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Stochastic differential games for which the open-loop equilibrium is subgame perfect

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Abstract It is generally admitted that a correct forecasting of uncertain variables needs Markov decision rules. In a dynamic game environment, this belief is reinforced if one focuses on credible actions of the players. Usually, subgame perfectness requires equilibrium strategies to be constructed on Markov rules. It comes as a surprise that there are interesting classes of stochastic differential games where the equilibrium based on open loop strategies is subgame perfect. This fact is well known for deterministic games. We explore here the stochastic case, not dealt with up to now, identifying different game structures leading to the subgame perfectness of the open-loop equilibrium.

Keywords Stochastic differential game · Open-loop strategies · Feedback strategies · Markov Perfect Nash Equilibrium

Mathematics Subject Classification (2000) 49N70 · 49N90 · 91A23

1 Introduction

The quality and type of information available to the players at the time of taking their decisions in dynamic games is of the utmost importance. The outcome of the game may have quite different properties. In general, the equilibrium based on open-loop strategies is not subgame perfect, as the information is not

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enough for the players to play optimally at intermediate stages of the game. On the contrary, closed-loop strategies, or more concretely, Markov strategies, are in many cases the correct tool to construct a subgame perfect Nash equilibrium[1]. Moreover, in a stochastic game, playing with Markov strategies seems to be the only way that the players may forecast the evolution of the uncertain environment. It comes as a surprise that, in some cases, open-loop equilibrium is subgame perfect, even in the presence of random shocks. Our purpose in this paper is to identify stochastic differential games with this property.

To our knowledge, the identification of games for which the open loop equilibrium is also a Markov Perfect Nash Equilibrium (MPNE henceforth), is studied here for the first time for stochastic differential games. Mehlmann [12] provides a nice synthesis of known deterministic games where the open loop equilibrium is also subgame perfect. They are classified into the classes of trilinear games, introduced by Clemhout and Wan [2], state redundant or separable games, Dockner et al. [3], Jorgensen et al. [10], and exponential games, see Reinganum [13]. It is the class of exponential games where the treatment given in Mehlmann [12], Theorem 4.9, is quite general, with arbitrary dimensions in the state space and in the strategy spaces of the players. This is due to the special structure of exponential games, and under a strong assumption, which is that the Hamiltonian game admits a unique equilibrium with a given functional dependence. This works well for the class of exponential games, but seems not to be applicable to other classes of games. We avoid the assumption on the Hamiltonian game and attack the problem directly, focusing on the structural aspects of the data defining the game.

Fershtman [5] introduced the concept of classes of equivalence of games to attack the problem for deterministic games. Two games are considered open-loop (or closed-loop) equivalent if they have the same open-loop (or closed-loop) Nash equilibria. It turns out that diffeomorphic transformations in the state variable give rise to games in the same equivalence class. Then the issue is to find a suitable transformation that leads the game into one where the open-loop equilibrium is independent of the initial state condition. Undisputedly, this is an elegant approach. To translate it to the stochastic case would be an interesting exercise in using stochastic calculus. However, one limitation with this approach is that the existence of solutions to the HJB equation is not provided, and neither is an expression for the value function of the players at hand. In consequence, the transformation approach does not indicate a method on how to find the open-loop MPNE. This fact is also stressed in Mehlmann [12]. We try a different approach, focusing on the structural properties of the objects defining the game. In this way, we are able to give the value function of the players in a quite explicit form. Our results hold for arbitrary dimensions in the state space, but restrict ourselves to one strategy foe each player. Under a typical concavity assumption on the Hamiltonian of the players, we cover the classical classes of deterministic games for which

[1] See Başar and Olsder [1], where the issue of information in dynamic games is thoroughly analyzed. Also, Fershtman [6] provides a clever analysis of the meaning of open–loop and closed–loop rules.

the open loop Nash equilibrium is subgame perfect, giving true extensions, providing testable hypothesis to sustain our findings, and more importantly, also dealing with stochastic differential games.

The paper is organized as follows. Section 2 defines the class of stochastic differential games we study and establishes the conditions for the game to admit an open-loop MPNE. We start from a separable structure in state and strategies, which allows us to cover most of the deterministic games already known to possess open-loop MPNE. Our main theorem not only provides testable assumptions on the game structure, but identifies both the open-loop MPNE and the value function of the players. The theory is applied to several games. Some of them come from deterministic games, for which it is well known that they admit open-loop MPNE. We show how the game could be extended to become stochastic—in several ways—by maintaining this feature. Others applications are new, even in the deterministic setting, where for instance we discover a new type of game with a power structure with an open-loop MPNE. An interesting application is given in Example 2, which studies the competition of oligopolistic firms operating under demand uncertainty. Section 3 studies the case where the game's uncertainty is correlated. Section 4 introduces a variation in the class of games studied in previous sections, by allowing an additive structure in the player's payoffs. Section 5 is devoted to some final remarks.

2 The game, conditions for open-loop MPNE and applications

Consider a stochastic differential game in which

(i) N is the number of players;
(ii) The time horizon is finite and fixed, $[0,T]$;
(iii) A generic strategy of player i, denoted u^i, takes values in a subset $U^i \subseteq \mathbb{R}$, that is, we assume that the strategy space of each player is one dimensional. A profile of strategies, $u = (u^1, \dots, u^N)$, consists of one strategy for each player, and takes values in $U = U^1 \times \cdots \times U^N$.
(iv) The vector of state variables $X(t) = (X_1(t), \dots, X_n(t)) \in \Omega \subseteq \mathbb{R}^n$ is driven by the SDE

$$dX_j(s) = F_j(s, X(s), u(s))\, ds + G_j(s, X(s), u(s)) dw_j(s), \quad X(t) = x, \tag{1}$$

$t \leq s \leq T$, for $j = 1, \dots, n$, where the functions $F_j, G_j : [0,T] \times \Omega \times U \longrightarrow \mathbb{R}$ are of class C^2, and $w_1, \dots, w_n$ are n independent standard Brownian motions defined in a suitable probabilistic space with the usual filtration. We will use lower case notation for the vectors $x = (x_1, \dots, x_n)$ of the state space Ω.
(v) The payoff of player i is defined as follows: given strategies for the rest of players $u_{-i} = (u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^N)$, player i, $i = 1, \dots, N$, seeks

to maximize

$$J^i(t,x,u_{-i};u^i) = \mathrm{E}_{tx}\left\{ \int_t^T e^{-\rho^i(s-t)}\, L^i(s,X(s),u^i(s),u_{-i}(s))\,ds \right. \\ \left. + e^{-\rho^i(T-t)} S^i(T,X(T)) \right\}, \tag{2}$$

where E_{tx} denotes the conditional expectation with respect to the initial condition (t,x) and where $L^i : [0,T]\times\Omega\times U \longrightarrow \mathbb{R}$ and $S^i : [0,T]\times\Omega \longrightarrow \mathbb{R}$ are of class C^2. The discount factor $\rho^i \geq 0$ is supposed to be constant[2].

(vi) We specify the following separable structure in strategies/state variables.

$$L^i(t,x,u^1,\ldots,u^N) = \ell^i(t,u^1,\ldots,u^N)h^i(x) + \beta^i(t), \quad i = 1,\ldots,N,$$
$$F_j(t,x,u^1,\ldots,u^N) = f_j(t,u^1,\ldots,u^N)k_j(x) + \gamma_j(t)m_j(x), \quad j = 1,\ldots,n,$$
$$G_j(t,x,u^1,\ldots,u^N) = g_j(t,u^1,\ldots,u^N)n_j(x), \quad j = 1,\ldots,n.$$

Definition 1 An *open-loop strategy* of player i is a time path function $u^i : [0,T] \longrightarrow U^i$. The set of all possible open-loop strategies of player i is denoted as $\mathcal{O}^i$.

Definition 2 A *Markov strategy* of player i is a decision rule function $u^i : [0,T]\times\Omega \longrightarrow U^i$ that is continuous in t and uniformly Lipschitz in x for each t. The set of all possible Markov strategies of player i is denoted as $\mathcal{M}^i$.

Note that $\mathcal{O}^i \subset \mathcal{M}^i$, that is, an open-loop strategy is a (degenerate) Markov strategy.

Definition 3 A *Markov Perfect Nash Equilibrium* (MPNE for short) is a profile $u^* \in \mathcal{M}^1\times\cdots\times\mathcal{M}^N$ such that for all $i = 1,\ldots,N$

$$J^i(t,x,u^*_{-i};(u^i)^*) \geq J^i(t,x,u^*_{-i};u^i)$$

for all $u^i \in \mathcal{M}^i$, for all initial condition $(t,x) \in [0,T]\times\Omega$.

Note that for a profile of path functions $\lambda(t) = (\lambda^1(t),\ldots,\lambda^N(t))$ to be a (degenerate) MPNE, $\lambda^i(t)$ must be stable against any deviation $u^i(t,x)$ in $\mathcal{M}^i$, that is

$$J^i(t,x,\lambda_{-i};\lambda^i) \geq J^i(t,x,\lambda_{-i};u^i).$$

So, given the equilibrium strategies $\lambda_{-i}(t)$ of the rest of the players, player i gains nothing by employing, instead of λ^i, any other decision rule that uses the information provided by X at the time of taking the decision. Of course this is why open-loop strategies are not, in general, subgame perfect.

