables the result obtained in Corollary 4.1: supplementary cost is greater when the investment is made only in the bond. Risky investment allows to get higher mean returns, making possible to diminish the expected amortization rate.

[INSERT TABLE 4 HERE]

[INSERT TABLE 5 HERE]

Table 6 is the corresponding one to Table 1, but showing the total expected contribution when the investment is in the bond only. When the inequality $q^\top\theta \geq 0$ holds, the total contribution with safe investment is higher than with investment in risky assets, see Corollary 4.1. When the opposite inequality holds, the property is not generally true. For example, in Table 3 are emphasized values where the optimal contributions with investment in risky assets is greater than the corresponding with investment only in the bond; of course in all of them q verifies $q^\top\theta < 0$.

[INSERT TABLE 6 HERE]

Finally, in all the cases of Tables 4, 5, 6 there is no dependency with respect to the correlations.

6 Conclusions

We have analyzed the management of a pension funding process of an aggregated defined benefit pension plan where the benefits are stochastic. The objective is to determine contributions and investments strategies maximizing the expected terminal fund and at the same time minimizing both the contribution risk and the variance of the unfunded actuarial liability. The problem is formulated as a modified mean variance optimization problem and has been solved by means of dynamic programming techniques.

The efficient frontier has a parabolic form, but it is not a perfect square because it is modified by a constant due to the randomness of benefits and to correlations between risky assets and benefits. This effect appears also in the optimal investment strategies, with a term depending on the current level of the actuarial liability modified by a factor involving the instantaneous variance of benefits and of risky assets, and its correlation. Other summand depends on the preferences of the sponsor, that is, of the relative importance of the objectives in the minimization functional. The remainder summand is proportional to the present expected value of debt reduction. We have also found what seems to be a new result in the literature of DB pension funds due to the stochastic character of the pension plan: there is a linear relationship between the optimal supplementary cost and the vector of optimal investment strategies, given in (16). A correction term is present due to the random behavior of benefits.

We have also proved that under suitable conditions about the sign of the correlations, the total expected contribution is lesser when the investment is in the mixed portfolio than when it is in the bond only. On the other hand, borrowing money provides the sponsor an efficient strategy with the minimal variance and minimal total contribution but, of course, the expected reduction of the actuarial liability is lesser than under the mixed portfolio.

A numerical illustration shows the analytical results proved in the paper, as well as other features of the model.

Further research should be directed to include: no–shortselling and no–borrowing restrictions, final bankruptcy prohibition, stochastic riskless rate of interest and other biobjective problems.

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A Appendix

Proof of Proposition 3.1. The proof relies in a standard separation argument for concave functions. It follows the arguments in Zhou and Li (2000), but we have an extra term in the form of the running cost giving the contribution risk. Let $(SC^*, \mathbf{\Lambda}^*)$ be an optimal solution of problem (1), (9), (11), with associated process X^*, where $\gamma = \mu^{-1}/2 + \mathbb{E}X^*(T)$. Let us suppose it is not optimal solution of (1), (9), (12). Then there exists an admissible strategy $(SC, \mathbf{\Lambda})$ such that the associated path X verifies $J(SC, \mathbf{\Lambda}) < J(SC^*, \mathbf{\Lambda}^*)$, that is to say,

$$\begin{aligned} &\mathbb{E}\int_0^T SC^2(t)dt - \mathbb{E}\int_0^T (SC^*)^2(t)dt \\ &\quad + \mathbb{E}X^2(T) - \mathbb{E}(X^*)^2(T) - 2\gamma\left(\mathbb{E}X(T) - \mathbb{E}X^*(T)\right) < 0. \end{aligned} \tag{18}$$

The function $g(y_1, y_2, y_3) = \mu(y_1 + y_3) - \mu y_2^2 - y_2$ is concave in $\mathbb{R}^3$ because the Hessian matrix is matrix is negative semi–definite. Observe

$$g\left(\mathbb{E}X^2(T), \mathbb{E}X(T), \mathbb{E}\int_0^T SC^2(t)dt\right) = J_1(SC, \mathbf{\Lambda}) + \mu J_2(SC, \mathbf{\Lambda}),$$

which is the objective function of problem (1), (9), (11).

