Notes for Macro II, course 2011-2012

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Summary:
The course has three aims: 1) get you acquainted with Dynamic Programming both deterministic and stochastic, a powerful tool for solving infinite horizon optimization problems; 2) analyze in detail the One Sector Growth Model, an essential workhorse of modern macroeconomics and 3) introduce you in the analysis of stability of discrete dynamical systems coming from Euler Equations.

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3 Sample Final Exam **61**

1 Deterministic Stationary Discounted Dynamic Programming

1.1 Motivation. Ramsey growth model

This section presents an schematic presentation of this classical model.

Setup

- Production Function

$$Y_t = F(K_t, N_t), \quad F : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+, \quad \begin{cases} Y & \text{output} \\ K & \text{capital} \\ N & \text{labor} \end{cases}$$

- Feasibility constraints

$$\begin{cases} C_t + I_t & \leq & Y_t, \\ I_t & = & K_{t+1} - (1-\delta)K_t, \\ C_t, K_t, N_t & \geq & 0 \end{cases}, \qquad \begin{cases} C & \text{consumption} \\ I & \text{investment} \\ \delta & \text{depreciation rate, } 0 \leq \delta \leq 1 \end{cases}$$

- Preferences

$$S(C_0, C_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u(C_t), \quad u : \mathbb{R}_+ \longrightarrow \mathbb{R}, \quad S : \mathbb{R}^\infty_+ \longrightarrow \mathbb{R}, \quad 0 < \beta < 1 \text{discount rate}$$

- Per capita variables

  $y = Y/N$, $k = K/N$, $c = C/N$

  Assume that F is homogeneous of degree one, hence $y_t = F(k_t, 1) = \tilde{f}(k_t)$ for some function f.
  From the feasibility constraints we get $c_t + \frac{N_{t+1}}{N_t} k_{t+1} - (1-\delta)k_t \leq \tilde{f}(k_t)$.
  Preferences are given by $S(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t N_t)$.
  Assume that $N_{t+1}/N_t = n$ and that $u(c) = c^\theta/\theta$. Then

$$S(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t N_t) = N_0^\theta \sum_{t=0}^{\infty} (n^\theta \beta)^t \frac{c_t^\theta}{\theta}$$

Planners' Problem

- Normalize

  $N_t = 1$ (leisure has no value)

- PP in per capita form

  Given $k_0 > 0$ and denoting $f(k_t) = \tilde{f}(k_t) + (1-\delta)k_t$

$$v(k_0) = \max_{\{c_t\}_0^\infty, \{k_{t+1}\}_0^\infty} \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \, : \, c_t + k_{t+1} \leq f(k_t), \; c_t, k_t \geq 0 \right\}$$

  The timing is as follows: c_t is chosen at the end of period $t+1$; the remaining capital, k_{t+1}, produces output $f(k_{t+1})$ at the end of period $t+2$ and then c_{t+1} is chosen, and so on

$$
\begin{tikzcd}
 & c_0 & c_1 & \\
 & f(k_0) \arrow[u] \arrow[d] & f(k_1) \arrow[u] \arrow[d] & \cdots \\
k_0 \arrow[ur] & k_1 \arrow[ur] & k_2 &
\end{tikzcd}
$$

- <u>Reduced form</u>

  Assume $c_t + k_{t+1} = f(k_t)$ for all $t \geq 1$

$$v(k_0) = \max\left\{ \sum_{t=0}^{\infty} \beta^t U(k_t, k_{t+1}) \,:\, 0 \leq k_{t+1} \leq f(k_t),\ t \geq 0 \right\},$$

  where $U(k_t, k_{t+1}) = u(f(k_t) - k_{t+1})$.

- <u>Finite horizon</u>

$$v(T, k_0) = \max\left\{ \sum_{t=0}^{T} \beta^t U(k_t, k_{t+1}) \,:\, 0 \leq k_{t+1} \leq f(k_t),\ 0 \leq t \leq T \right\}$$

  This is a finite dimensional programming problem, that can be handled with Kuhn-Tucker multipliers. In fact, it is a parametric optimization problem of the type we have studied in Math II (see the notes of the course)

- <u>Two-period problem</u> (we eliminate T from the notation)

$$v(k_0) = \max_{(k_1,k_2) \in G(k_0)} W(k_0, k_1, k_2)$$

  where $W : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ is $W(k_0, k_1, k_2) = U(k_0, k_1) + \beta U(k_1, k_2)$ and $G : \mathbb{R}_+ \rightrightarrows \mathbb{R}^2_+$ is

$$G(k_0) = \{(k_1, k_2) \in \mathbb{R}^2_+ \,:\, k_1 \leq f(k_0), k_2 \leq f(k_1)\}$$

  Assuming continuity of both u and f the Maximum Theorem applies, thus a solution exists, v is continuous and the optimal correspondence G^* has compact values and is uhc. If both U and f are concave, then v is concave and G^* has convex values. Moreover, if W is strictly concave, then the solution is unique and $G^*(k_0) = (k_1^*(k_0), k_2^*(k_0))$ is a continuous function

- <u>How to solve it?</u> Two methods

  Kuhn-Tucker multipliers: Consider the Lagrangian of the problem

$$L(k_0; k_1, k_2, \lambda_1, \lambda_2) = U(k_0, k_1) + \beta U(k_1, k_2) + \lambda_1(f(k_0) - k_1) + \lambda_2(f(k_2) - k_1).$$

Assuming smoothness, at the optimum we have that for $t = 1, 2$

$$\begin{cases} k_t L'_{k_t} = 0, \quad k_t \geq 0, \quad L'_{k_t} \leq 0 \\ \lambda_t L'_{\lambda_t} = 0, \quad \lambda_t \geq 0, \quad L'_{\lambda_t} \geq 0. \end{cases}$$

The usual procedure is to solve the system with equalities and then to check whether the solutions satisfy the set of inequalities. Under suitable convexity, K-T are sufficient for optimality.

Dynamic Programming: Find the optimal solution of the last stage of the problem, and then use backward induction to reach the beginning of the problem This requires to separate present and future utilities, what is possible due to ...

The Principle of Optimality: an optimal policy is such that whatever the initial capital and initial consumption decision are, the remaining decisions must constitute an optimal policy with regard to the capital resulting from the first decision.

Richard Bellman (1920-1984)

Let $v_n(k)$ be the maximum total utility when it remains n periods to the end and capital is k

In the two period problem

$$\begin{aligned} v_1(k_1) &= \max_{0 \leq k_2 \leq f(k_1)} U(k_1, k_2) = \max_{0 \leq k_2 \leq f(k_1)} u(f(k_1) - k_2) \\ v_2(k_0) &= \max_{0 \leq k_1 \leq f(k_0)} \{U(k_0, k_1) + \beta v_1(k_1)\} \\ &= \max_{0 \leq k_1 \leq f(k_0)} \{u(f(k_0) - k_1) + \beta v_1(k_1)\} \end{aligned}$$

Note that $v_2(k_0)$ is $v(k_0)$

In general, for a problem of T periods we get the Bellman equation

$$\begin{aligned} v_n(k) &= \max_{0 \leq k' \leq f(k)} \{u(f(k) - k') + \beta v_{n-1}(k')\}, \quad n = 1, 2, \ldots, T \\ v_0(k) &= 0 \end{aligned}$$

Let

$$y_n(k) \in \text{argmax}_{0 \leq k' \leq f(k)} \{u(f(k) - k') + \beta v_{n-1}(k')\}$$

An optimal policy is $k^*_{t+1}(k) = y_{T-t}(k)$, $t = 0, 1, \ldots, T-1$

The associated consumption policies are

$$c^*_t(k) = f(k^*_t(k)) - k^*_{t+1}(k)$$

Example 1.1. $u(c) = \sqrt{c},\ f(k) = Ak,\ k_0 > 0$

$$
\begin{aligned}
v_1(k) &= \max_{0 \le k' \le Ak} \sqrt{Ak - k'} = \sqrt{Ak} \Rightarrow k_2^*(k) = 0, c_1^*(k) = Ak \\
v_2(k) &= \max_{0 \le k' \le Ak} \{\sqrt{Ak - k'} + \beta\sqrt{Ak'}\} \Rightarrow k_1^*(k) = \frac{A}{1+\beta^2} k \\
&\qquad\qquad\qquad\qquad\qquad\qquad c_0^*(k) = \frac{A\beta^2}{1+\beta^2} k
\end{aligned}
$$

The consumption sequence is $c_0^* = \frac{A\beta^2}{1+\beta^2} k_0,\ c_1^* = \frac{A^2}{1+\beta^2} k_0$

The capital sequence is $k_0,\ k_1^* = \frac{A}{1+\beta^2} k_0,\ k_2^* = 0$

Example 1.2. *The introduction of a bequest function may complicate the problem.*
$u(c) = \sqrt{c},\ f(k) = Ak,\ k_0 > 0,$ *bequest function* $b(T, k) = B \ln k$
The problem becomes $\max_{(c_0, c_1)} u(c_0) + \beta u(c_1) + \beta^2 b(2, k_2)$
Now $v_0(k) = B \ln k$. *Symmetry is broken: the problem is hard to solve*

$$
\begin{aligned}
v_1(k) &= \max_{0 \le k' \le Ak} \sqrt{Ak - k'} + \beta v_0(k') \\
&= \sqrt{Ak - 2B^2(\sqrt{1 + Ak} - 1)} + \beta B \ln(2B^2\sqrt{1 + Ak} - 1) \\
k_2^*(k) &= 2B^2(\sqrt{1 + Ak} - 1) \\
v_2(k) &= \max_{0 \le k' \le Ak} \{\sqrt{Ak - k'} + \beta v_1(k')\} =? \\
k_1^*(k) &=?
\end{aligned}
$$

- <u>Infinite Horizon</u>

  Recall:

$$
\begin{aligned}
v_n(k) &= \max_{0 \le k' \le f(k)} \{u(f(k) - k') + \beta v_{n-1}(k')\}, \quad n = 1, 2, \ldots, T \\
v_0(k) &= 0
\end{aligned}
$$

When $T = \infty$ there is no a last time period to start backward induction.
Intuitively, the value function satisfies (this will be proved throughout the course)

$$
v(k) = \max_{0 \le k' \le f(k)} \{u(f(k) - k') + \beta v(k')\}
$$

How do we start computations?

Answer: Set this functional equation as a fixed point problem defined on a suitable set of functions and suitable metric so that Banach's contraction mapping theorem is applicable.

1.2 General development

The general setup is as follows.

1. t denotes time and it is supposed to be discrete, $t = 0, 1, 2, \ldots$.

2. X, the set of possible states, is a subset of the Euclidean space $\mathbb{R}^m$.

3. $D : X \rightrightarrows A \subseteq \mathbb{R}^p$, a correspondence that associates with state x a nonempty set $D(x)$ of feasible decisions $a \in D(x)$. We denote $A = \bigcup_{x \in X} D(x)$.

4. $q : X \times A \longrightarrow X$, the law of motion. Given $x \in X$ and $a \in D(x)$ (we say that the pair (x, a) is admissible), $y = q(x, a)$ is the next state of the system. More generally, we could consider $q : X \times A \rightrightarrows X$ being a correspondence and $y \in q(x, a)$. In stochastic problems q will be a conditional probability of reaching state y from state x if the action is a.

5. $U : X \times A \longrightarrow \mathbb{R}$, the one-period return function. For (x, a) admissible, $U(x, a)$ is the current return, utility or income if the current state is x and the current action taken is a.

6. β, the discount factor, $0 < \beta < 1$.

A Markov policy or decision rule is a sequence $\pi = \{\pi_t, t = 0, 1, \ldots\}$ such that $\pi_t(x) \in D(x)$ for all t and for all $x \in X$. Let Π the set of Markov policies. A policy is stationary if there exists ϕ such that $\pi_t = \phi$ for all t.