A word on notation: what refers to players is indexed with superscripts, whereas the rest of the vectors and functions are indexed with subscripts. Given a function h and the vector $x = (x_1,\ldots,x_n)$, h_x is the gradient of x

[2] The results shown in the paper remain valid for a variable discount factor $e^{-\int_t^s \rho^i(r)dr}$, where $\rho^i(r) \geq 0$ is a continuous function. The changes in the equations shown in the paper are straightforward.

and h_{xx} the Hessian matrix. When x is a real variable, derivatives are denoted with primes, so h' and h'' are the first and second derivatives of h with respect to x. Derivative with respect to time is denoted as $\frac{d}{dt}$. For two vectors x, y of the same dimension, $x \cdot y = \sum_{j=1}^{n} x_j y_j$ is the scalar product.

Consider the following assumptions (A1)–(A2) listed below.

(A1) For $(t, x, u, p^i) \in [0, \infty) \times \Omega \times U \times \mathbb{R}^n$ and $n \times n$ symmetric matrix P^i, define the Hamiltonian

$$\begin{aligned} H^i(t, x, u, p^i, P^i) = L^i(t, x, u) + \sum_{j=1}^{n} p^i_j F_j(t, x, u) \\ + \frac{1}{2} \operatorname{trace}\left(G(t, x, u) G(t, x, u)^\top P^i\right). \end{aligned}$$

We suppose that for all $i = 1, \ldots, N$, for any $(t, x) \in [0, \infty) \times \Omega$, $u_{-i} \in U_{-i}$, $p^1, \ldots, p^N$ and P^i, the function

$$u^i \mapsto H^i(t, x, u^i | u_{-i}, p^i, P^i)$$

is strictly concave.

(A2) There are constants a^i_j, b^i_j, c^i_j, d^i_j, r^i_j, q^i_j, s^i, e^i, such that for all $i = 1, \ldots, N$

$$\sum_{j=1}^{n} a^i_j \frac{\partial f_j}{\partial u^i} + \frac{1}{2} \sum_{j=1}^{n} d^i_j \frac{\partial (g_j^2)}{\partial u^i} \neq 0, \tag{3}$$

and for all $i = 1, \ldots, N$, for all $j = 1, \ldots, n$

$$\begin{cases} k_j(x) \dfrac{\partial h^i}{\partial x_j}(x) = a^i_j h^i(x) + q^i_j, \\ m_j(x) \dfrac{\partial h^i}{\partial x_j}(x) = b^i_j h^i(x) + c^i_j, \\ n_j^2(x) \dfrac{\partial^2 h^i}{\partial x_j^2}(x) = d^i_j h^i(x) + r^i_j, \\ S^i(T, x) = s^i h^i(x) + e^i. \end{cases} \tag{4}$$

Also, the following conditions hold

$$\sum_{j=1}^{n} q^i_j \frac{\partial f_j}{\partial u^i}(t, u) = 0 \tag{5}$$

$$\sum_{j=1}^{n} r^i_j \frac{\partial (g_j^2)}{\partial u^i}(t, u) = 0 \tag{6}$$

for all $i = 1, \ldots, N$.

Assumption (A1) is a usual concavity assumption. Regarding (A2), it establishes the structural form of the game that makes it possible for an open-loop equilibrium to be subgame perfect. It requires that all data can be reproduced with the help of the function h^i and its derivatives of first and second order. It turns out that this is the structure of most of the (deterministic) differential games identified in the literature as supporting an open-loop MPNE. The virtue of (A2) is to isolate the central role of function h^i and to allow for an effective guess of the player's value function. Once one makes the correct guess for the value function, to solve for the maximizers in the player's Hamiltonian is more or less straightforward, but without the knowledge of the functional form of the value function, solving the HJB system is often impossible. The first two lines in (4) deal with the deterministic game and the third one deals with the random shocks. Note that m_j could be identically null, indicating that the drift has no non-homogeneous term. Also, the third identity in (4) is trivial if h^i is linear. The case h^i constant is excluded when g_j is independent of u, since, for the first line in (4) and (5), this would imply $a^i_j = 0$ for all j, contradicting (3). We will deal with this case afterwards, in Section 4.

Finally, the two conditions (5) and (6) in (A2) are a way to give a unified treatment to the different characteristics of the game. The latter condition is obviously fulfilled for arbitrary $r^i_j \neq 0$ if the instantaneous variance does not depend on the player's strategies, $G_j(t,x,u) \equiv G_j(t,x)$. Otherwise, in general, we should take $r^i_j = 0$. A similar comment applies to the former one. Observe that in deterministic problems, $\frac{\partial f_j}{\partial u^i} \neq 0$ is a non-degeneracy condition, in the sense that the players influencing the evolution of the deterministic dynamics is the usual framework. Thus, in deterministic games, $q^i_j = 0$. However, in the stochastic case, f_j might be independent of u and then q^i_j could be taken non null in (A2).

A deterministic game satisfying the two first lines in (A2) does not need to be state-separable (see Dockner et al. [3] for the conditions imposed on a game for being state separable).

Let $G(t,x,u)$ be the diagonal matrix with diagonal elements of the form $G_j(t,x,u) = g_j(t,u)n_j(x)$. For a function $\varphi^i : [0,T] \times \Omega \longrightarrow \mathbb{R}$ of class C^2, denoting by φ^i_{xx} the Hessian matrix of φ^i, we obtain the following expression

$$\operatorname{trace}(GG^\top \varphi^i_{xx}) = \sum_{j=1}^n g_j^2 n_j^2 \frac{\partial^2 \varphi^i}{\partial x_j^2}. \tag{7}$$

Also, for $i = 1, \ldots, N$, let the functions

$$\psi^i(t,u) = \frac{-\frac{\partial \ell^i}{\partial u^i}(t,u)}{\sum_{j=1}^n a^i_j \frac{\partial f_j}{\partial u^i}(t,u) + \frac{1}{2}\sum_{j=1}^n d^i_j \frac{\partial (g_j^2)}{\partial u^i}(t,u)}, \tag{8}$$

that are well defined, by (3).

Theorem 1 *Suppose that (A1) and (A2) hold and that the system of differential equations*

$$\begin{aligned} 0 &= \frac{d}{dt}\psi^i(t,\lambda(t)) - \rho^i\psi^i(t,\lambda(t)) + \ell^i(t,\lambda(t)) \\ &+ \psi^i(t,\lambda(t))\Big(\sum_{j=1}^n a_j^i f_j(t,\lambda(t)) + \sum_{j=1}^n b_j^i\gamma_j(t) + \frac{1}{2}\sum_{j=1}^n d_j^i (g_j(t,\lambda(t)))^2\Big), \end{aligned} \tag{9}$$

with final condition

$$\psi^i(T,\lambda(T)) = s^i, \tag{10}$$

for $i = 1,\dots,N$, admits a unique C^1 solution $\lambda(t) = (\lambda^1(t),\dots,\lambda^N(t))$, for which the system of differential equations

$$\begin{aligned} 0 &= \frac{d}{dt}\zeta^i(t) - \rho^i\zeta^i(t) + \beta^i(t) \\ &+ \psi^i(t,\lambda(t))\Big(\sum_{j=1}^n q_j^i f_j(t,\lambda(t)) + \sum_{j=1}^n c_j^i\gamma_j(t) + \frac{1}{2}\sum_{j=1}^n r_j^i (g_j(t,\lambda(t)))^2\Big), \end{aligned} \tag{11}$$

with final condition

$$\zeta^i(T) = e^i, \tag{12}$$

for $i = 1,\dots,N$, admits a unique C^1 solution $(\zeta^1(t),\dots,\zeta^N(t))$. Then $\lambda(t)$ is an MPNE of the stochastic differential game (1), (2), *and the value function is*

$$V^i(t,x) = \psi^i(t,\lambda(t))h^i(x) + \zeta^i(t). \tag{13}$$

Proof Let $\lambda = (\lambda^1(t),\dots,\lambda^N(t))$ be the solution of (9) with final condition (10). Note that (9) is uncoupled with (11). Let $(\zeta^1(t),\dots,\zeta^N(t))$ be the solution of (11) with final condition (12). We will show that $(V^1,\dots,V^N)$, defined in (13), satisfies the associated HJB system of PDEs[3]

$$-\rho^i V^i(t,x) + V_t^i(t,x) + \max_{v\in U^i}\Big\{H^i(t,x,v|\lambda_{-i},V_x^i(t,x),V_{xx}^i(t,x))\Big\} = 0, \tag{14}$$

for $0 \le t < T$, and $V^i(T,x) = S^i(T,x)$.