The concavity[3] of g and (18) imply

$$\begin{aligned} &g\left(\mathbb{E}X^2(T), \mathbb{E}X(T), \mathbb{E}\int_0^T SC^2(t)dt\right) \\ &\quad \le g\left(\mathbb{E}(X^*)^2(T), \mathbb{E}X^*(T), \mathbb{E}\int_0^T (SC^*)^2(t)dt\right) \\ &\quad\quad + \mu\left(\mathbb{E}X^2(T) - \mathbb{E}(X^*)^2(T)\right) - \left(1 + 2\mu\mathbb{E}X^*(T)\right)\left(\mathbb{E}X(T) - \mathbb{E}X^*(T)\right) \\ &\quad\quad + \mu\left(\mathbb{E}\int_0^T SC^2(t)dt - \mathbb{E}\int_0^T (SC^*)^2(t)dt\right) \\ &\quad \le g\left(\mathbb{E}(X^*)^2(T), \mathbb{E}X^*(T), \mathbb{E}\int_0^T (SC^*)^2(t)dt\right) \\ &\quad\quad + \mu\Bigg(\mathbb{E}X^2(T) - \mathbb{E}(X^*)^2(T) - 2\gamma\left(\mathbb{E}X(T) - \mathbb{E}X^*(T)\right) \\ &\quad\quad\quad + \mathbb{E}\int_0^T SC^2(t)dt - \mathbb{E}\int_0^T (SC^*)^2(t)dt\Bigg) \\ &\quad < g\left(\mathbb{E}(X^*)^2(T), \mathbb{E}X^*(T), \mathbb{E}\int_0^T (SC^*)^2(t)dt\right). \end{aligned} \tag{19}$$

[3]If $g : \mathbb{R}^3 \to \mathbb{R}$ is a concave function of class $\mathcal{C}^1$, then $\forall \overline{x}, \overline{y} \in \mathbb{R}^3$, $g(\overline{x}) - g(\overline{y}) \le \nabla g(\overline{y})(\overline{x} - \overline{y})$, where $\nabla g(\overline{y})$ denotes the gradient vector of g at $\overline{y}$, i.e. $\nabla g(\overline{y}) = (g_{y_1}, g_{y_2}, g_{y_3})$.

Therefore $J_1(SC,\mathbf{\Lambda})+\mu J_2(SC,\mathbf{\Lambda}) < J_1(SC,\mathbf{\Lambda})+\mu J_2(SC^*,\mathbf{\Lambda}^*)$, by (19), that is to say, $(SC^*,\mathbf{\Lambda}^*)$ is not optimal for (1), (9), (11), which is a contradiction. $\square$

Proof of Theorem 4.1. In order to prove this result we use the dynamic programming approach, see Fleming and Soner (1993). Consider the value function of the control problem (1)–(9)–(12),

$$\widehat{V}(t,X,AL) = \min_{(SC,\mathbf{\Lambda})\in\mathcal{A}_{X,AL}} \{J((t,X,AL);SC,\mathbf{\Lambda}) : \text{s.t. } (1), (9)\}.$$

It is well known $\widehat{V}$ is solution of the HJB equation:

$$\begin{aligned} V_t + \min_{SC,\Lambda}\Big\{ & SC^2 + (rX + \mathbf{\Lambda}^\top(\mathbf{b}-r\mathbf{1}) + SC - \eta q^\top\theta AL)V_X \\ & + \kappa AL\, V_{AL} + \frac{1}{2}(\mathbf{\Lambda}^\top\Sigma\mathbf{\Lambda} - 2\eta AL\,\mathbf{\Lambda}^\top\sigma q + \eta^2 AL)V_{XX} \\ & + \frac{1}{2}\eta^2 AL^2 V_{AL,AL} + (\eta AL\,\mathbf{\Lambda}^\top\sigma q - \eta^2 AL^2)V_{X,AL}\Big\} = 0, \end{aligned} \tag{20}$$

$$V(T,X,AL) = X^2 - 2\gamma X. \tag{21}$$

Note that in (20) we have used (9) and the SDE of AL as function of the Brownian motions $\{w_i\}_{i=0}^n$, obtained from (1), that is

$$dAL(t) = \kappa AL(t)dt + \eta AL(t)\sqrt{1-q^\top q}\,dw_0(t) + \eta AL(t)q^\top dw(t).$$