Given $x_0 \in X$ we can associate with any policy the value

$$I(x_0, \pi) = \sum_{t=0}^{\infty} \beta^t U(x_t, \pi_t),$$

where $x_{t+1} = q(x_t, \pi_t)$, $t = 0, 1, \ldots$. The sequence $\{x_t\}_{t=0}^{\infty}$ such that $x_{t+1} = q(x_t, a_t)$ for some $a_t \in D(x_t)$ will we called an admissible path from x_0 associated to policy π. The set of such paths are denoted $X_\pi(x_0)$.

The problem is then to find a policy $\pi \in \Pi$ such that for any $x_0 \in X$, $I(x_0, \pi) \geq I(x_0, \pi')$ for every $\pi' \in \Pi$. We shall say that such π is an optimal policy. It is possible to show in this framework that it suffices to look for stationary policies, thus we reduce our exposition to this type. However, if the time period is finite, then usually the optimal policy depends on time.

The value function $v : X \longrightarrow \mathbb{R}$ is defined as

$$v(x_0) = \sup_{\pi \in \Pi} I(x_0, \pi).$$

We want to establish sufficient conditions such that this problem is well defined and to develop methods of resolution.

Example 1.3 (Ramsey Growth Model). *We have already seen this model. Here, $X = \mathbb{R}_+$ is the state space (capital, k), $A = \mathbb{R}_+$ is the action space (consumption, c) and $D(k) = [0, f(k)]$. The law of motion is $q(k, c) = f(k) - c$ and the utility function is $U(k, c) = u(c)$.*

Example 1.4 (Capacity expansion). *A monopolist has the following production technology. Given current capacity, Q, he can produce any amount of output, q, up to Q units at zero cost, but he cannot produce more than Q in the current period. Capacity can be increased over time but cannot be sold. Any nonnegative amount a of capacity can be added in any period at cost $c(a) = a^2$, but the new capacity cannot be used until the next period. The monopolist faces the same demand for his product each period given by $q = 1 - p$, where p is the price of output. The monopolist seeks to maximize the present value over an infinite horizon of the flow of profits.*

In this problem, the state space $X = \mathbb{R}_+$ is total capacity Q, the action space $A = \mathbb{R}^2_+$ is formed by pairs (output, capacity), (q, a). The constraints are $0 \le q \le Q$, thus $D(Q) = [0, Q] \times \mathbb{R}_+$, and the the law of motion is $q(Q, q, a) = Q + a$ (not to be confused with output!), thus $Q_{t+1} = Q_t + a_t$ for $t = 0, 1, \ldots$. The reward function is

$$U(Q, q, a) = pq - a^2 = q(1 - q) - a^2.$$

An example of Markov policy π is $(x_t, a_t) = (Q/2, 2^{-t})$, which is non stationary. Another one is $(x_t, a_t) = (\sqrt{Q}, Q/2)$, which is stationary. Actually, the second example is not quite a policy, since it is not feasible, as $q = \sqrt{Q} \nleq Q$ for $0 < Q < 1$. We will find the optimal policy with the tools developed below.

1.3 Bellman equation

The following two conditions will be in force.

- For all $x \in X$, $D(x) \neq \emptyset$.
- For all $x \in X$, $v(x)$ is finite-valued.

Lemma 1.5. *For any $\pi \in \Pi$*

$$I(x_0, \pi) = U(x_0, \pi_0) + \beta I(x_1, \pi^1),$$

where $x_1 = q(x_0, \pi_0)$ and $\pi^1 = (\pi_1, \ldots)$ is the continuation of policy π to period 1.

Proof. The result is evident from the identities.

$$I(x_0, \pi) = U(x_0, \pi_0) + \sum_{t=1}^{\infty} \beta^t U(x_t, \pi_t),$$

$$\beta I(x_1, \pi^1) = \beta \sum_{t=0}^{\infty} \beta^t U(x_{t+1}, \pi_{t+1}) = \sum_{t=1}^{\infty} \beta^t U(x_t, \pi_t).$$

□

The following result is fundamental.

Theorem 1.6 (Bellman equation)**.** *The value function satisfies for any $x \in X$*

$$v(x) = \sup_{\substack{y = q(x,a) \\ a \in D(x)}} \{U(x, a) + \beta v(y)\}. \tag{1}$$

Proof. Let $x_0 \in X$ arbitrary.

1. For all $a \in D(x_0)$, $v(x_0) \ge U(x_0, a) + \beta v(q(x_0, a))$.

   Let $a \in D(x_0)$ and let $x_1 = q(x_0, a)$. Since v is finite valued, given $\varepsilon > 0$ there exists $\pi \in \Pi$ such that

   $$I(x_1, \pi) \ge v(x_1) - \varepsilon.$$

   Now consider the policy $\pi' = (a, \pi)$, that is, the concatenation of action a with the policy π. By the definition of the value function we have

   $$v(x_0) \ge I(x_0, \pi') = U(x_0, a) + \beta I(x_1, \pi) \ge U(x_0, a) + \beta v(x_1) - \beta\varepsilon,$$

   where the equality is due to Lemma 1.5. Taking the supremum in a and letting $\varepsilon \to 0$, we have finished.

2. For all $\varepsilon > 0$ there exists $a_0 \in D(x_0)$ such that $v(x_0) < U(x_0, a_0) + \beta v(q(x_0, a_0)) + \varepsilon$.
   Since v is finite valued, given $\varepsilon > 0$ there exists $\pi \in \Pi$ such that

$$v(x_0) < I(x_0, \pi) + \varepsilon = U(x_0, a_0) + \beta I(x_1, \pi^1) + \varepsilon \leq U(x_0, a_0) + \beta v(x_1) + \varepsilon,$$

   where the equality is by Lemma 1.5 and π^1 is the continuation of policy π to period 1.

Hence, by 1 and 2 above, we have

$$\begin{aligned} U(x_0, a_0) + \beta v(q(x_0, a_0) &\leq \sup_{a \in D(x_0)} \{U(x_0, a) + \beta v(q(x_0, a))\} \\ &\leq v(x_0) \qquad \text{(by 1)} \\ &< U(x_0, a_0) + \beta v(q(x_0, a_0)) + \varepsilon \qquad \text{(by 2)}. \end{aligned}$$

Letting $\varepsilon \to 0$ we get the Bellman equation. □

Example 1.7.

Bellman equation of the Ramsey model:

$$v(k) = \sup_{c \in [0, f(k)]} \{u(c) + \beta v(f(k) - c)\} \equiv \sup_{k' \in [0, f(k)]} \left\{u(f(k) - k') + \beta v(k')\right\}. \tag{2}$$

Bellman equation of the Capacity Expansion model:

$$v(Q) = \sup_{0 \leq q \leq Q, a \geq 0} \{q(1-q) + \beta v(Q + a)\} \equiv \sup_{0 \leq q \leq Q, Q' \geq Q} \left\{q(1-q) + \beta v(Q')\right\}. \tag{3}$$

1.4 Finite horizon

Assume a finite horizon. That is, the problem ends at a fixed time T. The main change with respect to the formulation for $T = \infty$ is in the definitions of function S, since now it is important to know the periods that remain to the end, so that we define the value associated to π (of course, now π has only T components)

$$I_n(x_0, \pi) = \sum_{t=T-n}^{T} \beta^{t-T+n} U(x_t, a_t) + \beta^{n+1} B(x_{T+1}).$$

In the last summand we have considered a bequest function B that takes into account the final state x_{T+1}. Utilities are properly discounted to time $t = T - n$. Let $v_n(x_0)$ be the maximal utility that can be reached from state x_0 when it remains n periods to the end. Obviously, for any $x \in X$, $v_0(x) = \sup_{a \in D(x)} B(q(x, a))$. Thanks to the Principle of Optimality we have

Theorem 1.8 (Bellman equation, finite horizon)**.** *For any $x \in X$*

$$v_0(x) = \sup_{a \in D(x)} B(q(x, a)). \tag{4}$$

and $n = 1, 2, \ldots, T$ the following recursion

$$v_n(x) = \sup_{a \in D(x)} \{U(x, a) + \beta v_{n-1}(q(x, a))\}, \tag{5}$$

holds.

Example 1.9 (A Consumption-Savings problem with altruistic motives). *Consider someone with initial capital x_0 and let $x_1, x_2, \ldots$ be the levels of capital at the times $1, 2, \ldots$. At time t she decide to spends and amount c_t within the range $0 \leq c_t \leq x_t$ (hence, she cannot borrow) and the rest of the money is invested until time $t+1$ at an interest r, so the interest factor is $R = 1 + r$. She receives constant income $I \geq 0$ each period. The preferences of the consumer are given by $u(c_0) + \cdots + \beta^T u(c_T) + \beta^{T+1} B(x_{t+1})$. The term $B(x_{t+1})$ can be interpreted as the consumer having altruistic sentiments with respect to her descendants.*

In this problem

$$v_0(x) = B(R(I + x - c))$$

and for $n = 1, 2, \ldots, T$

$$v_n(x) = \sup_{\substack{y=R(I+x-c)\\ c\in[0,x]}} \{U(c) + \beta v_{n-1}(y)\}, \qquad .$$

Example 1.10 (Gambling). *In each play of a game, a gambler can bet any non-negative amount up to his current fortune and he will either win or lose that amount with probabilities p and $q = 1 - p$, respectively. He is allowed to make T bets in succession, and his objective is to maximize the expectation of the utility B of the final fortune (no discount is involved here). Suppose that utility is increasing in wealth.*

Although the problem involves probabilities, it is not difficult to extend our framework to cover this example. Let $v_n(x)$ the expected *maximal utility when the current fortune is x and remains n periods to the end. Denoting a the amount bet, we must have $0 \leq a < x$ and the next state or fortune is either $x + a$ with probability p, or $x - a$ with probability q. Hence, the Bellman recursion takes the form*

$$v_0(x) = B(x)$$

and for $n \geq 1$,

$$v_n(x) = \sup_{0\leq a<x} \{p v_{n-1}(x + a) + q v_{n-1}(x - a)\}.$$

1.5 The value function and the optimal policy from the Bellman equation

Why is useful the Bellman equation? Because, under mild conditions, it characterizes the value function (Theorem 1.12 below) and provides a method to find the optimal policy (Theorem 1.13 below).

Define an operator $\mathcal{B}$ defined over a suitable class of functions $f : X \longrightarrow \mathbb{R}$. For the moment we do not worry about the properties of f. The operator is defined as

$$(\mathcal{B}f)(x) = \sup_{\substack{y=q(x,a)\\ a\in D(x)}} \{U(x, a) + \beta f(y)\}, \tag{6}$$

and hope that the function $\mathcal{B}f$ is in the same class as f. The operator $\mathcal{B}$ has a nice interpretation: $(\mathcal{B}f)(x)$ is the value starting from state x of choosing an optimal action today given that the process terminates tomorrow with the receipt of $f(y)$, as a function of tomorrow's state, y. Observe that "optimal" has into account the influence of a on y.

Obviously, by Theorem 1.6 the value function is a fixed point of $\mathcal{B}$. What we want to explore now is the reverse implication: Is a fixed point of $\mathcal{B}$ the value function of the problem?

Example 1.11 (Multiple fixed points). *Let $X = A = \mathbb{R}_+$, $D(x) = [0, 2x]$, $q(x, a) = 2x - a$, $U(x, a) = -a$ and $1/2 < \beta < 1$. The Bellman equation is*

$$v(x) = \sup_{a\in[0,2x]} \{-a + \beta v(2x - a)\}.$$

It is easy to check that both $f_1(x) = 0$ and $f_2(x) = -2x$ are solutions. Which one, if any, is the value function?

Theorem 1.12. *Let f be a fixed point of $\mathcal{B}$ that satisfies*

$$\lim_{t\to\infty} \beta^t f(x_t) = 0$$

for all paths $\{x_t\}_{t=0}^{\infty}$ with $x_{t+1} = q(x_t, a_t)$, $a_t \in D(x_t)$. Then f is the value function of the problem.