Let us compute $H_{u^i}^i(t,x,u,V_x^i,V_{xx}^i)$. We substitute below the expression (7) for $\varphi^i = V^i$, $\operatorname{trace}(GG^\top V_{xx}^i) = \sum_{j=1}^n g_j^2(t,u)n_j^2(x)\frac{\partial^2 V^i}{\partial x_j^2}$. Obviously, each V^i is $C^{1,2}$ with $\frac{\partial^2 V^i}{\partial t\partial x_j} = \frac{\partial^2 V^i}{\partial x_j\partial t}$. Moreover, $\frac{\partial V^i}{\partial x_j} = \psi^i\frac{\partial h^i}{\partial x_j}$ and $\frac{\partial^2 V^i}{\partial x_j^2} = \psi^i\frac{\partial^2 h^i}{\partial x_j^2}$. In

[3] See Fleming and Rishel [7] or Dockner et al. [4] for details about the HJB equation.

the computations below we use $\psi^i(t)$ instead of $\psi^i(t,\lambda(t))$, to shorten notation. We have, by using assumption (A2)

$$
\begin{aligned}
&\frac{\partial H^i}{\partial u^i}(t,x,u^i|\lambda_{-i},V^i_x,V^i_{xx})\\
&\quad=\frac{\partial \ell^i}{\partial u^i}(t,u^i|\lambda_{-i})h^i(x)+\sum_{j=1}^n \frac{\partial f_j}{\partial u^i}(t,u^i|\lambda_{-i})k_j(x)\frac{\partial V^i}{\partial x_j}(t,x)\\
&\qquad+\frac{1}{2}\sum_{j=1}^n \frac{\partial (g_j^2)}{\partial u^i}(t,u^i|\lambda_{-i})n_j^2(x)\frac{\partial^2 V^i}{\partial x_j^2}(t,x)\\
&\quad=\frac{\partial \ell^i}{\partial u^i}(t,u^i|\lambda_{-i})h^i(x)+\psi^i(t)\sum_{j=1}^n \frac{\partial f_j}{\partial u^i}(t,u^i|\lambda_{-i})k_j(x)\frac{\partial h^i}{\partial x_j}(x)\\
&\qquad+\frac{1}{2}\psi^i(t)\sum_{j=1}^n \frac{\partial (g_j^2)}{\partial u^i}(t,u^i|\lambda_{-i})n_j^2(x)\frac{\partial^2 h^i}{\partial x_j^2}(x)\\
&\quad=\frac{\partial \ell^i}{\partial u^i}(t,u^i|\lambda_{-i})h^i(x)\\
&\qquad+\psi^i(t)\Big(\sum_{j=1}^n a_j^i\frac{\partial f_j}{\partial u^i}(t,u^i|\lambda_{-i})+\frac{1}{2}\sum_{j=1}^n d_j^i\frac{\partial (g_j^2)}{\partial u^i}(t,u^i|\lambda_{-i})\Big)h^i(x)\\
&\qquad+\psi^i(t)\cancel{\sum_{j=1}^n q_j^i\frac{\partial f_j}{\partial u^i}(t,u^i|\lambda_{-i})}+\frac{1}{2}\psi^i(t)\cancel{\sum_{j=1}^n r_j^i\frac{\partial (g_j^2)}{\partial u^i}(t,u^i|\lambda_{-i})}\\
&\quad=\left(\frac{\partial \ell^i}{\partial u^i}(t,u^i|\lambda_{-i})\right.\\
&\qquad\left.-\frac{\partial \ell^i}{\partial u^i}(t,\lambda)\frac{\sum_{j=1}^n a_j^i\frac{\partial f_j}{\partial u^i}(t,u^i|\lambda_{-i})+\frac{1}{2}\sum_{j=1}^n d_j^i\frac{\partial (g_j^2)}{\partial u^i}(t,u^i|\lambda_{-i})}{\sum_{j=1}^n a_j^i\frac{\partial f_j}{\partial u^i}(t,\lambda)+\frac{1}{2}\sum_{j=1}^n d_j^i\frac{\partial (g_j^2)}{\partial u^i}(t,\lambda)}\right)h^i(x).
\end{aligned}
$$

To reach the last equality, we have used (5) and (6), as well as the definition of ψ^i given in (8). The bottom line is identically 0 at $u^i=\lambda^i$. Since, by (A1), the Hamiltonian is strictly concave in u^i for all $i=1,\dots,N$, critical points are unique and global maximum of the Hamiltonian. Thus,

$$
H^i(t,x,\overbrace{\lambda^i|\lambda_{-i}}^{\lambda},V^i_x,V^i_{xx})=\max_{v\in U^i}H^i(t,x,v|\lambda_{-i},V^i_x,V^i_{xx}) \tag{15}
$$

Let us check that the HJB system (14) holds. Plugging the expression of the functions V^i and its derivatives into the HJB system, using the structural relations (A2) and (15), we have that the l.h.s of the HJB equation for player

i is (we omit the arguments of some functions)

$$
\begin{aligned}
& -\rho^i V^i(t,x) + V_t^i(t,x) + H^i(t,x,\lambda,V_x^i(t,x),V_{xx}^i(t,x)) \\
& = -\rho^i V^i(t,x) + \frac{\partial V^i}{\partial t}(t,x) + \beta^i(t) + \sum_{j=1}^n \gamma_j(t) m_j(x) \frac{\partial V^i}{\partial x_j}(t,x) \\
& \quad + \ell^i(t,\lambda) h^i(x) + \sum_{j=1}^n f_j(t,\lambda) k_j(x) \frac{\partial V^i}{\partial x_j}(t,x) \\
& \quad + \frac{1}{2}\sum_{j=1}^n g_j^2(t,\lambda) n_j^2(x) \frac{\partial^2 V^i}{\partial x_j^2}(t,x) \\
& = -\rho^i(\psi^i h^i + \zeta^i) + \left(\frac{d}{dt}\psi^i\right) h^i + \frac{d}{dt}\zeta^i + \beta^i + \sum_{j=1}^n \gamma_j m_j \frac{\partial h^i}{\partial x_j}\psi^i \\
& \quad + \ell^i h^i + \sum_{j=1}^n f_j k_j \frac{\partial h^i}{\partial x_j}\psi^i + \frac{1}{2}\sum_{j=1}^n g_j^2 n_j^2 \frac{\partial^2 h^i}{\partial x_j^2}\psi^i \\
& = h^i \left[\frac{d}{dt}\psi^i + \left(-\rho^i + \sum_{j=1}^n b_j^i \gamma_j + \sum_{j=1}^n a_j^i f_j + \frac{1}{2}\sum_{j=1}^n d_j^i g_j^2 \right) \psi^i + \ell^i \right] \\
& \quad + \left[-\rho^i \zeta^i + \frac{d}{dt}\zeta^i + \beta^i + \psi^i \sum_{j=1}^n q_j^i f_j + \psi^i \sum_{j=1}^n c_j^i \gamma_j + \frac{1}{2}\psi^i \sum_{j=1}^n r_j^i g_j^2 \right].
\end{aligned}
$$

Both summands are 0 by (9) and (11), respectively. Thus, the HJB system of PDEs holds. Now, the final condition. Note that $V^i(T,x) = \psi^i(T,\lambda(T))h^i(x) + \zeta^i(T) = s^i h^i(x) + e^i \equiv S^i(T,x)$, since $\lambda_i(T)$ is such that $\psi^i(T,\lambda(T)) = s^i$ by (10), and $\zeta^i(T) = e^i$ by (12). To finish the proof, we resort to a Verification Theorem in Dockner et al. [4] that establishes that V^i is the value function of player i and then, that $\lambda(t)$ is an MPNE. □

Remark 1 The existence of a unique solution of (9) and final condition (10) in the interval $[0,T]$ is not trivial, as the differential equations involved are non linear. Suitable global Lipschitz bounds on the functions could be imposed such that the existence of a unique solution is assured. This requires that the system is not degenerate, in the sense that it is possible to solve for $\frac{d}{dt}\lambda(t)$ from

$$
\frac{d}{dt}\psi^i(t,\lambda(t)) = \psi_t^i(t,\lambda(t)) + \sum_{j=1}^N \psi_{u^j}^i(t,\lambda(t)) \frac{d}{dt}\lambda_j(t),
$$

which needs the $N \times N$ matrix $(\psi_{u^j}^i(t,u))$ to be invertible.

In what follows, when the game has only one state variable, we will omit subscripts in j. So, for instance, when $n = 1$, we will use X for X_1, $g(t,u)$ for $g_1(t,u)$, or a^i for a_1^i in (A2), and the same for the rest of the constants and variables.

In all the examples below we will check that (A2) holds. The concavity assumption (A1) will be checked in specific models. The coefficients ψ^i and $\zeta^i(t)$ of the value function, as well as the open-loop MPNE $\lambda(t)$, are determined by (9), (10), (11) and (12). In order to save space, we will not write these equations for every example, but only for those games which we can solve explicitly. We take for granted the existence of solutions in the rest of examples.

Example 1 The following is Example 3.3 in Fershtman [5]. Consider the deterministic differential game with payoff

$$J^i = \int_t^T \delta^i(s)(l^i(s,u(s))(1-X(s)) + \epsilon^i(s)X(s))ds, \quad i = 1,\dots,N,$$

and the evolution of the state variable governed by

$$dX = f(s,u)(1-X)ds, \quad X(t) = x.$$

This game, introduced by Fershtman, is a generalization of a one-player problems studied in Kamien and Schwartz [11]. Here, the state variable $X(t)$ is the cumulative probability that an event occurred before time t; $l^i(t,u)$ is the payoff to player i if the event did not occur; and $\epsilon^i(t)$ is the payoff to player i if the event occurred prior to time t. Looking at its structure, the game is of the linear state class; see in Section 2.3 below for a more general case.