If there exists a smooth solution V of this equation, strictly convex with respect to X, then the minimizer values of the supplementary cost and investments are given by

$$\widehat{SC}(V_X) = -\frac{V_X}{2}, \qquad \widehat{\mathbf{\Lambda}}(V_X,V_{XX},V_{X,AL}) = -\Sigma^{-1}(\mathbf{b}-r\mathbf{1})\frac{V_X}{V_{XX}} + \eta AL\,\sigma^{-\top}q\left(1-\frac{V_{X,AL}}{V_{XX}}\right). \tag{22}$$

After substitution of these values in (20) we obtain $\widehat{V}$ satisfies

$$\begin{aligned} V_t + rXV_X - \frac{1}{4}V_X^2 - \frac{1}{2}\theta^\top\theta\frac{V_X^2}{V_{XX}} + \kappa AL\,V_{AL} + \frac{1}{2}\eta^2 AL^2 V_{AL,AL} + \frac{1}{2}\eta^2 AL^2(1-q^\top q)V_{XX} \\ - \eta^2 AL^2(1-q^\top q)V_{X,AL} - \eta AL\,\theta^\top q V_X\frac{V_{X,AL}}{V_{XX}} - \frac{1}{2}\eta^2 AL^2 q^\top q\frac{V_{X,AL}^2}{V_{XX}} = 0, \end{aligned}$$

with the final condition (21). We try a quadratic solution of the form

$$\widehat{V}(t,X,AL) = \beta_0(t) + \beta_X(t)X + \beta_{AL}(t)AL + \beta_{XX}(t)X^2 + \beta_{AL,AL}(t)AL^2 + \beta_{X,AL}(t)X\,AL\,,$$

so that from (22) the optimal controls must be

$$\begin{aligned}\mathbf{\Lambda} &= \Sigma^{-1}(\mathbf{b}-r\mathbf{1})\left(\frac{-\beta_X}{2\beta_{XX}} - X - \frac{\beta_{X,AL}}{2\beta_{XX}}AL\right) + \eta AL\,\sigma^\top q\left(1-\frac{\beta_{X,AL}}{\beta_{XX}}\right),\\ SC &= -\frac{1}{2}\left(\beta_X + 2\beta_{XX}X + \beta_{X,AL}AL\right) = \beta_{XX}\left(-\frac{\beta_X}{2\beta_{XX}} - X - \frac{\beta_{X,AL}}{2\beta_{XX}}AL\right).\end{aligned} \tag{23}$$

The following ordinary differential equations are obtained for the above coefficients:

$$\dot\beta_0 = \frac{\beta_X^2}{4} + \frac{\theta^\top\theta}{4}\beta_X^2\beta_{XX}, \quad \beta_0(T)=0,$$

$$\dot\beta_X = (-r+\theta^\top\theta)\beta_X + \beta_X\beta_{XX}, \quad \beta_X(T) = -2\gamma, \tag{24}$$

$$\dot\beta_{AL} = -\kappa\beta_{AL} + \frac{1}{2}\left(\theta^\top\theta + \eta\theta^\top q\right)\frac{\beta_X\beta_{X,AL}}{\beta_{XX}} + \frac{1}{2}\beta_X\beta_{X,AL}, \quad \beta_{AL}(T)=0,$$

$$\dot\beta_{XX} = (-2r+\theta^\top\theta)\beta_{XX} + \beta_{XX}^2, \quad \beta_{XX}(T)=1. \tag{25}$$

$$\begin{aligned}\dot\beta_{AL,AL} = &-(2\kappa+\eta^2)\beta_{AL,AL} + \left(\frac{\theta^\top\theta}{4} + \frac{\eta}{2}\theta^\top q + \eta^2 q^\top q\right)\frac{\beta_{X,AL}^2}{\beta_{XX}}\\ &+\eta^2(1-q^\top q)(\beta_{X,AL}-\beta_{XX}) + \frac{\beta_{X,AL}^2}{4}, \quad \beta_{AL,AL}(T)=0,\end{aligned}$$

$$\dot\beta_{X,AL} = (-r-\kappa+\eta+\theta^\top\theta)\beta_{X,AL} + \beta_{XX}\beta_{X,AL}, \quad \beta_{X,AL}(T)=0. \tag{26}$$