Proof. Let $x_0 \in X$ and let any $\pi \in \Pi$. Then

$$\begin{aligned} f(x_0) &\geq U(x_0, a_0) + \beta f(x_1) \\ &\geq U(x_0, a_0) + \beta U(x_1, a_1) + \beta^2 f(x_2) \\ &\vdots \\ &\geq \sum_{t=0}^{T} \beta^t U(x_t, a_t) + \beta^{T+1} f(x_{T+1}). \end{aligned}$$

Taking limits as $T \to \infty$ we have $f(x_0) \geq \sum_{t=0}^{\infty} \beta^t U(x_t, a_t) = S(x_0, \pi)$. Then, $f(x_0) \geq v(x_0)$.

On the other hand, for any x_t and $\varepsilon > 0$, there exists $a_t \in D(x_t)$ such that

$$f(x_t) < U(x_t, a_t) + \beta f(x_{t+1}) + \epsilon 2^{-t}$$

thus,

$$f(x_0) < \sum_{t=0}^{T} \beta^t U(x_t, a_t) + \beta^{T+1} f(x_{T+1}) + \epsilon \sum_{t=0}^{T} 2^{-t}.$$

Taking limits as $T \to \infty$ we have

$$f(x_0) < \sum_{t=0}^{\infty} \beta^t U(x_t, a_t) + 2\varepsilon,$$

thus $f(x_0) \leq \sum_{t=0}^{\infty} \beta^t U(x_t, a_t) \leq v(x_0)$. □

In Example 1.11, f_1 is the value function. Observe that $\{x_t\}_{t=0}^{\infty}$ defined as $x_t = 2^{t-1} x_0$ is an admissible path from x_0 and

$$\lim_{t\to\infty} \beta^t f_2(2^{t-1} x_0) = -x_0 \lim_{t\to\infty} \beta^t 2^t = \infty \qquad \forall x_0 > 0,$$

since $\beta > 1/2$.

Theorem 1.13. *Assume that a_t^* solves $\sup_{a_t \in D(x_t)} \{U(x_t, a_t) + \beta v(q(x_t, a_t))\}$ and that*

$$\limsup_{t\to\infty} \beta^t v(x_t^*) \leq 0,$$

where $x_{t+1}^* = q(x_t^*, a_t^*)$, x_0 given. Then, $\pi^* = (a_0^*, \ldots, a_t^*, \ldots)$ is an optimal policy.

Proof. Note that $v(x_t^*) = U(x_t^*, a_t^*) + \beta v(x_{t+1}^*)$ for all $t = 0, 1, \ldots$, hence for all $T \geq 1$

$$v(x_0) = \sum_{t=0}^{T} \beta^t U(x_t^*, a_t^*) + \beta^{T+1} v(x_{T+1}^*).$$

Letting $T \to \infty$ we have

$$v(x_0) \leq \sum_{t=0}^{\infty} \beta^t U(x_t^*, a_t^*) = S(x_0, \pi^*),$$

hence $v(x_0) = S(x_0, \pi^*)$ and π^* is optimal. □

To use the theorem we need to know the value function, which is not known in advance since it is characterized by the functional equation at the same time that the optimal policy. This problem does not appear in the finite horizon case, as the computation starts from the known value function v_0 and proceeds recursively. One method for solving the infinite horizon problem is to "guess" a functional form for the value function, then substituting it into de Bellman equation forcing to be a solution, and deriving after the optimal choice variable with Theorem 1.13. Of course, this method is of limited applicability, although it is really nice when it works at it gives the complete solution of the problem. Let us illustrate the method with an example.

Example 1.14 (Ramsey Growth Model). *Consider in the Ramsey Model full depreciation,* $\delta = 1$, *production function* $\tilde{f}(k) = Ak^\alpha$, *with* $A > 0$ *and* $0 < \alpha < 1$ *and utility* $u(c) = \ln c$. *The Bellman equation written in terms of capital (reduced form) is*

$$v(k) = \sup_{k' \in [0, Ak^\alpha]} \left\{ \ln\left(Ak^\alpha - k'\right) + \beta v(k') \right\}.$$

- *Guess that* $f(k) = a \ln k + b$ *for suitable constants* $a > 0$ *and* b *is a solution.*
- *Substitute into the Bellman equation*

$$a \ln k + b = \sup_{k' \in [0, Ak^\alpha]} \left\{ \ln\left(Ak^\alpha - k'\right) + \beta(a \ln k' + b) \right\}$$

- *Perform the maximization operation. To this end, note that in this example the maximizer, if exists, must be interior, thus*

$$\frac{\partial}{\partial k'}\{\cdot\} = 0 = -\frac{1}{Ak^\alpha - k'} + \beta \frac{a}{k'}.$$

  Solving we get $k' = \phi(k) = \frac{\beta a A}{1+\beta a} k^\alpha$. *As the function inside brackets in the Bellman equation is concave,* k' *is a truly maximizer.*

- *Plug the maximizer into the functional equation to determine the unknown parameters.*

$$\begin{aligned} a \ln k + b &= \ln\left(Ak^\alpha - \phi(k)\right) + \beta(a \ln \phi(k) + b) \\ &= \ln\left(\frac{a}{1+\beta a} k^\alpha\right) + \beta b + \beta a \ln\left(\frac{\beta a A}{1+\beta a} k^\alpha\right). \end{aligned}$$

  Thus,

$$\begin{aligned} a &= \frac{\alpha}{1-\beta\alpha}, \\ b &= \ln\left(\frac{A}{1+\beta a}\right) + \beta b + \beta a \ln\left(\frac{\beta a A}{1+\beta a}\right), \end{aligned}$$

  that represent an admissible solution since $\alpha\beta < 1$. *Note that substituting* a *into* $k' = \phi(k)$ *we get*

$$k_{t+1} = \alpha\beta A k_t^\alpha. \tag{7}$$

- *Check that both the solution found and the maximizer fulfill Theorem 1.12 and 1.13, respectively.*

  Note that the value function is increasing in capital, and that the maximal growth rate of the capital path is α. *Thus, it suffices to check that the hypothesis of the theorem holds for the path given by* $k_{t+1} = \alpha\beta A k_t^\alpha$, *with* k_0 *given. We have*

$$k_1 = \alpha\beta A k_0^\alpha, \quad k_2 = \alpha\beta A k_1^\alpha = (\alpha\beta A)^{1+\alpha} k_0^{\alpha^2}, \quad \ldots$$

and in general,

$$k_t = (\alpha\beta A)^{1+\alpha+\dots+\alpha^{t-1}} k_0^{\alpha^t} \to (\alpha\beta A)^{(1-\alpha)^{-1}} \qquad \text{as } t \to \infty.$$

Hence, for any path $\{k_t\}_{t=0}^{\infty}$

$$\lim_{t\to\infty} \beta^t f(k_t) = \lim_{t\to\infty} \beta^t(a \ln k_t + b) = \lim_{t\to\infty} \beta^t(a(1-\alpha)^{-1} \ln(\alpha\beta A) + b) = 0.$$

Thus, $f = v$.

We conclude that the optimal policy function is

$$\phi(k) = \alpha\beta A k^{\alpha}.$$

Notice that $\phi(k)$ *is the capital for the next period that characterizes the optimal consumption,* $c(k) = Ak^{\alpha} - \phi(k)$. *Given the initial* k_0, *these functions generate sequences of optimal capital and consumption* $k_{t+1}^* = \phi(k_t^*)$, $c_t^* = c(k_t^*)$, *respectively.*

1.6 Reduced form models

When the dimension of the action space is the same as the dimension of the state space and for any fixed x the mapping $x' = q(x, a)$ is a bijection, then it is possible to write the problem in an equivalent form. This consists in eliminating the decision variable and formulating the problem in terms of today's and tomorrow's states only. Let the correspondence $\Gamma : X \rightrightarrows X$ given by

$$\Gamma(x) = \{q(x, a) \,:\, a \in D(x)\},$$

that gives the set of possible tomorrow's states if the today's state is x. Suppose that from $x' = q(x, a)$ it is possible to solve for $a = \xi(x, x')$. Then, the Bellman equation becomes

$$v(x) = \sup_{x' \in \Gamma(x)} \left\{ W(x, x') + \beta v(x') \right\},$$

where $W(x, x') = U(x, \xi(x, x'))$. Thus, now the problem reduces to find the optimal next state of the system and the optimal action is implicit in the formulation. Reduced form models are less general than the model we are working with, thus all the results developed so far are obviously valid for reduced form models. The advantage of using this formulation is that cleaner results are obtained. Moreover most interesting economic models can be written in reduced form easily.

Note that now an optimal policy means an optimal path, since the decisions variables have been hidden in the formulation. Nevertheless, they can be found easily using $a = \xi(x, x')$.

1.7 Euler Equations

We say that a policy $\pi \in \Pi$ is interior if it prescribes at any t an action a_t such that there exists $\delta_t > 0$ such that both $B(a_t, \delta_t) \subseteq D(x_t)$ and $B(x_{t+1}, \delta_t) \subseteq \Gamma(x_t)$ for all t.

It happens that for interior optimal policies and smooth data is is possible to give first order conditions of optimality independent of the value function. In the result below, U_a (resp. U_x) is the gradient of U with respect to the decision variables a (resp. state variables x), and q_a (resp. q_x) is the Jacobian matrix of q with respect to a (resp. x). The symbol $^\top$ means transpose.

Theorem 1.15 (Euler Equations)**.** *Suppose that both* U *and* q. *If* π *is an interior optimal policy then there exists* $\{\lambda_t\}_{t=0}^{\infty}$, *such that for all* $t = 0, 1, \dots$

$$0 = U_a(x_t, a_t) + q_a^\top(x_t, a_t)\lambda_t, \tag{8}$$

$$\lambda_t = \beta U_x(x_{t+1}, a_{t+1}) + \beta q_x^\top(x_{t+1}, a_{t+1})\lambda_{t+1}, \tag{9}$$

where $\{x_t\}_{t=1}^{\infty} \in X_\pi(x_0)$.

Remark 1.16. *When the action space and the state space have the same dimension and* $\det(q_a) \neq 0$ *it is possible to eliminate* λ_t, λ_{t+1} *from the formulation to get*

$$0 = \beta U_x(x_{t+1}, a_{t+1}) + q_a^{-\top}(x_t, a_t) U_a(x_t, a_t) - \beta q_x^{\top}(x_{t+1}, a_{t+1}) q_a^{-\top}(x_{t+1}, a_{t+1}) U_a(x_{t+1}, a_{t+1}). \tag{10}$$

Proof. Given x_t, consider a slight variation of a_t, $\tilde{a}_t$, the associated $\tilde{x}_{t+1} = q(x_t, \tilde{a}_t)$ and an action $\tilde{a}_{t+1} \in D(\tilde{x}_{t+1})$ such that $x_{t+2} = q(\tilde{x}_{t+1}, \tilde{a}_{t+1})$. That is, after changing the optimal a_t, we return to the original optimal path in step $t+2$. This is possible since the optimal policy is interior and the deviation is small enough.

$$\begin{tikzcd}
x_t \arrow[r, "a_t"] \arrow[dr, "\tilde{a}_t"'] & x_{t+1} \arrow[r, "a_{t+1}"] & x_{t+2} \\
& \tilde{x}_{t+1} \arrow[ur, "\tilde{a}_{t+1}"'] &
\end{tikzcd}$$

Since the remainder terms in the sum defining $I(x_0, \pi)$ are the same, we center on the effect of the deviation in the summands t and $t+1$. An optimal policy must be such that it maximizes (Principle of Optimality)

$$\beta^t U(x_t, a_t) + \beta^{t+1} U(x_{t+1}, a_{t+1})$$

subject to $x_{t+1} = q(x_t, a_t)$ and $x_{t+2} = q(x_{t+1}, a_{t+1})$. Thus, we write the necessary conditions of optimality for this finite dimensional problem with the Lagrangian

$$L(a_t, x_{t+1}, \lambda_t, \lambda_{t+1}) = U(x_t, a_t) + \beta U(x_{t+1}, a_{t+1}) + \lambda_t^{\top}(q(x_t, a_t) - x_{t+1}) + \beta \lambda_{t+1}^{\top}(q(x_{t+1}, a_{t+1}) - x_{t+2}).$$

The first order conditions give

$$\begin{aligned}
L_{a_t} &= 0 = U_a(t) + q_a^{\top}(t) \lambda_t, \\
L_{x_{t+1}} &= 0 = \beta U_x(t+1) - \lambda_t + \beta q_x^{\top}(t+1) \lambda_{t+1},
\end{aligned}$$

where we have simplified notation. □

Example 1.17 (Production with labor choice)**.** *Recall this model studied in the Problems, where we found the Euler Equations using the Envelope Theorem of Benveniste and Scheinkman. Let us find now the EE using the result above. Equation* (8) *is*

$$\begin{pmatrix} u_c(t) \\ u_\ell(t) \end{pmatrix} + \lambda_t \begin{pmatrix} -1 \\ f_\ell(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $u_c(t) f_\ell(t) = -u_\ell(t)$ *and* (9) *is*

$$\lambda_t = \beta f_k(t+1) \lambda_{t+1}$$

or $u_c(t) = \beta f_k(t+1) u_c(t+1)$, *that coincide of course, with those found with the Envelope Theorem.*

Recall the definition given in Section 2.6 of a problem formulated in reduced form.