If we rewrite the payoff functional as $\delta^i(t)(l^i(t,u)-\epsilon^i(t))(1-x)+\delta^i(t)\epsilon^i(t)$, then we identify $\ell^i(t,u) = \delta^i(t)(l^i(t,u)-\epsilon^i(t))$, $h^i(x) = 1-x$, $\beta^i(t) = \delta^i(t)\epsilon^i(t)$; moreover, $k(x) = 1-x$, $m(x) = 0$, $n(x) = 0$, and also $\rho^i = 0$. It is clear that the first line in (4), $k(h^i)' = a^i h^i + q^i$, is fulfilled for $a^i = -1$ and $q^i = 0$. Analogously for the remainder conditions in (A2), with $b^i = c^i = d^i = r^i = 0$. Assuming condition (A1) is satisfied, then the value function is $V^i(t,x) = \psi^i(t,\lambda(t))(1-x) + \zeta^i(t)$, where

$$\psi^i(t,u) = \frac{\delta^i(t)\frac{\partial l^i}{\partial u^i}(t,u)}{\frac{\partial f}{\partial u^i}(t,u)},$$

by (8), and $\lambda(t) = (\lambda^1(t),\dots,\lambda^N(t))$ satisfies (9), (10), while ζ^i also satisfies (11), (12). Under these premises, the game admits an open-loop MPNE (already proved in Fershtman [5]).

2.1 Games with exponential structure in the state variable

Reinganum [13] introduced deterministic games with a linear-quadratic structure in the strategies and exponential in the state variable. Reinganum proved that the open-loop equilibrium is subgame perfect. Later, Jørgensen [8] showed the same property for games having an exponential structure also in the strategies. Ferhstman [5] generalized this result by allowing any structure in the strategies (taking for granted that an MPNE exists, although not explicitly stated). We analyze a stochastic extension of the game. It is defined by

$$J^i = E_{tx}\int_t^T e^{-\rho^i(s-t)}\ell^i(s,u(s))\exp\left(a^i\cdot X(s)\right)ds \\ + e^{-\rho^i(T-t)}s^i E_{tx}\exp\left(a^i\cdot X(T)\right),$$

where $a^i = (a^i_1, \ldots, a^i_n)$, and the dynamics is

$$dX_j = f_j(s,u)ds + g_j(s,u)dw_j(s), \quad j = 1,\ldots,n.$$

We identify $h^i(x) = \exp\left(a^i\cdot x\right)$, $n_j = 1$, $\beta^i = \gamma_j = m_j = 0$ and $k_j = 1$. Note that $S^i(x) = s^i h^i(x)$. It is easy to check that the rest of the conditions in (A2) holds, with the constants as indicated

$$k_j\frac{\partial h^i}{\partial x_j} = a^i_j\exp\left(a^i\cdot x\right) = a^i_j h^i, \qquad (a^i_j = a^i, q^i_j = 0)$$
$$n_j^2\frac{\partial^2 h^i}{\partial x_j^2} = (a^i_j)^2\exp\left(a^i\cdot x\right) = (a^i_j)^2 h^i \qquad (d^i_j = (a^i_j)^2, r^i_j = 0).$$

When the vector $a^i \equiv a = (a_1,\ldots,a_n)$ is the same for all players $i = 1,\ldots,N$, the dynamics can be made more general

$$dX_j = \left(f_j(s,u) + \gamma_j(s)(b_j + c_j\exp\left(-a\cdot X\right))\right)ds + g_j(s,u)dw_j(s),$$

$j = 1,\ldots,N$. We have

$$k_j\frac{\partial h^i}{\partial x_j} = a_j\exp\left(a\cdot x\right) = a_j h^i, \quad (a^i_j = a_j, q^i_j = 0)$$
$$m_j\frac{\partial h^i}{\partial x_j} = (b_j + c_j\exp\left(-a\cdot x\right))\exp\left(a\cdot x\right)a_j = a_j b_j h^i + a_j c_j$$
$$(b^i_j = a_j b_j, c^i_j = a_j).$$

Within this case with $a^i \equiv a$, if g_j does not depend on u for all $j = 1,\ldots,n$, then we can consider a more general diffusion coefficient, $n_j^2(x) = \sigma_{1j}^2 + \sigma_{2j}^2\exp\left(-a\cdot x\right)$, where σ_{ij} are constants, $i = 1,2$. Hence, the dynamics is now

$$dX_j = \left(f_j(s,u) + \gamma_j(s)(b_j + c_j\exp\left(-a\cdot X\right))\right)ds \\ + g_j(s)\sqrt{\sigma_{1j}^2 + \sigma_{2j}^2\exp\left(-a\cdot X\right)}dw_j(s).$$

Let us check that (A2) holds.

$$n_j^2\frac{\partial^2 h^i}{\partial x_j^2} = a_j^2\left(\sigma_{1j}^2 + \sigma_{2j}^2\exp\left(-a\cdot x\right)\right)\exp\left(a\cdot x\right) = a_j^2\sigma_{1j}^2 h^i + a_j^2\sigma_{2j}^2,$$

so we identify $d^i_j = a_j^2\sigma_{1j}^2$, $r^i_j = a_j^2\sigma_{2j}^2$ for all $i = 1,\ldots,N$, $j = 1,\ldots,n$. Observe that $r^i_j \neq 0$ is possible since $\frac{\partial g_j}{\partial u^i} = 0$ for all i,j, by (6).

According to Theorem 1, if condition (A1) holds and (9), (10), (11), (12) admit solution, then there is an open-loop MPNE.

2.2 Games with power structure in the state variable

We may extend Fershtman's Example 3.4 in several dimensions and not only because we consider the stochastic game. We can consider different powers of x in the payoff functional for each player, and a more general drift term in the dynamics.

2.2.1 Case 1

Let $h^i(x) = x_1^{a_1^i} \cdots x_n^{a_n^i}$ and let

$$dX_j = f_j(s,u)X_j ds + g_j(s,u)X_j dw_j(s), \quad j = 1, \ldots, N.$$

Hence, $k_j(x) = n_j(x) = x_j$. It is easy to check that (A2) holds.

$$k_j \frac{\partial h^i}{\partial x_j} = a_j^i x_j h^i x_j^{-1} = a_j^i h^i, \qquad (q_j^i = 0)$$
$$n_j^2 \frac{\partial^2 h^i}{\partial x_j^2} = a_j^i(a_j^i - 1)x_j^2 h^i x_j^{-2} = a_j^i(a_j^i - 1)h^i, \qquad (d_j^i = a_j^i(a_j^i - 1),\ r_j^i = 0).$$

2.2.2 Case 2

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $x = (x_1, \ldots, x_n)$, and $h^i(x) = (\alpha_1 x_1 + \cdots + \alpha_n x_n)^{a^i} = (\alpha \cdot x)^{a^i}$. Let

$$dX_j = f_j(s,u)(\alpha \cdot X)dt + g_j(s,u)(\alpha \cdot X)dw_j(s), \quad j = 1, \ldots, N.$$

(A2) holds

$$k_j \frac{\partial h^i}{\partial x_j} = a^i \alpha_j (\alpha \cdot x)(\alpha \cdot x)^{a^i - 1} = a^i \alpha_j h^i, \quad (a_j^i = a^i \alpha_j,\ q_j^i = 0)$$
$$n_j^2 \frac{\partial^2 h^i}{\partial x_j^2} = a^i(a^i - 1)\alpha_j^2 (\alpha \cdot x)^2 (\alpha \cdot x)^{a^i - 2} = a^i(a^i - 1)\alpha_j^2 h^i,$$
$$(d_j^i = a^i(a^i - 1)\alpha_j^2,\ r_j^i = 0).$$

There are more possibilities in the case that $a_1 = \cdots = a_N = a$. Then $h^i(x) = (\alpha \cdot X)^a$ for all $i = 1, \ldots, N$. Now let the dynamics be

$$dX_j = \big(f_j(s,u)(\alpha \cdot X) + \gamma_j(s)(\mu_j(\alpha \cdot X) + \nu_j(\alpha \cdot X)^{1-a})\big)\, ds$$
$$+ g_j(s,u)(\alpha \cdot x)dw_j(s),$$

$j = 1, \ldots, N$, where μ_j and ν_j are constants. Assumption (A2) is fulfilled.

$$k_j \frac{\partial h^i}{\partial x_j} = a\alpha_j(\alpha \cdot X)(\alpha \cdot X)^{a-1} = a\alpha_j h^i, \quad (a_j^i = a\alpha_j,\ q_j^i = 0)$$
$$m_j \frac{\partial h^i}{\partial x_j} = a\alpha_j(\mu_j(\alpha \cdot X) + \nu_j(\alpha \cdot X)^{1-a})(\alpha \cdot X)^{a-1} = a\alpha_j\mu_j h^i + a\alpha_j\nu_j,$$
$$(b_j^i = a\alpha_j\mu_j,\ c_j^i = a\alpha_j\nu_j).$$

Within the case with $a_1 = \cdots = a_N = a$, assume further that g_j is independent of u for all j. Then a more general diffusion coefficient is possible. Now let the dynamics be

$$\begin{aligned} dX_j &= \big(f_j(s,u)(\alpha \cdot X) + \gamma_j(s)(\mu_j(\alpha \cdot X) + \nu_j(\alpha \cdot X)^{1-a})\big)\, ds \\ &\quad + g_j(s)\sqrt{\sigma_{1j}^2(\alpha \cdot X)^2 + \sigma_{2j}^2(\alpha \cdot X)^{2-a}}dw_j(s), \end{aligned}$$

$j = 1, \ldots, N$, where σ_{ij} are constants, $i = 1, 2$. Hence, $n_j^2(x) = \sigma_{1j}^2(\alpha \cdot x)^2 + \sigma_{2j}^2(\alpha \cdot x)^{2-a}$. Let us check that the condition regarding the diffusion coefficient in (A2) holds (the others have just been tested).

$$\begin{aligned} n_j^2 \frac{\partial^2 h^i}{\partial x_j^2} &= (\sigma_{1j}^2(\alpha \cdot X)^2 + \sigma_{2j}^2(\alpha \cdot X)^{2-a})a(a-1)\alpha_j^2(\alpha \cdot X)^{a-2} \\ &= a(a-1)\sigma_{1j}^2\alpha_j^2 h^i + a(a-1)\sigma_{2j}^2\alpha_j^2. \end{aligned}$$

Here $d_j^i = a(a-1)\sigma_{1j}^2\alpha_j^2$ and $r_j^i = a(a-1)\sigma_{2j}^2\alpha_j^2$. As above, note that now $r_j^i \neq 0$ is admissible, by (6).