The method of resolution of this system is standard. The solution of the equation (25), of Ricatti type, can be found for example in Kloeden–Platen (1999), p. 572,

$$\beta_{XX}(t) = f(t),$$

and using this explicit expression of β_{XX} we can obtain from (24) (see Arnold (1974), p. 139)

$$\beta_X(t) = -2\gamma e^{-r(T-t)}f(t).$$

Substituting in (26), β_{XAL} is given by

$$\dot\beta_{X,AL} = (-r-\kappa+\eta+\theta^\top\theta+f(t))\beta_{X,AL}, \quad \beta_{X,AL}(T)=0,$$

that is to say $\beta_{X,AL}=0$. Plugging these expressions into (23) we obtain (13) and (14), respectively. □

Proof of Theorem 4.2. Under the optimal feedback control (13)–(14), the stochastic differential equation for process X, (9), is:

$$dX(t) = \left((r-\theta^\top\theta - f(t))X(t) + (\theta^\top\theta + f(t))\gamma e^{-r(T-t)}\right) dt$$
$$- \eta\sqrt{1-q^\top q} AL(t) dw_0(t) + \theta^\top \left(\gamma e^{-r(T-t)} - X(t)\right) dw(t),$$

with $X(0) = X_0$. Applying the Ito's formula to X^2 we obtain

$$\begin{aligned} dX^2(t) =& 2\Big((r-\theta^\top\theta/2 - f(t))X^2(t) + f(t)\gamma e^{-r(T-t)}X(t) + (1/2)\theta^\top\theta\gamma^2 e^{-2r(T-t)} \\ &+(1/2)\eta^2(1-q^\top q)AL^2(t)\Big) dt \\ &- 2\eta\sqrt{1-q^\top q}AL(t)X(t)dw_0(t) + 2\theta^\top\left(\gamma e^{-r(T-t)}X(t) - X^2(t)\right) dw(t), \end{aligned}$$

with $X^2(0) = X_0^2$. Taking expectations on both previous stochastic differential equations we obtain that functions $m_1(t) = \mathbb{E}X(t)$ and $m_2(t) = \mathbb{E}X^2(t)$ satisfy the linear ordinary differential equations

$$\begin{aligned} \dot{m}_1(t) =& (r-\theta^\top\theta - f(t))m_1(t) + (\theta^\top\theta + f(t))\gamma e^{-r(T-t)}, \quad m_1(0) = X_0, \\ \dot{m}_2(t) =& (2r-\theta^\top\theta - 2f(t))m_2(t) + 2f(t)\gamma e^{-r(T-t)}m_1(t) \\ &+ \theta^\top\theta\gamma^2 e^{-2r(T-t)} + \eta^2(1-q^\top q)AL_0^2 e^{(2\kappa+\eta^2)t}, \quad m_2(0) = X_0^2, \end{aligned} \tag{27}$$

where in (27) we have used that $\mathbb{E}AL^2(t) = AL_0^2 e^{(2\kappa+\eta^2)t}$, by (1).

Following Arnold (1974), p. 139,

$$m_1(t) = \mathbb{E}X(t) = e^{\int_0^t(r-\theta^\top\theta - f(s))ds}\left(X_0 + (\theta^\top\theta + f(t))\gamma\int_0^t e^{-\int_0^s(r-\theta^\top\theta-f(v))dv}e^{-r(T-s)}ds\right),$$

that, after some calculations it is

$$\begin{aligned} \mathbb{E}X(t) =& e^{(r-\theta^\top\theta)t}\left(1 - c_1 e^{(2r-\theta^\top\theta)(T-t)}\right) \\ &\times\left(\frac{X_0}{1-c_1e^{(2r-\theta^\top\theta)T}} + \gamma e^{-(r-\theta^\top\theta)T}\left(\frac{e^{-\theta^\top\theta(T-t)}}{1-c_1e^{(2r-\theta^\top\theta)(T-t)}} - \frac{e^{-\theta^\top\theta T}}{1-c_1e^{(2r-\theta^\top\theta)T}}\right)\right), \end{aligned} \tag{28}$$

for all $t \in [0,T]$. For $t = T$ we have

$$\mathbb{E}X(T) = \alpha X_0 + \beta\gamma, \tag{29}$$

where

$$\beta = 1 - e^{-\theta^\top \theta T} \frac{1-c_1}{1-c_1 e^{(2r-\theta^\top\theta)T}} = 1 - e^{2rT} f(0),$$