Corollary 1.18 (Euler Equations. Reduced form models)**.** *Suppose a reduced form model. Assume that* W *is differentiable and let* $\{x_t\}_{t=0}^{\infty}$ *be an optimal policy (or path) such that* x_{t+1} *is interior to* $\Gamma(x_t)$ *for all* $t = 0, 1, \ldots$. *Then*

$$W_y(x_t, x_{t+1}) + \beta W_x(x_{t+1}, x_{t+2}) = 0, \quad t = 0, 1, \ldots. \tag{11}$$

Proof. Since the model is in reduced form, there exists ξ such that $a_t = \xi(x_t, x_{t+1})$. Consider the equality $q(x_t, \xi(x_t, x_{t+1})) = x_{t+1}$ and derive with respect to x_t and with respect to x_{t+1} to get

$$\begin{aligned} q_a \xi_y &= I, \\ q_x + q_a \xi_x &= 0, \end{aligned}$$

so that $\xi_y = q_a^{-1}$ and $\xi_x = -q_a^{-1} q_x$. Now, from the definition of W we have

$$\begin{aligned} W_x &= U_x + \xi_x^\top U_a = U_x - q_x^\top q_a^{-\top} U_a, \\ W_y &= \xi_y^\top U_a = q_a^{-\top} U_a. \end{aligned}$$

Adding and using our shorthand notation above we get

$$W_y(x_t, x_{t+1}) + \beta W_x(x_{t+1}, x_{t+2}) = q_a^{-\top} U_a(t) + \beta U_x(t+1) - \beta q_x^\top q_a^{-\top} U_a(t+1),$$

which is null by (10). $\square$

Note that (11) is a second order system of difference equations. Thus, to get the full optimal path we need two initial conditions, at $t = 0$ and at $t = 1$. However, only x_0 is known in advance; x_1 has to be determined in the optimization process. It happens that a boundary condition at infinite substitute the initial unknown value x_1. It is called the transversality condition, and in general it is only a sufficient condition for optimality.

We establish the sufficient condition of optimality for interior paths only for reduced form models. Recall that $x_{t+1} \in \Gamma(x_t)$ subsumes now the two conditions $x_{t+1} = q(x_t, a_t)$, $a_t \in D(x_t)$, where Γ is an appropriated correspondence. Let us denote $X(x_0)$ the set of admissible path from x_0 in a reduced form model, and let $S(x_0, \{x_t\}_{t=1}^\infty) = \sum_{t=0}^\infty \beta^t W(x_t, x_{t+1})$.

Theorem 1.19 (Sufficient conditions for optimality. Reduced form models)**.** *Suppose that X is convex and that Γ has a convex graph. Suppose also that W is concave and differentiable in the interior of Ω. If for $x_0 \in X$, $\{x_t^*\}_{t=1}^\infty$ is an interior path satisfying* (11) *and the transversality condition*

$$\lim_{t\to\infty} \beta^t W_y(x_t^*, x_{t+1}^*)(x_{t+1} - x_{t+1}^*) \le 0, \tag{12}$$

for every other admissible path $\{x_t\}_{t=0}^\infty \in X(x_0)$, then $\{x_t^*\}_{t=1}^\infty$ is optimal.

Proof. For any $\{x_t\} \in X(x_0)$ consider the difference $D = S(x_0, \{x_t\}) - S(x, \{x_t^*\})$, which is well defined. Because W is concave and differentiable, we have

$$\begin{aligned} D &= \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t (W(x_t, x_{t+1}) - W(x_t^*, x_{t+1}^*)) \\ &\le \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \Big(W_x(x_t^*, x_{t+1}^*)(x_t - x_t^*) + W_y(x_t^*, x_{t+1}^*)(x_{t+1} - x_{t+1}^*) \Big) \equiv \lim_{T\to\infty} D_T. \end{aligned}$$

But rearranging terms, taking into account that $x_0^* = x_0$, we note that

$$D_T = \sum_{t=0}^{T-1} \beta^t \big(W_y(x_t^*, x_{t+1}^*) + \beta W_x(x_{t+1}^*, x_{t+2}^*) \big)(x_{t+1} - x_{t+1}^*) + \beta^T W_y(x_T^*, x_{T+1}^*)(x_{T+1} - x_{T+1}^*).$$

If (11) and (12) hold, then

$$D = \lim_{T\to\infty} D_T = \lim_{T\to\infty} \beta^T W_y(x_T^*, x_{T+1}^*)(x_{T+1} - x_{T+1}^*) \le 0,$$

hence $\{x_t^*\}_{t=0}^\infty$ is an optimal path. $\square$

Remark 1.20. *Note that* (12) *can be also written as*

$$\lim_{t\to\infty} \beta^t W_x(x^*_{t+1}, x^*_{t+2})(x^*_{t+1} - x_{t+1}) \le 0,$$

for a path that satisfies (11). *In many economic models, one of the cases being the Ramsey growth model, W is increasing with respect to the variables x and the state space is $X = \mathbb{R}^m_+$. Then*

$$D = \lim_{t\to\infty} \beta^t W_x(x^*_{t+1}, x^*_{t+2})(x^*_{t+1} - x_{t+1}) \le \lim_{t\to\infty} \beta^t W_x(x^*_{t+1}, x^*_{t+2}) x^*_{t+1},$$

and so with these assumptions the transversality condition can be expressed as

$$\lim_{t\to\infty} \beta^t W_x(x^*_t, x^*_{t+1}) x^*_t = 0, \tag{13}$$

the advantage been that the limit concerns only the path $\{x^*_t\}$.

Example 1.21 (Ramsey Model)**.** *Using the reduced form of the Ramsey Model in* (2) *we get $W(k, k') = u(f(k) - k')$, thus $W_k = u'(c)f'(k)$ and $W_{k'} = -u'(c)$. Hence the EE is*

$$-u'(f(k_t) - k_{t+1}) + \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) = 0, \qquad t = 0, 1, 2, \dots \tag{14}$$

$k_0 > 0$ given.

For instance, suppose that $u(c) = \ln c$, $f(k) = Ak^\alpha$. Then the EE is

$$\frac{-1}{Ak_t^\alpha - k_{t+1}} + \beta \frac{\alpha A k_{t+1}^{\alpha-1}}{Ak_{t+1}^\alpha - k_{t+2}} = 0,$$

which is a non-linear difference equation of second order, with only one initial condition.

It is easy to check that the solution (7) *found in Example 1.14 satisfies the Euler Equation. To show this, substitute k_{t+1} and $k_{t+2} = (\alpha\beta A)^{1+\alpha} k_t^{\alpha^2}$ into the Euler Equation above to get*

$$\begin{aligned}
&\frac{-1}{A\cancel{k_t^\alpha} - \alpha\beta A\cancel{k_t^\alpha}} + \beta \frac{\alpha A(\alpha\beta A)^{\alpha-1}\cancel{k_t^{\alpha^2-\alpha}}}{A(\alpha\beta A)^\alpha \cancel{k_t^{\alpha^2}} - (\alpha\beta A)^{1+\alpha}\cancel{k_t^{\alpha^2}}} \\
&= \frac{-1}{A(1-\alpha\beta)} + \beta \frac{\alpha A\cancel{(\alpha\beta A)^{\alpha-1}}}{A(\alpha\beta A)^{\cancel{\alpha}} - \cancelto{(\alpha\beta A)^2}{(\alpha\beta A)^{1+\alpha}}} \\
&= \frac{-1}{A(1-\alpha\beta)} + \cancel{\beta} \frac{\cancel{\alpha A}}{\cancel{\alpha A}\cancel{\beta} A(1-\alpha\beta)} = 0
\end{aligned}$$

Finally, let us check that the transversality condition (13) *holds for this model. We have*

$$\beta^t W_x(k_t, k_{t+1})k_t = \beta^t \frac{\alpha A k_t^{\alpha-1}}{Ak_t^\alpha - k_{t+1}} k_t = \beta^t \frac{\alpha A k_t^{\alpha-1}}{Ak_t^\alpha - \alpha\beta A k_t} k_t = \beta^t \frac{\alpha}{1-\alpha\beta} \to 0 \qquad \textit{as } t \to \infty.$$

1.8 Steady States

Suppose that in a problem we have found a policy function $a = \phi(x)$ so that we can compute the successive optimal states

$$x_{t+1} = h(x_t) \equiv q(x_t, \phi(x_t)), \qquad x_0 \text{ given.}$$

This is a first order system of differences equations giving the optimal dynamics of the state variable, so it is very important to study its properties, specially its long–run behavior and asymptotic properties.

We shall say that $x^0 \in X$ is an optimal equilibrium point, stationary point or steady state of the system is x^0 is a fixed point of h, $h(x^0) = x^0$. Assuming h is continuous, fixed points of g are the only possible limits of sequences $\{x_t\}_{t=0}^{\infty}$ generated with h:

$$x^0 = \lim_{t\to\infty} x_{t+1} = \lim_{t\to\infty} h(x_t) = h(\lim_{t\to\infty} x_t) = h(x^0).$$

This motivates the definition. One problem with this approach is that in general the function h is not known in advance. However, we know that interior solutions to the optimization problem satisfy Euler Equations. Center on the reduced models case, this means

$$0 = W_y(x_t, x_{t+1}) + \beta W_x(x_{t+1}, x_{t+2}), \qquad t = 0, 1, \dots.$$

Hence if x^0 is interior to $\Gamma(x^0)$, a necessary condition for x^0 to be a stationary point is

$$0 = W_y(x^0, x^0) + \beta W_x(x^0, x^0).$$

If W is concave, this condition is also sufficient, that is, $x^0 \in \Gamma^*(x^0)$, where $\Gamma^*(x)$ denotes the optimal choice correspondence at x. In general, points x^0 satisfying the Euler Equations will be called steady states, without the word "optimal".

Example 1.22 (The Ramsey growth model)**.** *In the Ramsey model the steady states are solutions of the equation:*

$$0 = -u'(f(k) - k) + \beta u'(f(k) - k) f'(k)$$

or $u'(f(k) - k)(-1 + \beta f'(k)) = 0$*. Since we are assuming* $u' > 0$*, the* interior *steady states are points* $k^0 > 0$ *satisfying* $f'(k^0) = 1/\beta$*. Obviously this equation has only one solution if* f *is strictly concave. Notice that other, noninterior steady states could exist, and in fact the optimal policy in the Ramsey model has 0 as steady state, since* $\Gamma(0) = \{0\}$*. Thus, in principle, we could have either* $k_t \to k^0$*,* $k_t \to 0$ *or none of the above, as* $t \to \infty$*. We will see that under suitable conditions on the production function, the sequence converges to* k^0*.*

1.9 Existence and uniqueness of a fixed point for the Bellman operator

We will prove here that the Bellman operator

$$(\mathcal{B}f)(x) = \sup_{\substack{y=q(x,a)\\ a\in D(x)}} \{U(x,a) + \beta f(y)\},$$

is a contraction mapping on the metric space of bounded functions if the one step reward function is itself bounded. Let $B(X)$ be the space of bounded functions with the supremum norm, $\|f\| = \sup_{x\in X} |f(x)|$. We know that $B(X)$ is a complete metric space.