According to Theorem 1, if condition (A1) holds and (9), (10), (11), (12) admit solution, then there is an open-loop MPNE.

Example 2 Consider a resource extraction game where N firms extract a resource at rate u_i, $i = 1, \ldots, N$, in oligopolistic competition. The unitary price of the resource is

$$p(x_1, x_2, u) = x_2^b\Big(x_1 \sum_{i=1}^N u^i\Big)^{-a},$$

with $0 < a < 1$, $b \geq 0$, and where $x_1 > 0$ is the resource stock, $x_2 > 0$ is a parameter defining the demand function and u is the strategy's profile of the players. Under exploitation, the resource follows the SDE

$$dX_1 = \Big(\theta X_1 - \Big(\sum_{j=1}^N u^j\Big)X_1\Big)ds + \sigma_1 X_1 dw_1(s), \qquad X_1(t) = x_1 > 0,$$

with $\theta \geq 0$ being the natural rate of growth of the population. Note that $f_1(u) = \theta - \sum_{j=1}^N u^j$. The effort rate u^i translates into an effective rate of captures of $x_1 u^i$ when the stock level of the resource is $X_1 = x_1$. Thus, the lower the resource, the harder it is to maintain the effective extraction rate. Demand varies stochastically, which is reflected in the parameter $x_2 = X_2$, which follows the SDE

$$dX_2 = \Big(\mu - \nu\Big(\sum_{j=1}^N u^j\Big)\Big)X_2 ds + \sigma_2 X_2 dw_2(s), \quad X_2(t) = x_2 > 0.$$

The interpretation of the term $f_2(u) = \mu - \nu\Big(\sum_{j=1}^N u^j\Big)$, where μ and ν are constants, is that extraction effort affects future sales prices, maybe due to the possibility that the produced good can be stored through time—maybe by using a suitable technology. So, the higher the global extraction effort, the lower the expected future price of the good will fall. Under time path strategies, both processes X_1 and X_2 almost surely remain positive.

Each firm seeks to maximize in u^i

$$
\begin{aligned}
J^i(t, x_1, x_2, u_{-i}; u^i) & \\
&= E_{tx_1x_2} \int_t^T e^{-\rho^i(s-t)} p(X_1(s), X_2(s), u(s)) u^i(s) X_1(s) ds \\
&\quad + e^{-\rho^i(T-t)} E_{tx_1x_2} \left(p(X_1(T), X_2(T), \overline{u}) X_1(T) \overline{u}^i \right) \\
&= E_{tx_1x_2} \int_t^T e^{-\rho^i(s-t)} X_1(s)^{1-a} X_2(s)^b u^i(s) \Big(\sum_{j=1}^N u^j(s) \Big)^{-a} ds \\
&\quad + e^{-\rho^i(T-t)} s^i E_{tx_1x_2} \left(X_1(T)^{1-a} X_2(T)^b \right),
\end{aligned}
$$

subject to the evolution of X_1 and X_2. The cost of extraction of the resource is zero. The bequest function $S^i(T, x_1, x_2) = p(x_1, x_2, \overline{u}) x_1 \overline{u}^i$ accounts for the value of the resource at the termination time T. The parameter $\overline{u}^i \equiv u^i(T)$ stands for the extraction rate of the players at T—unknown so far—and $\overline{u} = (\overline{u}^1, \ldots, \overline{u}^N)$. This unknown value will be fixed in equilibrium. We think that this is one of the possible ways to fix the bequest at the terminal time, something that is not always straightforward to do[4]. The constant s^i is $s^i = \overline{u}^i \Big(\sum_{j=1}^N \overline{u}_j \Big)^{-a}$. We identify $\ell^i(u) = u^i \Big(\sum_{j=1}^N u^j(s) \Big)^{-a}$, $h^i(x_1, x_2) = x_1^{1-a} x_2^b$, $i = 1, 2$, f_1 and f_2 as defined above, $k_1(x_1, x_2) = x_1$, $k_2(x_1, x_2) = x_2$, $m_1(x_1, x_2) = m_2(x_1, x_2) = 0$, $n_1(x_1, x_2) = x_1$, $n_2(x_1, x_2) = x_2$. Moreover, $\beta^1(t) = \ldots = \beta^N(t) = 0$, $\gamma_1(t) = \gamma_2(t) = 0$, $g_1(t, u) = \sigma_1$, and $g_2(t, u) = \sigma_2$. The game is one of power structure in the state variables studied above. In fact

$$
\begin{aligned}
k_1 \frac{\partial h^i}{\partial x_1} &= (1-a) h^i, & k_2 \frac{\partial h^i}{\partial x_2} &= b h^i, \\
n_1^2 \frac{\partial^2 h^i}{\partial x_1^2} &= -a(1-a) h^i, & n_2^2 \frac{\partial^2 h^i}{\partial x_2^2} &= -b(1-b) h^i.
\end{aligned}
$$

The function ψ^i in (8) is

$$
\psi^i(u) = \frac{\Big(\sum_{j=1}^N u^j \Big)^{-(1+a)} \Big(\Big(\sum_{j=1}^N u^j \Big) - a u^i \Big)}{1 - a + b\nu}.
$$

Since the game is symmetric, we look for a symmetric equilibrium. Thus, we denote each individual strategy with $\lambda = \lambda^1 = \cdots = \lambda^N$ (not to be confused

[4] If no bequest function is imposed, then the extraction rate equilibrium becomes unbounded at the terminal time. The model is similar to that studied in Reinganum and Stokey [14]. Uncertainty does not mitigate the problem.

with a profile of strategies). Then $\psi(u) = \psi^i(u,\dots,u)$ is given by

$$\psi(u) = \frac{N^{-(a+1)}(N-a)u^{-a}}{1-a+b\nu}.$$

Moreover, $\ell(u) = \ell^1(u) = \cdots = \ell^N(u) = N^{-a}u^{1-a}$. Also, $s = s^1 = \cdots = s^N = N^{-a}\overline{u}^{1-a}$. The ODE (9) is

$$\begin{aligned} 0 = \frac{d}{dt}\psi + N^{-a}u^{1-a} - \rho\psi + \psi\Big(& -(1-a)Nu + b(\mu - \nu Nu) + \theta(1-a) \\ & + \frac{1}{2}\big(-\sigma_1^2 a(1-a) - \sigma_2^2 b(1-b)\big)\Big). \end{aligned}$$

Let the constants

$$\begin{aligned} A &= \rho - \theta(1-a) + \frac{1}{2}\sigma_1^2 a(1-a) + \frac{1}{2}\sigma_2^2 b(1-b) - b\mu, \\ B &= \Big((1-a)N + b\nu N\Big)D^{\frac{1}{a}} - N^{-a}D^{\frac{1}{a}-1}, \\ D &= \frac{N^{-(a+1)}(N-a)}{1-a+b\nu}. \end{aligned}$$

Using the equality $\psi = Du^{-a}$, (9) can be written solely in terms of ψ as

$$\dot{\psi} = A\psi + B\psi^{1-\frac{1}{a}}.$$

It can be explicitly integrated

$$\ln\left(\frac{A\psi(t)^{\frac{1}{a}} + B}{As^{\frac{1}{a}} + B}\right) = \frac{A}{a}(t-T),$$

using $\psi(T) = s$. Noting that $\psi^{\frac{1}{a}} = D^{\frac{1}{a}}\lambda(t)^{-1}$, plugging in this value and solving for $\lambda(t)$, we get

$$\lambda(t) = \frac{AD^{\frac{1}{a}}}{(As^{\frac{1}{a}} + B)e^{\frac{A}{a}(t-T)} - B}.$$

Now, in equilibrium, it must be $\lambda(T) = \overline{u}$, thus $(D/s)^{\frac{1}{a}} = \overline{u}$. Since $s = N^{-1}\overline{u}^{1-a}$, we can solve for $\overline{u} = DN^{\frac{a}{1-a}}$, thus $s = (D/N)^{1-a}$, a value that must be substituted into the expression of $\lambda(t)$ given above to obtain

$$\lambda(t) = \frac{AD^{\frac{1}{a}}}{(AD^{\frac{1}{a}-1}N^{1-\frac{1}{a}} + B)e^{\frac{A}{a}(t-T)} - B}.$$

On the other hand, the game satisfies the concavity condition (A1). To prove this, compute the second derivative of ℓ^i with respect to u^i

$$\ell^i_{u^iu^i}(u^1,\dots,u^N) = -a\left(\sum_{j=1}^N u^j\right)^{-a-2}\left(2\sum_{j=1}^N u^j - (a+1)u^i\right),$$

which is negative, since $0 < a < 1$ and $2\sum_{j=1}^N u^j - (a+1)u^i = 2\sum_{j\neq i} u^j + (1-a)u^i > 0$. As $x_1^{1-a}x_2^b > 0$ and both f_1 and f_2 are linear in u^i, the Hamiltonian H^i is strictly concave with respect to u^i.