$$\alpha = e^{rT}(1-\beta).$$

Analogously,

$$m_2(t) = \mathbb{E}X^2(t) = e^{\int_0^t (2r-\theta^\top\theta-2f(s))ds} \Bigg(X_0^2 + 2\gamma \int_0^t e^{-\int_0^s (2r-\theta^\top\theta-2f(v))dv} f(s) e^{-r(T-s)} m_1(s) ds$$
$$+\theta^\top\theta\gamma^2 \int_0^t e^{-\int_0^s (2r-\theta^\top\theta-2f(v))dv} e^{-2r(T-s)} ds$$
$$+\eta^2(1-q^\top q) A L_0^2 \int_0^t e^{-\int_0^s (2r-\theta^\top\theta-2f(v))dv} e^{(2\kappa+\eta^2)s} ds \Bigg),$$

that, after some calculations it is transformed in

$$\mathbb{E}X^2(T) = \delta X_0^2 + 2\alpha\gamma\epsilon X_0 + \gamma^2(\beta - (1-\beta)\epsilon) + \nu, \tag{30}$$

where

$$\delta = \alpha^2 e^{\theta^\top\theta T},$$

$$\epsilon = 1 - (1-\beta) e^{\theta^\top\theta T},$$

$$\nu = \eta^2(1-q^\top q) A L_0^2 e^{(2\kappa+\eta^2)T} \int_0^T \frac{e^{(2r-\theta^\top\theta-2\kappa-\eta^2)t}}{(1-c_1 e^{(2r-\theta^\top\theta)t})^2} dt.$$

In order to find the mean–variance efficient frontier we obtain the terminal variance:

$$\begin{aligned}
\mathbb{V}\mathrm{ar}X(T) &= \mathbb{E}X^2(T) - (\mathbb{E}X(T))^2 \\
&= \delta X_0^2 + 2\alpha\gamma\epsilon X_0 + \gamma^2(\beta-(1-\beta)\epsilon) + \nu - (\mathbb{E}X(T))^2 \\
&= \delta X_0^2 + 2\alpha\frac{1}{\beta}(\mathbb{E}X(T) - \alpha X_0)\epsilon X_0 + \frac{1}{\beta^2}(\mathbb{E}X(T)-\alpha X_0)^2(\beta-(1-\beta)\epsilon) + \nu - (\mathbb{E}X(T))^2 \\
&= \frac{1-\beta}{\beta}\frac{\beta-\epsilon}{\beta}\left(\mathbb{E}X(T) - e^{rT}X_0\right)^2 + \nu \\
&= \left(\frac{1-\beta}{\beta}\right)^2 \left(e^{\theta^\top\theta T} - 1\right)\left(\mathbb{E}X(T) - e^{rT}X_0\right)^2 + \nu,
\end{aligned}$$

where in the second equality we have used (30) and in the third one we have used (29). □

Proof of Proposition 4.1. By (13),

$$\begin{aligned}\mathbb{E}\int_0^T e^{-rt}SC^*(t,X(t))dt &= \int_0^T e^{-rt}f(t)\left(\gamma e^{-r(T-t)} - \mathbb{E}X(T)\right)dt\\ &= \frac{(1-c_1)e^{-\theta^\top\theta T}}{1-c_1e^{(2r-\theta^\top\theta)T}}\left(\gamma e^{-rT} - X_0\right)\int_0^T e^{2rt}dt\\ &= \pi\left(\mathbb{E}X(T) - e^{rT}X_0\right),\end{aligned}$$

where the second equality is due to (15) and (28), and the third to the definition of β and (29).

□

Proof of Corollary 4.1. We suppose the manager wishes an expected terminal fund $\mathbb{E}X(T) = z$ with both investment possibilities, in the mixed portfolio and in the fixed rent.

a) Consider the total supplementary cost as a function of the Sharpe ratio of the portfolio, $\theta^\top\theta$:

$$\overline{SC}(y) = \frac{e^{2rT}-1}{2r}e^{-rT}\left(z - e^{rT}X_0\right)\left(\frac{1}{\beta(y)} - 1\right), \quad y \in [0,T],$$

where

$$\beta(y) = 1 - e^{-Ty}\frac{1-c_1(y)}{1-c_1(y)e^{-T(-2r+y)}},$$

with $c_1(y) = 1/(-2r+y+1)$. The expected total supplementary cost in the first situation is $\overline{SC}(\theta^\top\theta)$ and in the second $\overline{SC}(0)$.