Theorem 1.23 (Blackwell (1965))**.** *Let* $T : B(X) \longrightarrow B(X)$ *be an operator satisfying*

1. $f \leq g \Rightarrow Tf \leq Tg$ *(*T *is monotonous);*
2. *For some* $0 < \beta < 1$*,* $T(f + c) \leq Tf + \beta c$*, where* c *is any constant function (*T *discounts).*

Then, T *is a contraction on* $B(X)$ *of modulus* β*.*

Proof. Given $f, g \in B(X)$, note that

$$f \leq g + \|f - g\|, \qquad g \leq f + \|f - g\|.$$

By monotonicity and discounting

$$Tf \leq T(g + \|f - g\|) \leq Tg + \beta\|f - g\|, \qquad Tg \leq T(f + \|f - g\|) \leq Tf + \beta\|f - g\|.$$

Hence

$$Tf - Tg \le \beta \|f - g\|, \qquad -(Tf - Tg) \le \beta \|f - g\|,$$

that is, for any $x \in X$

$$|Tf(x) - Tg(x)| \le \beta \|f - g\|.$$

Taking the supremum in X we get

$$\|Tf - Tg\| \le \beta \|f - g\|.$$

$\square$

Let us define the function $\psi : X \longrightarrow X$

$$\psi(x) = \sup_{a \in D(x)} U(x, a).$$

Theorem 1.24. *Suppose that $\psi \in B(X)$. Then the Bellman operator is a contraction on $B(X)$.*

Proof. It is clear that $\mathcal{B}$ is monotonous because it is defined as a supremum. Discounting is also easily checked, as in fact $\mathcal{B}(f + c) = \mathcal{B}f + \beta c$.

Let us show that $\mathcal{B} : B(X) \longrightarrow B(X)$. Let $f \in B(X)$. By monotonicity we have

$$\mathcal{B}(-\|f\|) \le \mathcal{B}f \le \mathcal{B}(\|f\|),$$

hence, using the discount property with $c = \pm\|f\|$ we have

$$\psi - \beta\|f\| \le \mathcal{B}f \le \psi + \beta\|f\|.$$

It is thus clear that

$$\|\mathcal{B}f\| \le \|\psi\| + \beta\|f\| < \infty,$$

$\square$

Thus, from the Theorem of Banach, we conclude that the Bellman equation has a unique bounded solution, which is clearly the value function of the optimization problem, see Theorem 1.12. It can be approached by successive iterations of $\mathcal{B}$ applied to any initial bounded function f. That is, $\mathcal{B}^n f v$ uniformly as $n \to \infty$. This provides a first approximation method for solving the original problem. Note, however, that without further assumptions existence of an optimal policy is not guaranteed.

Theorem 1.25. *Suppose that ψ is bounded, that both U and q are continuous functions and that the correspondence D is continuous and compact-valued. Then an optimal stationary Markov policy exists. If the solution of $\max_{\substack{y=q(x,a)\\ a \in D(x)}} \{U(x, a) + \beta v(y)\}$ is unique, it is continuous on X.*

Proof. Let $f \in B(X)$ be continuous. Then, by the Theorem of the Maximum, $\mathcal{B}f$ is continuous. Since $C_b(X) \subseteq B(X)$ is closed, the fixed point of $\mathcal{B}$ belongs to $C_b(X)$, that is, the value function v is continuous. It follows that we can change sup by max in the definition of the Bellman operator and that for any $x \in X$ the optimal correspondence $D^*(x)$ is uhc and compact-valued. Selecting $a^*(x) \in D^*(x)$ we can apply Theorem 1.13, thus an optimal policy exists. If the solution is unique, it is continuous. $\square$

Now we establish new properties of the value function and the optimal correspondence for *reduced form* models (see Section 2.6). We assume that W is continuous and bounded and that Γ is continuous and compact-valued.

Theorem 1.26. *Assume that W is concave and that the graph of Γ is convex. Then the unique bounded solution v of the Bellman equation is concave and the optimal correspondence is convex-valued. Moreover, if W is strictly concave with respect to y, the unique policy function is continuous.*

Proof. Observe that the set of bounded, continuous and concave functions is closed in the supremum metric, and that under the assumptions, the Bellman operator maps this set into itself (please, consult the notes of Math II). Then, as the fixed point is $v = \lim_{n\to\infty} \mathcal{B}^n f$ for any continuous and bounded function f, we pick one that is concave, so that the fixed point is also concave. $\square$

Theorem 1.27. *Assume that for each y, $W(\cdot, y)$ is strictly increasing in each of its first m arguments and that Γ is monotone in the sense that $x \leq \tilde{x}$ (componentwise) implies $\Gamma(x) \subseteq \Gamma(\tilde{x})$. Then the unique bounded solution v of the Bellman equation is strictly increasing.*

Proof. Observe that the set of bounded, continuous and strictly increasing functions is closed in the supremum metric, and that under the assumptions, the Bellman operator maps this set into itself. $\square$

Example 1.28. *[The Ramsey growth model] Recall the problem of optimal growth in a one-good economy, expressed in reduced form*

$$\begin{aligned} &\max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t W(k_t, k_{t+1}) \\ &\quad s.t. \quad 0 \leq k_{t+1} \leq f(k_t), \quad t = 0, 1, \ldots \\ &\qquad\quad \text{given } k_0 \geq 0. \end{aligned}$$

, where $W(k, k') = U(f(k) - k')$ for some utility function U. Corresponding to this problem we have the Bellman equation

$$v(k) = \sup_{0 \leq k' \leq f(k)} \{W(f(k) - k') + \beta v(k')\}.$$

Consider the following assumptions.

- *U is continuous.*
- *f is strictly increasing, continuous and there exists $\hat{k} > 0$ such that $f(\hat{k}) = \hat{k}$, $f(k) < k$ for all $k > \hat{k}$ and $k \leq f(k) \leq \hat{k}$ for all $0 \leq k \leq \hat{k}$. This means that $\hat{k}$ is the largest capital stock that can be maintained.*

Note that Theorem 1.25 does no apply since U is allowed to unbounded in $\mathbb{R}_+$. This is the reason to impose a maximum sustainable capital condition. As we will prove, there exists a constant M such that for any $k_0 \leq M$, the optimal path $k_t^* \in [0, M]$ for all t. Thus, we restrict the state space to the compact interval $[0, M]$, on which U is bounded as it is continuous.

Let us show our claim. In fact, let $M = \max(k_0, \hat{k})$. If $k_0 \leq \hat{k}$, then for any path $\{k_{t+1}\}$ we have

$$\begin{aligned} &0 \leq k_1 \leq f(k_0) \leq f(\hat{k}) = \hat{k}, \\ &0 \leq k_2 \leq f(k_1) \leq f(\hat{k}) = \hat{k}, \\ &\vdots \end{aligned}$$

that proves $k_t \leq \hat{k} \leq M$ for all t. If $k_0 > \hat{k}$, then $k_1 \leq f(k_0) < k_0$; if $k_1 > \hat{k}_0$, then $k_2 \leq f(k_1) < k_1 < k_0$. If $k_1 \leq \hat{k}_0$, then $k_2 \leq f(k_1) \leq \hat{k}$, and hence for all $t \geq 2$, $k_t \leq \hat{k}$. Thus we have proved that is $k_0 > \hat{k}$, then $k_t \leq \max(k_0, \hat{k}) = M$.

It is easy to check that Theorem 1.25 is now applicable, and thus the Bellman equation admits a unique solution in $X = [0, M]$, which is continuous and is in fact the value function of the problem *for initial $k_0 \in [0, M]$. The optimal policy correspondence is non empty valued, compact valued and u.h.c. Hence an optimal policy of the sequence problem exists.*

Important properties (will be proved in a problem set):

- *If f is (strictly) concave, then the graph of Γ is convex, so that if U is (strictly) concave, then v is (strictly) concave. If U is strictly concave, then the optimal policy h is unique, thus continuous. Here, $k_{t+1} = h(k_t)$ is next period's capital stock along the optimal path if this period's stock is k_t.*

- *If U is strictly increasing, then v is strictly increasing.*

- *Assume U is twice continuously differentiable and satisfies $U(0) = 0$, $U' > 0$, $U'' < 0$ and the Inada Condition $U'(0+) = \infty$. Function f is twice continuously differentiable and satisfies $f(0) = 0$. Its derivative satisfy $f' > 0$, $f'' < 0$, $\lim_{x\to\infty} f'(x) < 1$.*

  Then the optimal policy is interior: $\forall t$, $c_t > 0$, $k_t > 0$, if $k_0 > 0$. Hence the Euler Equation (14) *is satisfied. Moreover, the optimal policy h and the optimal consumption policy $\phi(k) = f(k) - h(k)$ are both monotone increasing. Consequently, $\{k_t^*\}$ defined as $k_{t+1}^* = h(k_t^*)$, $t \geq 1$, $k_0^* = k_0$ is either a monotone increasing or decreasing sequence.*

  If $f'(0+) > 1/\beta$, then the unique k^0 satisfying $f'(k^0) = 1/\beta$ is the unique fixed point of h on $(0, M)$. Moreover, the optimal path $\{k_t^*\}$ converges to the optimal steady state k^0. If $f'(0+) \leq 1/\beta$, then the optimal path $\{k_t^*\}$ converges to zero.

To show some of the properties above, it is useful to employ the Envelope Theorem of Benveniste and Scheinkman, that we now state for a general model in reduced form. Notice that interiority of the optimal path is a crucial assumption. We say that a point x is interior to a subset X if $x \in X$ is not at the boundary of X, that is, if there exists $\delta > 0$ such that $B(x, \delta) \subseteq X$.

Theorem 1.29 (Envelope Theorem)**.** *Let a reduced form model and let x_0 an interior point of X. Suppose that the optimal policy $y_0 = h(x_0)$ is interior to $\Gamma(x_0)$ and that the one-step reward function W is concave in Ω and differentiable at (x_0, y_0). Then, the value function is differentiable at x_0 and the derivative is*

$$v_x(x_0) = W_x(x_0, h(x_0)).$$

1.10 Value Iteration

Theorem 1.25 provides a computational method to find the value function v: since the value function is the unique fixed point of the Bellman operator $\mathcal{B}$ and for any $f \in B(X)$, $\|\mathcal{B}^n f - v\| \to 0$ as $n \to \infty$, this suggest the following value iteration algorithm:

```
pick any     f ∈ B(X)
repeat
|            compute Bf
|            set e = ||Bf − f||
|            set f = Bf
until        e <tolerance
solve for a  f-greedy policy φ
```

Where, given $f \in B(X)$, we say that the policy ϕ is f-greedy if for any $x \in X$

$$\phi(x) \in \mathrm{argmax}_{a \in D(x)} \{U(x, a) + \beta f(q(x, a))\}.$$

Please consult the excellent book of J. Stachurski "Economic Dynamics: Theory and Computation" (The MIT Press, 2009), from which this section is inspired.

Note that value function iteration is given by the recursion

$$v_n(x) = \max_{a \in D(x)} \{U(x,a) + \beta v_{n-1}(q(x,a))\}, \qquad n = 1, 2, \ldots, \qquad v_0 \in B(X).$$

Thus, each v_n is the value function of a finite-horizon problem (with bequest function v_0) that approximates the infinite-horizon problem.

Contrary to the guess and verify method, the value function iteration always works, at least theoretically.