Consider the following variation regarding the demand parameter X_2. Suppose now that the extraction effort also affects the volatility of X_2 according to

$$dX_2 = \Big(\mu - \nu\Big(\sum_{j=1}^N u^j\Big)\Big)X_2 ds + \sigma_2\Big(\sum_{j=1}^N u^j\Big)^{\frac{1}{2}} X_2 dw_2(s), \quad X_2(t) = x_2 > 0.$$

Now $g_2(u) = \sigma_2(\sum_{j=1}^N u^j)^{\frac{1}{2}}$ and (6) holds. Thus, the greater the effort, the higher the volatility in the demand function. The choice of this particular functional form is due to the aim of solving explicitly for the MPNE, but other choices would work as long as they respect the concavity assumption (A1) and the existence of a unique solution of (9). This equation changes to

$$\begin{aligned} 0 = \frac{d}{dt}\psi + N^{-a}u^{1-a} \; - \rho\psi + \psi\Big(- (1-a)Nu + b(\mu - \nu Nu) \\ + \theta(1-a) + \frac{1}{2}\big(- \sigma_1^2 a(1-a) - \sigma_2^2 b(1-b)Nu\big)\Big) \end{aligned} \tag{16}$$

which can be rewritten as $\dot{\psi} = \tilde{A}\psi + \tilde{B}\psi^{1-\frac{1}{a}}$, where

$$\begin{aligned} \tilde{A} &= \rho - \theta(1-a) - b\mu + \frac{1}{2}\sigma_1^2 a(1-a), \\ \tilde{B} &= \Big((1-a)N + b\nu N\Big)D^{\frac{1}{a}} - N^{-a}D^{\frac{1}{a}-1} + \frac{1}{2}\sigma_2^2 b(1-b)ND^{\frac{1}{a}}. \end{aligned}$$

The solution is obtained as in the above case, replacing the constants A and B by the new ones, as well as computing the correct $\overline{u}$. Moreover, the required concavity of H^i is also fulfilled. It is not the aim of this paper to go further into the study of this model.

Example 3 This is a modification of Yeung [15]. The game models a resource extraction problem in a competitive environment. Firms try to maximize profits in an oligopoly game with non-linear costs. In the original formulation, there are multiple branching processes together with the continuous time random fluctuations that drive the game's stock dynamics and payoffs. We eliminate the branching process but generalize some other aspects of the game as it is explained below. Yeung's game is obtained when $\tau = 2$ and $C^1 = \cdots = C^N = C$, $s^1 = \cdots = s^N = s$, that is, when the game is symmetric. Throughout the example, a, b, τ, σ and C^i, s^i, $i = 1, \ldots, N$, are positive constants, with $\tau > 1$.

$$\begin{aligned} &\max_{v^i} E_{tx}\int_t^T e^{-\rho^i(s-t)}\left(\frac{v^i(s)}{\big(\sum_{j=1}^N v^j(s)\big)^{\frac{1}{\tau}}} - C^i\frac{v^i(s)}{(X(s))^{\frac{1}{\tau}}}\right)ds + e^{-\rho^i(T-t)}s^i E_{tx}X(T)^{1-\frac{1}{\tau}} \\ &\text{s.t.:}\quad dX = \Big(aX^{\frac{1}{\tau}} - bX - \textstyle\sum_{j=1}^N v^j\Big)ds + \sigma X dw(s),\ X(t) = x. \end{aligned}$$

Upon the substitution $v^i = u^i x$, the game is transformed into

$$\begin{aligned}
&\max_{u^i} E_{tx} \int_t^T e^{-\rho^i(s-t)} X^{1-\frac{1}{\tau}} \left(\frac{u^i(s)}{\left(\sum_{j=1}^N u^j(s)\right)^{\frac{1}{\tau}}} - C^i u^i(s) \right) ds \\
&\qquad + e^{-\rho^i(T-t)} s^i E_{tx} X(T)^{1-\frac{1}{\tau}} \\
&\text{s.t.:} \quad dX = \left(aX^{\frac{1}{\tau}} - bX - \left(\sum_{j=1}^N u^j \right) X \right) ds + \sigma X dw(s), \; X(t) = x.
\end{aligned}$$

Using our notation and letting $0 < \kappa = 1 - \frac{1}{\tau} < 1$

$$\begin{aligned}
\ell^i(u^1, \ldots, u^N) &= \frac{u^i}{\left(\sum_{j=1}^N u^j\right)^{1-\kappa}} - C^i u^i, \\
f(u^1, \ldots, u^N) &= -\sum_{j=1}^N u^j.
\end{aligned}$$

Also, $h^i(x) = x^\kappa$, $k(x) = x$, $m(x) = ax^{1-\kappa} - bx$, $n(x) = \sigma x$, and $S^i(x) = s^i x^\kappa$. Moreover, $\beta^i(t) = 0$ for all i and $\gamma(t) = g(t,u) = 1$. Assumption (A2) holds, with the constants given in the curved parenthesis below

$$\begin{aligned}
k(x)(h^i)'(x) &= \kappa x^\kappa = \kappa h^i(x), \quad (a^i = \kappa, \; q^i = 0) \\
m(x)(h^i)'(x) &= (ax^{1-\kappa} - bx)\kappa x^{\kappa-1} = -\kappa b h^i(x) + a\kappa, \\
&\quad (b^i = -\kappa b, \; c^i = a\kappa) \\
n^2(x)(h^i)''(x) &= \sigma^2 x^2 \kappa(\kappa-1) x^{\kappa-2} = \sigma^2 \kappa(\kappa-1) h^i(x), \\
&\quad (d^i = \sigma^2 \kappa(\kappa-1), \; r^i = 0) \\
S^i(x) &= s^i x^\kappa = s^i h^i(x),
\end{aligned}$$

for all $i = 1, \ldots, N$. We do not carry out the computations to find explicitly the open-loop MPNE (in the symmetric case, as the asymmetric one is hard), as the problem is similar to the previous example.

2.3 Games with linear structure in the state variable

See Jorgensen *et al* [9] and Jorgensen *et al* [10] for an exhaustive analysis of linear state deterministic games. Here we consider a special case. Let $a = (a_1, \ldots, a_n)$ and let $h^i(x) = a \cdot x + \mu$, where $\mu \in \mathbb{R}$. Let $k_j(x) = g(x) = a \cdot x + \mu$, and $m_j(x) = G(x) = a \cdot x + \mu_j$, $\mu_j \in \mathbb{R}$ for all $j = 1, \ldots, n$. Hence, the payoff functional of player i is

$$J^i = E_{tx} \int_t^T e^{-\rho^i(s-t)} \ell^i(s, u(s))(a \cdot X(s) + \mu) ds + e^{-\rho^i(T-t)} E_{tx}(a \cdot X(T)),$$

and the dynamics of the game is

$$dX_j = (f_j(s,u)(a \cdot X + \mu) + \gamma_j(s)(a \cdot X + \mu_j)) \, ds + g_j(s,u) n_j(X) dw_j(s),$$

where n_j is any function. Note that n_j satisfies the third line of (4) in Assumption (A2). This is because the second derivative of h^i vanishes. It is easy to check that the rest of conditions in (A2) hold.

$$k_j \frac{\partial h^i}{\partial x_j} = a_j(a \cdot x + \mu) = a_j h^i, \qquad (a^i_j = 0, q^i_j = 0)$$
$$m_j \frac{\partial h^i}{\partial x_j} = a_j(a \cdot x + \mu_j) = a_j h^i + a_j(\mu_j - \mu) \qquad (b^i_j = a_j, c^i_j = a_j(\mu_j - \mu)).$$

There is a more ample class of linear state games that do not correspond to our structural assumptions. This is due to the special characteristics of linearity in the state variable. To cover this class of games, we should modify our approach. As it is a rather particular case, and the class has been extensively studied in the literature, both in theoretical and applied papers, we will not pursue this issue any further and instead refer the interested reader to the papers cited above.