It is very easy to check

$$\beta'(y) = Te^{-Ty}\frac{(-2r+y)^2 + (-2r+y) + e^{-(-2r+y)T}/T}{\left(-2r+y+1-e^{-(-2r+y)T}\right)^2} > 0$$

by Assumption 3. Using $z \geq e^{rT}X_0$ we obtain $\overline{SC}'(y) < 0$, that is to say, $\overline{SC}$ is a strictly decreasing function in $\mathbb{R}$. Therefore $\overline{SC}(\theta^\top\theta) < \overline{SC}(0)$.

b) The expected total contribution is in the first case

$$\overline{C} = \frac{1-e^{-(r-\kappa)T}}{r-\kappa}NC_0 + \overline{SC}(\theta^\top\theta).$$

By Proposition 2.1 and Assumption 2, $NC_0 = P_0 + (\kappa - r - \eta q^\top \theta)AL_0$, that is smaller or equal to $P_0 + (\kappa - r)AL_0$, because $q^\top \theta \geq 0$. Total contribution in the second case is obtained making $\theta = 0$, so the proof finishes applying a). $\square$

Justification of Assumption 2. Consider first only one worker, with age x. Once the liability is valued for one worker, the aggregated case is easily obtained as it is shown below. The sponsor wishes to value at the current time t the asset $Y^x(t, P)$ consisting in paying P monetary units at the age of retirement d, where P is a geometric Brownian motion according to Assumption 1. Since P is not tradeable and we suppose the existence of two independent sources of uncertainty, to value Y^x we resort to equilibrium arguments, following the approach of Constantinides (1978). To simplify the exposition, let us consider that only a risky asset S exists, which is freely traded in the market. The multidimensional case is straightforward. The method considers that the risk uncorrelated with S is not priced. Consider the asset at any intermediate time, $Y^x(t+\tau, P)$, $0 \leq \tau \leq d - x$. Forming a portfolio with one unit of asset Y^x and ξ units of S, $R = Y^x + \xi S$, and applying Itô's Lemma, we have

$$\begin{aligned} dR &= dY^x + \xi dS \\ &= \left(Y_p^x \kappa P + \frac{1}{2} Y_{pp}^x \eta^2 P^2 + Y_\tau^x + \xi b S \right) dt + Y_p^x \eta P \, dB + \xi \sigma S \, dw \\ &= \left(Y_p^x \kappa P + \frac{1}{2} Y_{pp}^x \eta^2 P^2 + Y_\tau^x + \xi b S \right) dt + Y_p^x \eta P \sqrt{1 - q^2} dw_0 + \left(Y_p^x \eta P q + \xi \sigma S \right) dw. \end{aligned}$$

The first equality is the self–financing condition, the second one follows from Itô's Lemma, using that P is geometric Brownian motion, and the last equality uses $B = \sqrt{1 - q^2}\, w_0 + qw$. The selection $\xi = -Y_p^x \eta P q / \sigma S$ eliminates the risk related with Brownian w. We also disregard the risk orthogonal to it, that is, the risk related with w_0 is not priced. The total return of the hedge portfolio must be equal to the risk free rate of interest, $r(Y^x + \xi S)$. Thus we obtain the pricing PDE

$$rY^x = Y_\tau^x + Y_p^x P \left(\kappa - \frac{q\eta}{\sigma}(b - r) \right) + \frac{1}{2} Y_{pp}^x \eta^2 P^2$$

with boundary conditions $Y^x(t + d - x, P) = P$, $Y^x(t + \tau, 0) = 0$. The solution is

$$Y^x(t + \tau, P) = Pe^{-(r - \kappa + q\eta\theta)(d - x - \tau)},$$

hence at time of valuation t $(\tau = 0)$, $Y^x(t, P(t)) = P(t)e^{-(r-\kappa+q\eta\theta)(d-x)}$. Now, to obtain the total liability $AL\,(t)$ we aggregate the result for any age $x \in [a, d]$ having into account the way benefits accumulates depending on age, to obtain

$$AL\,(t) = \int_a^d Y^x(t, P(t))M(x)\,dx = P(t)\int_a^d e^{-(r-\kappa+q\eta\theta)(d-x)}M(x)\,dx.$$

On the other hand, the actuarial definition of AL given in Section 2 and Assumption 1 provide

$$AL\,(t) = P(t)\int_a^d e^{-(\delta-\kappa)(d-x)}M(x)\,dx.$$

See the proof of Proposition 2.1 in Josa–Fombellida and Rincón–Zapatero (2004). Comparing the expressions obtained, we conclude that δ must be chosen equal to $r + q\eta\theta$ in order to attain a risk–neutral valuation of the liabilities.