Example 1.30. *We have used the guess method in Example 1.14, where we assumed full depreciation, $\delta = 1$. Let us now apply the value function iteration*[1]

- *Start off with an initial v_0, say $v_0(k) = 0$.*
- *Set $v_1(k) = \sup_{k' \in [0, f(k)]} \{\ln(Ak^\alpha - k') + \beta 0\} = \ln A + \alpha \ln k$.*
- *Set $v_2(k) = \sup_{k' \in [0, f(k)]} \{\ln(Ak^\alpha - k') + \beta(\ln A + \alpha \ln k')\}$.*

  The v_2-greedy policy is $k' = \frac{\beta\alpha}{1+\beta\alpha} Ak^\alpha$, so

$$\begin{aligned} v_2(k) &= \ln\left(Ak^\alpha - \frac{\beta\alpha}{1+\beta\alpha}Ak^\alpha\right) + \beta\left(\ln A + \alpha \ln\left(\frac{\beta\alpha}{1+\beta\alpha}Ak^\alpha\right)\right) \\ &= \ln\left(\frac{A}{1+\beta\alpha}\right) + \beta \ln A + \alpha\beta \ln\left(\frac{\beta\alpha A}{1+\beta\alpha}\right) + (\alpha + \alpha^2\beta)\ln k. \end{aligned}$$

- *Repeat until one sees the pattern of the sequence. It converges to*

$$v(k) = \frac{1}{1-\beta}\left(\ln(A(a - \beta\alpha)) + \frac{\beta\alpha}{1-\beta\alpha}\ln(A\beta\alpha)\right) + \frac{\alpha}{1-\alpha\beta}\ln k.$$

Usually, the value function algorithm is used to find numerical approximations of the value function. Let us show the method over the above problem, with data: $\delta = 1$, $U(c) = \ln c$, $f(k) = 10k^{0.3}$ and $\beta = 0.95$. The algorithm considers a discretization of the state space and computes the value function at the grid points. The grid should contain the steady state. Let us calculate for this model:

$$f'(k^0) = \frac{1}{\beta} \quad \Rightarrow \quad 3(k^0)^{-0.7} = \frac{1}{0.95} \quad \Rightarrow \quad k^0 \approx 5.73.$$

We also want to find $\hat{k}$, the maximal sustainable capital stock.

$$f(\hat{k}) = \hat{k} \quad \Rightarrow \quad 3\hat{k}^{0.3} = \hat{k} \quad \Rightarrow \quad \hat{k} \approx 26.7.$$

The minimum grid point should be larger than 0, and the maximum grid point minimum than $\hat{k}$. In fact, let us take the grid $K = \{1, 2, 3, 4, 5, 6\}$. We compute

$$v_n(k) = \max_{\substack{0 \le k' \le 10k^{0.3} \\ k' \in K}} \{\ln(10k^{0.3} - k') + 0.95 v_{n-1}(k')\}.$$

[1] Yes, a flagrant contradiction is here. The one-step reward function is $U(c) = \ln c$, thus $\psi(k) = \sup_{c \in [0, Ak^\alpha]} \ln c = \ln A + \alpha \ln k$, that is not bounded in none interval of the form $(0, M]$. Thus, we cannot apply Theorem 1.25 and we do not know whether the value function iteration converges to the solution we have found by the guessing method. Other approaches exists that analyze this problem.

$$
\begin{array}{lll}
h_0(1): & k'=1: & \ln(10-1) = \mathbf{2.1972} \\
& k'=2: & \ln(10-2) = \\
& k'=3: & \ln(10-3) = \\
& | & |
\end{array}
$$

$$
\begin{array}{lll}
h_0(2): & k'=1: & \ln(10\cdot 2^{0.3}-1) = \mathbf{2.4258} \\
& k'=2: & \ln(10\cdot 2^{0.3}-2) = \\
& k'=3: & \ln(10\cdot 2^{0.3}-3) = \\
& | & |
\end{array}
$$

$$\vdots$$

$$
\begin{array}{lll}
h_1(1): & k'=1: & \ln(10-1) + 0.95\cdot 2.1972 = 4.2846 \\
& k'=2: & \ln(10-2) + 0.95\cdot 2.4258 = \mathbf{4.3839} \\
& k'=3: & \ln(10-3) + 0.95\cdot 2.5575 = 4.3755 \\
& k'=4: & \ln(10-4) + 0.95\cdot 2.6502 = 4.3094 \\
& k'=5: & \ln(10-5) + 0.95\cdot 2.7217 = 4.1951 \\
& k'=6: & \ln(10-6) + 0.95\cdot 2.7799 = 4.0272
\end{array}
$$

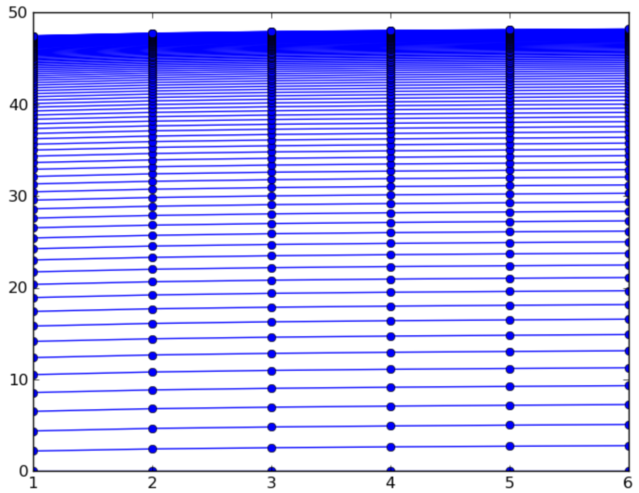
$$
\begin{array}{lll}
h_1(2): & k'=1: & \ln(10\cdot 2^{0.3}-1) + 0.95\cdot 2.1972 = 4.5132 \\
& k'=2: & \ln(10\cdot 2^{0.3}-2) + 0.95\cdot 2.4258 = 4.6377 \\
& k'=3: & \ln(10\cdot 2^{0.3}-3) + 0.95\cdot 2.5575 = \mathbf{4.6609} \\
& k'=4: & \ln(10\cdot 2^{0.3}-4) + 0.95\cdot 2.6502 = 4.6353 \\
& k'=5: & \ln(10\cdot 2^{0.3}-5) + 0.95\cdot 2.7217 = 4.5751 \\
& k'=6: & \ln(10\cdot 2^{0.3}-6) + 0.95\cdot 2.7799 = 4.4833
\end{array}
$$

Continuing in this way and selecting tolerance $e = 0.01$, one finds after 108 iterations

Numerical Implementation

k	$v_0(k)$	$h_0(k)$	$v_1(k)$	$h_1(k)$	$v_2(k)$	...	$v_{108}(k)$
1	0	1	2.1972	2	4.3839		47.4738
2	0	1	2.4258	3	4.6609		47.7591
3	0	1	2.5575	3	4.8187		47.9340
4	0	1	2.6502	4	4.9298		48.0531
5	0	1	2.7217	4	5.0197		48.1430
6	0	1	2.7799	4	5.0803		48.2211

The approximations are shown in the following graph



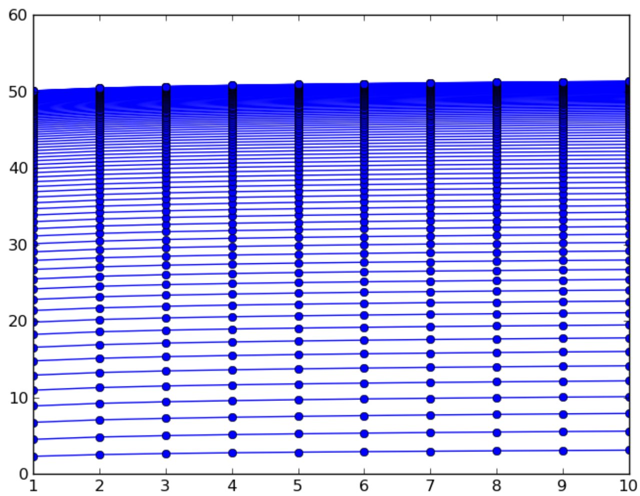
The greedy policy is $h_{108}(k) = (3, 3, 4, 4, 4, 5)$.

Consider now $\delta = 0.7$*, so that the steady state solves*

$$3k^{-0.7} + (1 - 0.7) = \frac{1}{0.95} \quad \Rightarrow \quad k^0 \approx 7.2097.$$

Let the state space be $K = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. *Now*

$$h_{108}(k) = (3, 4, 5, 6, 6, 7, 7, 7, 8, 8).$$



1.11 Policy Function Iteration or Howard improvement algorithm

One of the most computationally demanding part of value function iteration is the computation of the policy function. Moreover, the convergence of v_n to the value function can be very slow. The policy improvement algorithm is an alternative that proceeds as follows. Given a policy ϕ, compute the value v^ϕ of this policy, that is

$$v^\phi(x) = \sum_{t=0}^{\infty} \beta^t U(x_t, \phi(x_t)), \qquad x_{t+1} = q(x_t, \phi(x_t)).$$

For practical purposes, it is usual to evaluate the right hand side with large T, working with an approximated value of v^ϕ. Then compute a v^ϕ-greedy policy ϕ' and continue in this fashion computing the value $v^{\phi'}$ until $\phi' = \phi$.

pick any	ϕ
repeat	
	compute v^ϕ
	compute a v^ϕ-greedy policy ϕ'
	set $e = \phi - \phi'$
	set $\phi = \phi'$
until	$e = 0$

Example 1.31. *Consider again Example 1.14.*

- *Pick a feasible policy function:* $h_0(k) = \frac{1}{2}Ak^\alpha$ *and consider the path* $k_{t+1} = h_0(k_t)$.
- *Compute*

$$\begin{aligned} v^{h_0}(k) = S(k, \{k_{t+1}\}) &= \sum_{t=0}^{\infty} \beta^t \ln\left(Ak_t^\alpha - \frac{1}{2}Ak_t^\alpha\right) \\ &= \sum_{t=0}^{\infty} \beta^t \ln\left(\frac{1}{2}Ak_t^\alpha\right) \\ &= \sum_{t=0}^{\infty} \beta^t \left(\ln\left(\frac{1}{2}A\right) + \alpha \ln k_t\right). \end{aligned}$$

Note that

$$k_t = \frac{1}{2}Ak_{t-1}^\alpha = \frac{1}{2}A\left(\frac{1}{2}Ak_{t-2}^\alpha\right)^\alpha = \left(\frac{1}{2}A\right)^{1+\alpha} A^{1+\alpha} k_{t-2}^{\alpha^2},$$

hence $k_t = Dk_0^{\alpha^t}$ *for a given constant* D. *Thus plugging this into the expression for* v^{h_0} *we have*

$$v^{h_0}(k) = \sum_{t=0}^{\infty} \beta^t \left(\ln\left(\frac{1}{2}A\right) + \alpha \ln D + \alpha^{t+1} \ln k_0\right) = E + \frac{\alpha}{1-\beta\alpha} \ln k_0.$$

- *Compute*

$$\max_{k'} \left\{ \ln\left(Ak^\alpha - k'\right) + \beta\left(E + \frac{\alpha}{1-\beta\alpha} \ln k'\right) \right\}.$$

Taking the first-order condition yields

$$\frac{-1}{Ak^\alpha - k'} + \frac{\beta\alpha}{A - \beta\alpha} \frac{1}{k'} = 0.$$

thus

$$k' = \alpha\beta Ak^\alpha.$$

The policy improvement algorithm converges in a single step.

1.12 Deterministic Dynamics

Consider the Ramsey model of Example 1.28

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$
$$\text{s.t.} \quad k_{t+1} = f(k_t) - c_t$$
$$c_t, k_{t+1} \geq 0, \quad k_0 \text{ given.}$$

where $f(k_t) = \tilde{f}(k_t) + (1-\delta)k_t$.