Remark 2 We have supposed throughout this section that the diffusion matrix is diagonal. To analyze the more general case where the diffusion matrix is not diagonal is straightforward but cumbersome to write. Let us suppose that the state variables $X(t) = (X_1(t), \dots, X_n(t)) \in \Omega \subseteq \mathbb{R}^n$ now are driven by the SDE

$$dX_j(s) = F_j(s, X(s), u(s))\, ds + \sum_{k=1}^{n} G_{jk}(s, X(s), u(s)) dw_k(s), \quad X(t) = x,$$

$t \le s \le T$, for $j = 1, \dots, n$, where the functions $F_j, G_{jk} : [0,T] \times \Omega \times U \longrightarrow \mathbb{R}$ are of class C^2 and the standard Brownian motions $w_1, \dots, w_n$ are independent. Then (7) becomes

$$\begin{aligned} \operatorname{trace}(GG^\top \varphi^i_{xx}) &= \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} G^2_{jk} \Big) \frac{\partial^2 \varphi^i}{\partial x_j^2} + 2 \sum_{l<j} \Big(\sum_{j=1}^{n} \sum_{k=1}^{n} G_{jk} G_{lk} \Big) \frac{\partial^2 \varphi^i}{\partial x_j x_l} \\ &= \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} g^2_{jk} (n_{jk})^2 \Big) \frac{\partial^2 \varphi^i}{\partial x_j^2} + 2 \sum_{l<j} \Big(\sum_{j=1}^{n} \sum_{k=1}^{n} g_{jk} n_{jk} g_{lk} n_{lk} \Big) \frac{\partial^2 \varphi^i}{\partial x_j x_l}, \end{aligned} \tag{17}$$

for all $i = 1, \dots, N$, $j, k, l = 1, \dots, n$. It would be easy to update (A2) by using the expression above.

3 The case of correlated Brownian motion

This section extends the previous results to the case where $w_1, \dots, w_n$ are correlated and the diffusion matrix is diagonal. The general matrix case with correlated Brownian motions would give rise to a notationally more complex situation that conceptually does not add much to the problem, hence it will be avoided. Correlation means that there are constants $-1 \le \delta_{jk} \le 1$ such that $E(w_j(t)w_k(t)) = \delta_{jk}t$ for $j, k = 1, \ldots, n$. If w_j and w_k are independent, then $\delta_{jk} = 0$. Of course, $\delta_{jk} = \delta_{kj}$ and $\delta_{jj} = 1$, for all j, k. The modifications are straightforward to do. It is only necessary to take into account that the expression (7) becomes

$$\mathrm{trace}(GG^\top \varphi^i_{xx}) = \sum_{j=1}^n g_j^2 n_j^2 \frac{\partial^2 \varphi^i}{\partial x_j^2} + 2\sum_{j=1}^n \sum_{k<j} \delta_{jk} g_j g_k n_j n_k \frac{\partial^2 \varphi^i}{\partial x_j \partial x_k},$$

with the corresponding change in the HJB equation.

Assumption (A1) remains in force and (A2) changes to

(A2)' There are constants a^i_j, b^i_j, c^i_j, d^i_j, r^i_j, q^i_j, , $\tilde{d}^i_{jk}$, $\tilde{r}^i_{jk}$, s^i, e^i, such that for all $i = 1, \ldots, N$

$$\sum_{j=1}^n a^i_j \frac{\partial f_j}{\partial u^i} + \frac{1}{2}\sum_{j=1}^n d^i_j \frac{\partial (g_j^2)}{\partial u^i} + \sum_{j=1}^n \sum_{k<j} \tilde{d}^i_{jk} \delta_{jk} \frac{\partial (g_j g_k)}{\partial u^i} \neq 0,$$

and for all $i = 1, \ldots, N$, for all $j, k = 1, \ldots, n$

$$\begin{cases} k_j(x)\dfrac{\partial h^i}{\partial x_j}(x) = a^i_j h^i(x) + q^i_j, \\ m_j(x)\dfrac{\partial h^i}{\partial x_j}(x) = b^i_j h^i(x) + c^i_j, \\ n_j^2(x)\dfrac{\partial^2 h^i}{\partial x_j^2}(x) = d^i_j h^i(x) + r^i_j, \\ n_j(x) n_k(x)\dfrac{\partial^2 h^i}{\partial x_j \partial x_k}(x) = \tilde{d}^i_{jk} h^i(x) + \tilde{r}^i_{jk}, \qquad k<j, \\ S^i(T, x) = s^i h^i(x) + e^i. \end{cases}$$

Also, the following conditions hold

$$\sum_{j=1}^n q^i_j \frac{\partial f_j}{\partial u^i}(t, u) = 0,$$

$$\sum_{j=1}^n r^i_j \frac{\partial (g_j^2)}{\partial u^i}(t, u) + 2\sum_{j=1}^n \sum_{k<j} \tilde{r}^i_{jk} \delta_{jk} \frac{\partial (g_j g_k)}{\partial u^i}(t, u) = 0,$$

for all $i = 1, \ldots, N$.

Also, it is necessary to redefine

$$\psi^i(t, u) = \frac{-\frac{\partial \ell^i}{\partial u^i}(t, u)}{\sum_{j=1}^n a^i_j \frac{\partial f_j}{\partial u^i}(t, u) + \frac{1}{2}\sum_{j=1}^n d^i_j \frac{\partial (g_j^2)}{\partial u^i}(t, u) + \sum_{j=1}^n \sum_{k<j} \tilde{d}^i_{jk} \delta_{jk} \frac{\partial (g_j g_k)}{\partial u^i}(t, u)}.$$

The proof of the following result is along the lines of Theorem 1, so it is omitted. Modifications in the equations obtained are immediate.

Theorem 2 *Suppose that (A1) and (A2)' hold and that the system of differential equations*

$$
\begin{aligned}
0 =& \frac{d}{dt}\psi^i(t,\lambda(t)) - \rho^i\psi^i(t,\lambda(t)) + \ell^i(t,\lambda(t)) + \psi^i(t,\lambda(t))\Big(\sum_{j=1}^n a_j^i f_j(t,\lambda(t)) \\
&+ \sum_{j=1}^n b_j^i\gamma_j(t) + \frac{1}{2}\sum_{j=1}^n d_j^i g_j^2(t,\lambda(t)) + \sum_{j=1}^n\sum_{k<j}\tilde{d}_{jk}^i\delta_{jk}(g_jg_k)(t,\lambda(t))\Big),
\end{aligned}
\tag{18}
$$

with final condition

$$\psi^i(T,\lambda(T)) = s^i,$$

for $i = 1,\dots,N$, admits a unique C^1 solution $\lambda(t) = (\lambda_1(t),\dots,\lambda_N(t))$, for which the system of differential equations

$$
\begin{aligned}
0 =& \frac{d}{dt}\zeta^i(t) - \rho^i\zeta^i(t) + \beta^i(t) + \psi^i(t,\lambda(t))\Big(\sum_{j=1}^n q_j^i f_j(t,\lambda(t)) \\
&+ \sum_{j=1}^n c_j^i\gamma_j(t) + \frac{1}{2}\sum_{j=1}^n r_j^i g_j^2(t,\lambda(t)) + \sum_{j=1}^n\sum_{k<j}\tilde{r}_{jk}^i\delta_{jk}(g_jg_k)(t,\lambda(t))\Big),
\end{aligned}
$$

with final condition

$$\zeta^i(T) = e^i,$$

for $i = 1,\dots,N$, admits a unique C^1 solution $(\zeta_1(t),\dots,\zeta_N(t))$. Then $\lambda(t)$ is an MPNE of the stochastic differential game and the value function is

$$V^i(t,x) = \psi^i(t,\lambda(t))h^i(x) + \zeta^i(t).$$

Example 4 Consider Example 2 above, where there is correlation $-1 \le \delta_{12} \le 1$ between w_1 and w_2. The additional condition in (A2)'

$$n_1n_2\frac{\partial h^i}{\partial x_1\partial x_2} = (1-a)bh^i,$$

holds. Equation (18) now reads

$$
\begin{aligned}
0 =& \frac{d}{dt}\psi + N^{-a}u^{1-a} - \rho\psi + \psi\big(-(1-a)Nu + b(\mu - \nu Nu) + \gamma(1-a)\big) \\
&+ \frac{1}{2}\psi\big(-g_1^2a(1-a) - g_2^2b(1-b)\big) + \psi\delta_{12}g_1g_2(1-a)b.
\end{aligned}
$$

It has an additional term $\psi\delta_{12}\sigma_1\sigma_2(1-a)b$ in comparison to (16), but the structure of the equation is the same.

4 The case with additive structure in the payoffs

As promised, we now consider the case where h^i is constant for all $i = 1, \ldots, N$. We fix the value $h^i = 1$. This case was excluded in assumption (A2). We suppose uncorrelated Brownian motions, in order to simplify the development. The structure of the game is the same for the functions F_j and G_j, but L^i can be generalized somewhat to

$$L^i(t, x, u^1, \ldots, u^N) = \ell^i(t, u^1, \ldots, u^N) + \chi^i(t) z^i(x),$$

where we have included the term $\beta^i(t)$ in ℓ^i and where $\chi^i(t) z^i(x)$ is an extra term. Of course, χ^i could be identically null. Assumption (A2) is substituted by (A2)" below

(A2)" There are constants a^i_j, b^i_j, c^i_j, d^i_j, r^i_j, q^i_j, s^i, e^i, such that for all $i = 1, \ldots, N$

$$\sum_{j=1}^{n} q^i_j \frac{\partial f_j}{\partial u^i} + \frac{1}{2} \sum_{j=1}^{n} r^i_j \frac{\partial (g_j^2)}{\partial u^i} \neq 0,$$

and for all $i = 1, \ldots, N$, for all $j = 1, \ldots, n$

$$\begin{cases} k_j(x) \dfrac{\partial z^i}{\partial x_j}(x) = a^i_j z^i(x) + q^i_j, \\ m_j(x) \dfrac{\partial z^i}{\partial x_j}(x) = b^i_j z^i(x) + c^i_j, \\ n_j^2(x) \dfrac{\partial^2 z^i}{\partial x_j^2}(x) = d^i_j z^i(x) + r^i_j, \\ S^i(T, x) = s^i z^i(x) + e^i. \end{cases} \tag{19}$$

Also, the following conditions hold

$$\sum_{j=1}^{n} a^i_j \frac{\partial f_j}{\partial u^i}(t, u) = 0 \tag{20}$$

$$\sum_{j=1}^{n} d^i_j \frac{\partial (g_j^2)}{\partial u^i}(t, u) = 0 \tag{21}$$

for all $i = 1, \ldots, N$.