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Table 1

Terminal standard deviation $\sigma_{X(T)}$

Expected debt	uncorrelation $q^\top q = 0$				intermediate correlation $q^\top q = 0.5$				perfect correlation $q^\top q = 1$			
$z = \mathbb{E}X(T)$	$T = 1$	$T = 2$	$T = 5$	$T = 10$	$T = 1$	$T = 2$	$T = 5$	$T = 10$	$T = 1$	$T = 2$	$T = 5$	$T = 10$
-0.15	2.0029	2.7013	5.1545	14.1069	1.4163	1.9101	3.6448	9.9751	0.0184	0.0144	0.0112	0.0093
-0.10	2.0031	2.7013	5.1545	14.1069	1.4166	1.9102	3.6448	9.9751	0.0331	0.0240	0.0159	0.0115
-0.05	2.0034	2.7014	5.1546	14.1069	1.4170	1.9104	3.6448	9.9751	0.0478	0.0336	0.0206	0.0137
0	2.0038	2.7016	5.1546	14.1069	1.4176	1.9105	3.6449	9.9751	0.0626	0.0431	0.0253	0.0159

Table 2

Initial investment in the risky assets $(\lambda_1(0, X_0, AL_0) + \lambda_2(0, X_0, AL_0))/F_0$

Expected debt	$q = (0,0)^\top$				$q = (1/2, 1/2)^\top$				$q = (-1/2, -1/2)^\top$			
$z = \mathbb{E}X(T)$	$T=1$	$T=2$	$T=5$	$T=10$	$T=1$	$T=2$	$T=5$	$T=10$	$T=1$	$T=2$	$T=5$	$T=10$
−0.15	0.308	0.265	0.287	0.355	0.512	0.470	0.491	0.559	0.104	0.061	0.082	0.151
−0.10	0.555	0.441	0.406	0.438	0.759	0.645	0.610	0.642	0.351	0.237	0.202	0.234
−0.05	0.802	0.617	0.526	0.521	1.006	0.821	0.730	0.725	0.598	0.413	0.321	0.317
0	1.049	0.793	0.645	0.604	1.253	0.997	0.849	0.808	0.845	0.588	0.441	0.399

Expected debt	$q = (1/2, -1/2)^\top$				$q = (-1/2, 1/2)^\top$				$q = (\sqrt{2}/2, \sqrt{2}/2)^\top$			
$z = \mathbb{E}X(T)$	$T=1$	$T=2$	$T=5$	$T=10$	$T=1$	$T=2$	$T=5$	$T=10$	$T=1$	$T=2$	$T=5$	$T=10$
−0.15	0.215	0.173	0.194	0.262	0.401	0.358	0.380	0.448	0.597	0.554	0.575	0.644
−0.10	0.462	0.348	0.313	0.345	0.648	0.534	0.499	0.531	0.844	0.730	0.695	0.727
−0.05	0.709	0.524	0.433	0.428	0.895	0.710	0.618	0.614	1.091	0.906	0.814	0.810
0	0.956	0.700	0.552	0.511	1.142	0.886	0.738	0.696	1.338	1.081	0.934	0.892

Expected debt	$q = (-\sqrt{2}/2, -\sqrt{2}/2)^\top$				$q = (\sqrt{2}/2, -\sqrt{2}/2)^\top$				$q = (-\sqrt{2}/2, \sqrt{2}/2)^\top$			
$z = \mathbb{E}X(T)$	$T=1$	$T=2$	$T=5$	$T=10$	$T=1$	$T=2$	$T=5$	$T=10$	$T=1$	$T=2$	$T=5$	$T=10$
−0.15	0.019	−0.023	−0.002	0.066	0.177	0.134	0.155	0.224	0.439	0.397	0.418	0.486
−0.10	0.266	0.152	0.117	0.149	0.424	0.310	0.275	0.307	0.686	0.572	0.537	0.569
−0.05	0.513	0.328	0.237	0.232	0.671	0.486	0.394	0.390	0.933	0.748	0.657	0.652
0	0.760	0.504	0.356	0.315	0.918	0.661	0.514	0.472	1.181	0.924	0.776	0.735