The Euler Equation

$$U'(c_t) = \beta f'(k_{t+1}) U'(c_{t+1}) \tag{15}$$

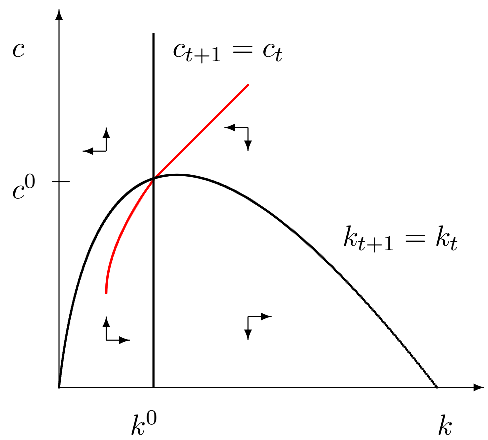
and the constraint

$$k_{t+1} = f(k_t) - c_t \tag{16}$$

form a system of two nonlinear difference equations that govern the dynamics of optimal consumption and capital.

Graphic analysis. Find the steady state (c^0, k^0):

$$c_{t+1} = c_t \Leftrightarrow f'(k) = \frac{1}{\beta},$$
$$k_{t+1} = k_t \Leftrightarrow c = f(k) - k.$$



The direction of the arrows in the picture is obtained as follows. Below (above) the graph of $c = f(k) - k$ we have, using (16)

$$k_{t+1} - k_t = f(k_t) - k_t - c_t > 0 \quad (< 0),$$

thus k increases (decreases). At the right (left) of vertical line $\beta f'(k) = 1$ we have, using (15)

$$\frac{U'(c_t)}{U'(c_{t+1})} = \beta f'(k_{t+1}) < (>) \beta f'(k^0) = 1,$$

since f' is strictly decreasing. Thus $c_{t+1} < (>) c_t$ since U' is strictly decreasing.

Saddle path. The optimal accumulation path is given by the red line segments; it is called saddle path, since the steady state (k^0, c^0) has this property: initial conditions on that locus generate optimal paths c_t^*, k_{t+1}^* that converge to the steady state as $t \to \infty$. Given k_0, consumption must take the value c_0 such that the system is in the saddle path and converges to the steady state. All other time paths diverge from the steady state and they are not optimal: either k_{t+1} becomes negative in finite time or the transversality condition is violated. If not, suppose suppose that $\tilde{c}_0 > c_0$. Then, the path violate the non-negativity constraint on k_{t+1} in finite time since the trajectory only can move only towards the boundary of the feasible region, as indicated by the arrows. Suppose now that $\tilde{c}_0 < c_0$. Then, along the new path consumption tends to zero and all investment goes to replace depreciated capital. The transversality condition is violated: to see this, note that for $k_0 > k^0$ we have $k_{t+1} > k^0$, thus $1/f'(k_{t+1}) > 1/f'(k_0)$ and $\alpha \equiv 1/f'(k_0) > 1/\beta > 1$. By (15)

$$\frac{U'(c_{t+1})}{U'(c_t)} = \frac{1}{\beta f'(k_{t+1})} > \frac{1}{\beta f'(k^0)} = \frac{\alpha}{\beta}.$$

Then

$$\frac{U'(c_{t+1})}{U'(c_t)} \frac{U'(c_t)}{U'(c_{t-1})} \cdots \frac{U'(c_1)}{U'(c_0)} > \frac{\alpha^{t+1}}{\beta^{t+1}},$$

hence

$$\beta^t U'(c_t) k_{t+1} \geq \alpha^t k_{t+1} \to \infty$$

as $t \to \infty$ since $\alpha > 1$ and $k_{t+1} > k^0 > 0$ for all t.

Linearization of non-linear systems. In general, the global dynamics of non-linear systems is difficult or impossible to analyze. The usual method is to make a local analysis, based on the behavior of a linear system that approximates the original one around the steady state. Then, it is needed to know what characterizes stability of linear systems (eigenvalues) and under what conditions one can infer local properties of stability for the non-linear system from the properties of the linear system.

Linear systems. A linear system of difference equations is

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + \mathbf{b},$$

where $\mathbf{x}_t$ is $m \times 1$ and A is a $m \times m$ matrix such that $I - A$ is not singular. Let $\mathbf{x}^0$ be the unique steady state

$$\mathbf{x}^0 = A\mathbf{x}^0 + b \Leftrightarrow \mathbf{x}^0 = (I - A)^{-1}\mathbf{b}.$$

Considering the new variable $\mathbf{y}_t = \mathbf{x}_t - \mathbf{x}^0$ (deviations from the steady state) one leads tot he homogeneous system $\mathbf{y}_{t+1} = A\mathbf{y}_t$.

$$\mathbf{y}_{t+1} = \mathbf{x}_{t+1} - \mathbf{x}^0 = A\mathbf{x}_t + \mathbf{b} - \mathbf{x}^0 = A(\mathbf{x}_t - \mathbf{x}^0) - (I - A)\mathbf{x}^0 + b = A(\mathbf{x}_t - \mathbf{x}^0) = A\mathbf{y}_t.$$

Thus, one can focus on the stability properties of the null vector for homogeneous systems. The solution of such a system is

$$\mathbf{y}_t = A^t \mathbf{y}_0, \qquad \mathbf{y}_0 = \mathbf{x}_0 - \mathbf{x}^0.$$

Hence, if we can compute A^t, we have an explicit solution of the linear system. However, to obtain A^t can be difficult. For this reason we introduce here the Jordan canonical form of a matrix, J. It is related with the original A by

$$J = P^{-1}AP$$

where P is non singular, and

$$J = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & J_k \end{pmatrix},$$

and each J_i is the finite union of Jordan boxes corresponding to λ_i, of the form

$$J_{i,1} = (\lambda_i), \quad J_{i,2} = \begin{pmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{pmatrix}, \dots, \quad J_{i,m_i} = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & 0 & \lambda_i \end{pmatrix}.$$

Here, $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of matrix A, that is, the distinct roots of the characteristic polynomial $|A - \lambda I| = 0$. The dimension of each submatrix depends of the number of independent generalized eigenvectors associated to each λ_i (a concept that is not needed in this course). If A is diagonalizable, then $m_i = 1$ for each $i = 1, \dots, m$ and $J_i = (\lambda_i)$ for each i.

Let $\mathbf{z}_t = P^{-1}\mathbf{y}_t$. Then

$$\mathbf{z}_{t+1} = P^{-1}\mathbf{y}_{t+1} = P^{-1}A\mathbf{y}_t = JP^{-1}\mathbf{y}_t \Rightarrow \mathbf{z}_{t+1} = J\mathbf{z}_t \Rightarrow \mathbf{z}_t = J^t\mathbf{z}_0,$$

where

$$J^t = \begin{pmatrix} J_1^t & 0 & \dots & 0 \\ 0 & J_2^t & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & J_m^t \end{pmatrix},$$

and

$$J_{i,j} = \begin{pmatrix} \lambda_i^t & t\lambda_i^{t-1} & \frac{t(t-1)}{2!}\lambda_i^{t-2} & \dots & \dots \\ 0 & \lambda_i^t & t\lambda_i^{t-1} & \dots & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_i^t & t\lambda_i^{t-1} \\ 0 & 0 & 0 & 0 & \lambda_i^t \end{pmatrix}.$$

The solution of the original system is thus

$$\mathbf{x}_t = \mathbf{x}^0 + PJ^tP^{-1}(\mathbf{x}_0 - \mathbf{x}^0). \tag{17}$$

Results on stability of linear systems. Looking at the structure of the matrices $J_{i,j}$ and to the form of the solution (17) it is easy to prove the following.

- (Globally asymptotically stable steady state).

$$\lim_{t\to\infty} \mathbf{x}_t = \mathbf{x}^0 \text{ for any initial condition } \mathbf{x}_0 \quad \Leftrightarrow \quad |\lambda_i| < 1 \ \forall i = 1, \dots, k.$$

- (Saddle point stability). Suppose that for $1 \le r < m$

$$|\lambda_1|, \dots, |\lambda_r| < 1, \quad |\lambda_{r+1}|, \dots, |\lambda_m| \ge 1.$$

(now we display all the eigenvalues, possibly with repetitions, depending of their algebraic multiplicity). Then $\mathbf{x}_t$ cannot converge to $\mathbf{x}^0$ unless $\mathbf{x}_0$ is such that the vector $P^{-1}(\mathbf{x}_0 - \mathbf{x}^0)$ has null components corresponding to the Jordan boxes of eigenvalues $|\lambda_{r+1}|, \dots, |\lambda_n|$. Equivalently,

$$\lim_{t\to\infty} \mathbf{x}_t = \mathbf{x}^0 \quad \Leftrightarrow \quad \mathbf{x}_0 = \mathbf{x}^0 + P\mathbf{z}_0, \text{ where } \mathbf{z}_0 \text{ has null components } z_{r+1}^0 = \dots = z_n^0 = 0.$$

In this case, the set of initial conditions $\{\mathbf{x}_0 : \mathbf{x}_0 = \mathbf{x}^0 + P\mathbf{z}_0\}$ is called the stable manifold.

Example 1.32. *Let the system*

$$x_{t+1} = x_t - \frac{1}{2}y_t + 1,$$
$$y_{t+1} = x_t - 1.$$

The matrix of the system is $\begin{pmatrix} 1 & -1/2 \\ 1 & 0 \end{pmatrix}$, with characteristic equation $\lambda^2 - \lambda + 1/2 = 0$. The (complex) roots are $\lambda_{1,2} = 1/2 \pm i/2$ of modulus $\rho = \sqrt{1/4 + 1/4} = 1/\sqrt{2} < 1$, hence the system is g.a.s. and the limit of any trajectory is the equilibrium point,

$$\mathbf{x}^0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Example 1.33. *Let the system*

$$x_{t+1} = x_t + 3y_t,$$
$$y_{t+1} = x_t/2 + y_t/2.$$

The matrix of the system is $\begin{pmatrix} 1 & 3 \\ 1/2 & 1/2 \end{pmatrix}$, *with characteristic equation* $\lambda^2 - (3/2)\lambda - 1 = 0$. *The roots are* $\lambda_1 = -1/2$ *and* $\lambda_2 = 2$. *The system is not g.a.s. However, there are initial conditions* $\mathbf{x}_0$ *such that the trajectory converges to the fixed point* $\mathbf{x}^0 = (0, 0)$. *The stable manifold is*

$$\mathbf{x}_0 = P \begin{pmatrix} a \\ 0 \end{pmatrix}, \qquad a \in \mathbb{R},$$

where the columns of P *are the independent eigenvectors of the matrix system. The eigenspaces are* $S(-1/2) =< (2, -1) >$ *and* $S(2) =< (3, 1) >$, *thus*

$$P = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}.$$

Hence the stable manifold is formed by vectors of the form $\mathbf{x}_0 = (2a, -a)$, *with* $a \in \mathbb{R}$. *This is the line* $x + 2y = 0$, *that coincides with* $S(-1/2)$.

Example 1.34. *Let he system*

$$x_{t+1} = y_t,$$
$$y_{t+1} = -x_t.$$

The matrix system has eigenvalues $\pm i$, *with modulus* $|\pm i| = 1$. *We cannot apply the above results. In fact the solution oscillates around the steady state* $(0, 0)$. *The path shows a cycle of period 4, that is,* $\mathbf{x}_{t+4} = \mathbf{x}_t$ *for every* $t \geq 0$.

Nonlinear systems. Given a function $g = (g_1, \ldots, g_m) : \mathbb{R}^m \longrightarrow \mathbb{R}^m$, a nonlinear system is

$$\mathbf{x}_{t+1} = g(\mathbf{x}_t).$$

Let $\mathbf{x}^0$ a steady state of the system, i.e., $g(\mathbf{x}^0) = x^0$. We suppose that each $g_i \in C^2(N(\mathbf{x}^0))$, where $N(\mathbf{x}^0)$ is a neighborhood of x^0. Let the matrix of partial derivatives of $g_1, \ldots, g_m$ (Jacobian) at the steady state

$$A = \left(\frac{\partial g_i}{\partial x_j}(\mathbf{x}^0) \right)_{i,j=1,\ldots,m}.$$

Replace the original nonlinear system by the linear one

$$\mathbf{x}_{t+1} = A\mathbf{x}_t.$$

Let $\lambda_1, \ldots, \lambda_k$ the distinct eigenvalues of matrix A.