Note that if $\chi^i = 0$, then the conditions involve the unknown functions z^i instead of h^i (that in this case is identically 1) as in the case studied in the previous section. If z^i is not given, then it has to be found from the conditions above, if possible.

Let the function ψ^i be defined for $i = 1, \ldots, N$, by

$$\psi^i(t, u) = \frac{-\frac{\partial \ell^i}{\partial u^i}(t, u)}{\sum_{j=1}^{n} q^i_j \frac{\partial f_j}{\partial u^i}(t, u) + \frac{1}{2} \sum_{j=1}^{n} r^i_j \frac{\partial (g_j^2)}{\partial u^i}(t, u)},$$

which is a bit different from that defined in (8). We have the following result, whose proof mimics that of Theorem 1, so it is omitted.

Theorem 3 *Suppose that (A1) and (A2)" hold and that the system of differential equations*

$$\begin{aligned} 0 &= \frac{d}{dt}\psi^i(t,\lambda(t)) - \rho^i\psi^i(t,\lambda(t)) + \chi^i(t) \\ &+ \psi^i(t,\lambda(t))\Big(\sum_{j=1}^n a^i_j f_j(t,\lambda(t)) + \sum_{j=1}^n b^i_j\gamma_j(t) + \frac{1}{2}\sum_{j=1}^n d^i_j (g_j(t,\lambda(t)))^2\Big), \end{aligned} \tag{22}$$

with final condition

$$\psi^i(T,\lambda(T)) = s^i, \tag{23}$$

for $i = 1,\dots,N$, admits a unique C^1 solution $\lambda(t) = (\lambda^1(t),\dots,\lambda^N(t))$, for which the system of differential equations

$$\begin{aligned} 0 &= \frac{d}{dt}\zeta^i(t) - \rho^i\zeta^i(t) + \ell^i(t,\lambda(t)) \\ &+ \psi^i(t,\lambda(t))\Big(\sum_{j=1}^n q^i_j f_j(t,\lambda(t)) + \sum_{j=1}^n c^i_j\gamma_j(t) + \frac{1}{2}\sum_{j=1}^n r^i_j (g_j(t,\lambda(t)))^2\Big) \end{aligned} \tag{24}$$

with final condition

$$\zeta_i(T) = e^i, \tag{25}$$

for $i = 1,\dots,N$, admits a unique C^1 solution $(\zeta^1(t),\dots,\zeta^N(t))$. Then $\lambda(t)$ is an MPNE of the stochastic differential game and the value function is

$$V^i(t,x) = \psi^i(t,\lambda(t))z^i(x) + \zeta^i(t).$$

Example 5 This is a stochastic version of the Eskimoean game; see Mehlmann [12], p. 106, for the deterministic case. In this fishery game, the inverse demand function is $p(u) = \frac{1}{x\sum_{j=1}^N u^j}$, the marginal cost is $C^i \neq 1$ for all $i = 1,\dots,N$, and the fish population obeys

$$dX = \Big(X(\mu - \delta\ln X) - X\sum_{j=1}^N u^j\Big)ds + \sigma X dw(s).$$

The growth function $m(x) = x(\mu - \delta\ln x)$ is the Gompertz law of population growth. The payoff of player i is

$$J^i = E_{tx}\int_t^T e^{-\rho^i(s-t)}(p(u(s))X(s) - C^i)u^i(s)ds + s^i e^{-\rho^i(T-t)}E_{tx}\ln X(T).$$

Hence $\ell^i(u) = \frac{u^i}{\sum_{j=1}^N u^j} - C^i u^i$, which is strictly concave in u^i, as it is easily checked because $\ell^i_{u^iu^i}(u) = -2\frac{\sum_{j\neq i} u^j}{(\sum_{j=1}^N u^j)^3} < 0$. The Hamiltonian H^i is strictly concave with respect to u^i because f is linear in u^i. Thus the game satisfies the concavity condition (A1). Moreover, $k(x)=x$, $f(u)=-\sum_{j=1}^N u^j$, $\gamma(t)=1$ and m was defined above. Function $n(x)=\sigma x$, $g(t,u)=1$ and $S^i(x)=s^i\ln x$. Finally, $h^i=1$ and $\chi^i=0$.

The function $z^i(x)$ is unknown and it must be determined from conditions (19). Observe that (19), first line, holds for $z^i(x)=\ln x$: $k(z^i)'=a^iz^i+q^i$, where $a^i=0$ and $q^i=1$, for $i=1,\dots,N$. The rest of the conditions are also fulfilled

$$m(z^i)'=x(\mu-\delta\ln x)\frac{1}{x}=b^iz^i+c^i,$$

for $c^i=\mu$ and $b^i=-\delta$, $i=1,\dots,N$. Also,

$$n^2(z^i)''=x^2(-x)^{-2}=d^iz^i+r^i,$$

for $d^i=0$ and $r^i=-\sigma^2$, $i=1,\dots,N$. The fourth line holds for $e^i=0$, and finally the first condition of (A2)', (20) and (21) hold. Thus, (A1) and (A2)' are fulfilled. We postulate the value function

$$V^i(t,x)=\psi^i(t,\lambda(t))\ln x+\eta^i(t).$$

Note that $V^i(T,x)=s^i\ln x$ by (23) and (25).

The function $\psi^i(u)=\ell^i_{u^i}(u)=\frac{Q-u^i}{Q^2}-C^i$, where $Q=\sum_{j=1}^N u^j$. Moreover, the ODE (22) is

$$\frac{d}{dt}\psi^i-(\rho^i+\delta)\psi^i=0,\qquad \psi^i(T)=s^i,$$

where we are abusing notation, identifying $\psi(t)$ with $\psi(\lambda(t))$. The solution is

$$\psi^i(t)=s^i\exp\left(-(\rho^i+\delta)(T-t)\right).$$

Solving for $\lambda^1(t),\dots,\lambda^N(t)$ is easy. From $\psi^i(\lambda)=\frac{Q-\lambda^i}{Q^2}-C^i$, $Q=\sum_{j=1}^N\lambda^j$, we obtain

$$\lambda^i(t)=Q(t)-(C^i+s^i\exp\left(-(\rho^i+\delta)(T-t)\right))Q(t)^2,$$

where

$$Q(t)=\frac{N-1}{\sum_{j=1}^N(C^j+s^j\exp\left(-(\rho_j+\delta)(T-t)\right))}.$$

We do not make the computations needed for getting $\eta^i(t)$ from (24) because, at least theoretically, it is straightforward.

A possible extension is to consider $g(t,u)=\sqrt{\sum_{j=1}^N u^j}$, that is to say,

$$dX=\Big(X(\mu-\delta\ln X)-X\sum_{j=1}^N u^j\Big)ds+\sigma X\sqrt{\sum_{j=1}^N u^j}dw(s).$$

Condition (A1) holds because $g^2(t,u)$ is a linear function of u^i; also, the first line in (A2) is satisfied because $\frac{1}{2}\sigma^2\neq-1$, as well as condition (21), since $d^i=0$. The functions ζ^i are computed the same as above.

5 Conclusions

We have identified stochastic differential games where an MPNE exists based on open-loop strategies. This is a natural extension of the corresponding results for deterministic games obtained in the past decades. Maybe, the belief that to construct an equilibrium that correctly forecasts the uncertain future needs Markov rules, has deterred researchers from addressing this problem in the stochastic case. We show in this paper that most of the known deterministic games presenting open-loop MPNE may be extended to stochastic games—in several ways—maintaining this feature. We obtain our results by focusing on the structural form of the functions defining the game, which is a novel approach. We based our results on the HJB equations, giving explicitly the value functions of the players, as well as optimality conditions in the form of ordinary differential equations that the open-loop equilibrium must satisfy. Hence, our approach departs from that of—on the other hand elegant—state-variable transformation of Fershtman [5] or Mehlmann [12], or those based on the optimality conditions of the Maximum Principle. Although we concentrate on known structures for the deterministic case: exponential, power and linear structures, translating them to the stochastic case, our assumption (A2), that specifies the relationship between the functions defining the games, would allow us to construct other game models with this property. It is not that we are recommending the construction of artificial *ad hoc* models. Rather, our findings would play the role of testing whether a particular game reflects an interesting model from economics, or operations research satisfies our assumptions, so that a solution is readily available. Future research should focus on finding general conditions guaranteeing the existence and uniqueness of solutions of the Cauchy problems (9)-(10) and (11)-(12).

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