Table 3

Total expected discounted contribution $\overline{C}$

Expected debt	$q=(0,0)^\top$				$q=(1/2,1/2)^\top$				$q=(-1/2,-1/2)^\top$			
$z=\mathbb{E}X(T)$	$T=1$	$T=2$	$T=5$	$T=10$	$T=1$	$T=2$	$T=5$	$T=10$	$T=1$	$T=2$	$T=5$	$T=10$
−0.15	0.210	0.399	1.145	3.333	0.202	0.382	1.091	3.171	0.218	$\underline{0.416}$	$\underline{1.199}$	$\underline{3.495}$
−0.10	0.249	0.434	1.170	3.347	0.241	0.417	1.116	3.185	0.257	0.451	$\underline{1.223}$	$\underline{3.509}$
−0.05	0.288	0.469	1.194	3.361	0.280	0.451	1.140	3.199	0.296	0.486	1.248	$\underline{3.523}$
0	0.328	0.503	1.219	3.375	0.320	0.486	1.165	3.213	0.336	0.521	1.273	$\underline{3.537}$

Expected debt	$q=(1/2,-1/2)^\top$				$q=(-1/2,1/2)^\top$				$q=(\sqrt{2}/2,\sqrt{2}/2)^\top$			
$z=\mathbb{E}X(T)$	$T=1$	$T=2$	$T=5$	$T=10$	$T=1$	$T=2$	$T=5$	$T=10$	$T=1$	$T=2$	$T=5$	$T=10$
−0.15	0.208	0.394	1.130	3.288	0.212	0.404	1.160	3.379	0.199	0.375	1.069	3.104
−0.10	0.247	0.429	1.155	3.302	0.251	0.438	1.185	3.393	0.238	0.409	1.094	3.118
−0.05	0.286	0.464	1.179	3.316	0.291	0.473	1.209	3.406	0.277	0.444	1.118	3.132
0	0.325	0.499	1.204	3.330	0.330	0.508	1.234	3.420	0.316	0.479	1.143	3.146

Expected debt	$q=(-\sqrt{2}/2,-\sqrt{2}/2)^\top$				$q=(\sqrt{2}/2,-\sqrt{2}/2)^\top$				$q=(-\sqrt{2}/2,\sqrt{2}/2)^\top$			
$z=\mathbb{E}X(T)$	$T=1$	$T=2$	$T=5$	$T=10$	$T=1$	$T=2$	$T=5$	$T=10$	$T=1$	$T=2$	$T=5$	$T=10$
−0.15	$\underline{0.221}$	$\underline{0.423}$	$\underline{1.221}$	$\underline{3.562}$	0.207	0.392	1.124	3.269	0.213	0.406	1.166	3.397
−0.10	0.260	$\underline{0.458}$	$\underline{1.246}$	$\underline{3.576}$	0.246	0.427	1.148	3.283	0.252	0.440	1.191	3.411
−0.05	0.300	0.493	$\underline{1.270}$	$\underline{3.590}$	0.285	0.462	1.173	3.297	0.292	0.475	1.216	3.425
0	0.339	0.528	$\underline{1.295}$	$\underline{3.604}$	0.324	0.497	1.198	3.311	0.331	0.510	1.240	3.440

Table 4

Total expected discounted supplementary cost

Expected debt	$\overline{SC}$			
$z = \mathbb{E}X(T)$	$T = 1$	$T = 2$	$T = 5$	$T = 10$
-0.15	0.049	0.053	0.059	0.060
-0.10	0.088	0.087	0.084	0.074
-0.05	0.127	0.122	0.108	0.088
0	0.167	0.157	0.133	0.102

Table 5

Total expected discounted supplementary cost:

safe investment

Expected debt	$\overline{SC}^0$			
$z = \mathbb{E}X(T)$	$T = 1$	$T = 2$	$T = 5$	$T = 10$
-0.15	0.059	0.067	0.089	0.118
-0.10	0.106	0.111	0.126	0.145
-0.05	0.153	0.156	0.163	0.173
0	0.200	0.200	0.200	0.200

Table 6

Total expected discounted contribution:

safe investment

Expected debt	$\overline{C}^0$			
$z = \mathbb{E}X(T)$	$T = 1$	$T = 2$	$T = 5$	$T = 10$
-0.15	0.220	0.413	1.175	3.391
-0.10	0.267	0.458	1.212	3.419
-0.05	0.314	0.502	1.249	3.446
0	0.361	0.546	1.286	3.473