Results on stability for nonlinear systems.

- (Local asymptotically stable steady state).

  $|\lambda_i| < 1 \; \forall i = 1, \ldots, k \quad \Rightarrow \quad \exists \tilde{N}(\mathbf{x}^0) \subseteq N(\mathbf{x}^0), \quad \lim_{t\to\infty} \mathbf{x}_t = \mathbf{x}^0$ for any initial condition $\mathbf{x}_0 \in \tilde{N}(\mathbf{x}^0)$.

- (Local saddle point stability). Suppose that for $1 \le r < m$

$$|\lambda_1|, \ldots, |\lambda_r| < 1, \quad |\lambda_{r+1}|, \ldots, |\lambda_m| \ge 1.$$

  Then there exists $\tilde{N}(\mathbf{x}^0) \subseteq N(\mathbf{x}^0)$ and a $C^2(\tilde{N}(\mathbf{x}^0))$ function $\phi : \tilde{N}(\mathbf{x}^0) \longrightarrow \mathbb{R}^{m-r}$ with a Jacobian matrix of rank $m-r$ such that for all $\mathbf{x}_0 \in \tilde{N}(\mathbf{x}^0)$ with $\phi(\mathbf{x}^0) = 0$, $\lim_{t\to\infty} \mathbf{x}_t = \mathbf{x}^0$.

  In this case, the set

$$\{\mathbf{x}_0 \in \tilde{N}(\mathbf{x}^0) \, : \, \phi(\mathbf{x}^0) = 0\}$$

  is called the stable manifold.

  Note that by the Implicit function Theorem, we can solve for $(x^0_{r+1}, \ldots, x^0_m)$ given $(x^0_1, \ldots, x^0_r)$ if the submatrix

$$\begin{pmatrix} \frac{\partial \phi_1}{\partial x_{r+1}}(\mathbf{x}^0) & \cdots & \frac{\partial \phi_1}{\partial x_m}(\mathbf{x}^0) \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_{m-r}}{\partial x_{r+1}}(\mathbf{x}^0) & \cdots & \frac{\partial \phi_{m-r}}{\partial x_m}(\mathbf{x}^0) \end{pmatrix}$$

  has maximal rank.

Example 1.35. *Let he system*

$$\begin{aligned} x_{t+1} &= x_t^2 - y_t, \\ y_{t+1} &= x_t. \end{aligned}$$

It has two steady states, $(0,0)$ *and* $(2,2)$*. The Jacobian matrix at any* (x,y) *is*

$$\begin{pmatrix} 2x & -1 \\ 1 & 0 \end{pmatrix}.$$

At point $(0,0)$ *the matrix is*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues $\lambda_{1,2} = \pm i$*, of modulus 1, hence we cannot deduce nothing for the nonlinear system. At* $(2,2)$ *the matrix is*

$$\begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix},$$

with $\lambda_{1,2} = 2 \pm \sqrt{3}$*. Thus the original system has a local saddle at* $(2,2)$*.*

Euler equations. (We maintain here the boldface notation for vectors).

If we know the optimal policy function h, so that $\mathbf{x}_{t+1} = h(\mathbf{x}_t)$ gives the optimal path, then assuming that it is o class C^2 we could apply the above results directly on h. However, h is in general not known and even it is not of class C^2 unless it is interior and the second derivatives of the utility functions satisfy some bounds (see the classical paper of M.S. Santos in Econometrica (1993)).

To overcome these difficulties, and assuming that the optimal policy is interior, we resort to the Euler equations

$$0 = W_y(\mathbf{x}_t, \mathbf{x}_{t+1}) + \beta W_x(\mathbf{x}_{t+1}, \mathbf{x}_{t+2}), \qquad t = 0, 1, \ldots$$

As we note, the above system is not in the usual form, that is, it is not solve with respect to $\mathbf{x}_{t+2}$. Nevertheless we can linearize the system in a straightforward way.

- (Linearization). Assume that $W \in C^2(\tilde{N}(\mathbf{x}^0, \mathbf{x}^0))$. Consider the truncated Taylor expansion of the Euler equations around the steady state. The notation $W^0_\cdot$ means that the corresponding partial derivative of W is evaluated at the steady state.

$$
\begin{aligned}
0 &= W_y(\mathbf{x}_t, \mathbf{x}_{t+1}) + \beta W_x(\mathbf{x}_{t+1}, \mathbf{x}_{t+2}) \\
&\approx \overbrace{W_y(\mathbf{x}^0, \mathbf{x}^0) + \beta W_x(\mathbf{x}^0, \mathbf{x}^0)}^{=0} \\
&\quad + (W^0_{xy})^\top(\mathbf{x}_t - \mathbf{x}^0) + W^0_{yy}(\mathbf{x}_{t+1} - \mathbf{x}^0) + \beta W^0_{xx}(\mathbf{x}_{t+1} - \mathbf{x}^0) + \beta W^0_{xy}(\mathbf{x}_{t+2} - \mathbf{x}^0) \\
&= (W^0_{xy})^\top \mathbf{y}_t + (W^0_{yy} + \beta W^0_{xx})\mathbf{y}_{t+1} + \beta W^0_{xy}\mathbf{y}_{t+2}.
\end{aligned}
$$

  In the above, note that $(W^0_{xy})^\top = W^0_{yx}$

- (Steady state). Uniqueness of the steady state requires that the matrix

$$(W^0_{xy})^\top + (W^0_{yy} + \beta W^0_{xx}) + \beta W^0_{xy}$$

  be not singular (put $\mathbf{y}_t = \mathbf{y}_{t+1} = \mathbf{y}_{t+2}$ and ask for a unique solution of the homogeneous system).

- (Normal form).
  Solving for $\mathbf{y}_{t+2}$ requires W^0_{xy} be non singular.

$$\mathbf{y}_{t+2} = -(\beta W^0_{xy})^{-1}(W^0_{xy})^\top \mathbf{y}_t - (\beta W^0_{xy})^{-1}(W^0_{yy} + \beta W^0_{xx})\mathbf{y}_{t+1}.$$

- (Transform into a first order system). Let $Y^\top = (\mathbf{y}^\top_{t+1}, \mathbf{y}^\top_t)$. The system can be rewritten as

$$\begin{pmatrix} \mathbf{y}_{t+2} \\ \mathbf{y}_{t+1} \end{pmatrix} = Y_{t+1} = AY_t = \begin{pmatrix} -(\beta W^0_{xy})^{-1} W^0_{yx} & -(\beta W^0_{xy})^{-1}(W^0_{yy} + \beta W^0_{xx}) \\ (0)_m & I_m \end{pmatrix} \begin{pmatrix} \mathbf{y}_{t+1} \\ \mathbf{y}_t \end{pmatrix},$$

  where A is a $2m \times 2m$ matrix.

The following result characterizes the saddle point behavior of steady states of Euler equations satisfying that both $(W^0_{xy})^\top + (W^0_{yy} + \beta W^0_{xx}) + \beta W^0_{xy}$ and W^0_{xy} are nonsingular. It says that the $2m$ eigenvalues of A satisfy the following property: if λ is an eigenvalue of A, then so is $(\beta\lambda)^{-1}$ (recall that it cannot be $\lambda = 0$).

This means that if $|\lambda| < 1$, then the eigenvalue $\beta^{-1}|\lambda|^{-1} > \beta^{-1} > 1$. Thus, no more that m eigenvalues can be smaller that 1 in modulus, and we have that the most we can have is saddle stability.

Ramsey model The Euler equation is

$$0 = -U'(f(k_t) - k_{t+1}) + \beta f'(k_{t+1})U'(f(k_{t+1}) - k_{t+2}).$$

Recall that we denote $W(x, y) = U(f(x) - y)$, and then $W_x = f'(x)U'(f(x) - y)$, $W_y = -U'(f(x) - y)$, so that

$$
\begin{aligned}
W_{xy} &= W_{yx} = -f'(x)U''(f(x) - y), \\
W_{xx} &= f''(x)U'(f(x) - y) + (f'(x))^2 U''(f(x) - y), \\
W_{yy} &= U''(f(x) - y).
\end{aligned}
$$

Then the stacked system with $\hat{k}_t = k_t - k^0$ is

$$\begin{pmatrix} \hat{k}_{t+2} \\ \hat{k}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 + \beta^{-1} + \left(\frac{f''/f'}{U''/U'}\right)^0 & -\beta^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{k}_{t+1} \\ \hat{k}_t \end{pmatrix}.$$

The matrix system has eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(1+\beta^{-1}+\left(\frac{f''/f'}{U''/U'}\right)^0\right) \pm \frac{1}{2}\sqrt{\left(1+\beta^{-1}+\left(\frac{f''/f'}{U''/U'}\right)^0\right)^2 - \frac{4}{\beta}}.$$

Note that $\lambda_1\lambda_2 = \beta^{-1}$. So if $|\lambda_1| < 1$, then the other is greater than one, as we already now. As an example, consider

$$\begin{gathered} U(c) = \ln c, \quad f(k) = Ak^\alpha + (1-\delta)k, \\ A = 10, \quad \alpha = 0.33, \quad \beta = 0.95, \quad \delta = 0.8. \end{gathered}$$

The steady state is $k^0 \approx 7.5378$ and the steady consumption is $c^0 = f(k^0) - k^0 \approx 13.4454$. Then $\lambda_1 \approx 0.0531$.

Returning to the general Ramsey model, and assuming that $|\lambda_1| < 1$, there exists a stable manifold associated to λ_1 that is tangent to the stable manifold of the linear system. It is given by $\phi(k', k) = 0$ with ϕ differentiable, passing through the steady state, $\phi(k^0, k^0) = 0$. By the Implicit Function Theorem, there exists a differentiable function h such that $h(k^0) = k^0$, $\phi(h(k), k) = 0$ for all k in a neighborhood of k^0, and $h'(k^0) = -(\phi_k/\phi_{k'})^0$. It happens, of course, that this function h coincides with the optimal policy function in a neighborhood of the steady state.

Since ϕ is tangent to the linear stable manifold, it must be that $\phi^0_{k'}u + \phi^0_k v = 0$ for any $(u, v) \in S(\lambda_1)$, hence

$$h'(k^0) = -\left(\frac{\phi_k}{\phi_{k'}}\right)^0 = \frac{u}{v}.$$

Therefore we can find the speed of convergence of the saddle path to the steady state simply obtaining an eigenvector (u, v) associated to λ_1. We can proceed in the usual way to get $(\lambda_1, 1) \in S(\lambda_1)$, so we can conclude that

$$h'(k^0) = \lambda_1.$$

In the numerical example above, the derivative of the optimal policy at the steady state is thus ≈ 0.0531.

Remembering the formula of λ_1, one can conclude that the speed of convergence decreases as the ratio

$$\frac{f''/f'}{U''/U'}$$

increases. This is because the function $x \longmapsto (a+x) - \sqrt{(a+x)^2 - b}$ is strictly decreasing.

1.13 Problems

1. *Suppose that in a dynamic programming problem there are constants M, N, k, α such that $\forall x \in X, \forall a \in D(x)$ the following bounds are satisfied.*

$$\begin{aligned} \|U(x,a)\| &\le M(\|x\| + \|a\|) + N, \\ \|q(x,a)\| &\le k(\|x\| + \|a\|), \\ \|a\| &\le \alpha\|x\|. \end{aligned}$$

Show that v is finite-valued if $\beta k(1+\alpha) < 1$.

2. *Consider the following planner's problem:*

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t \ln(C_t/N_t)$$

$$\begin{aligned} s.t. \quad & C_t + K_{t+1} - (1-\delta)K_t = AK_t^{\alpha} N_t^{1-\alpha}, \\ & C_t, K_t \geq 0, \quad K_0 \textit{ given}, \\ & N_t = \eta^t N_0, \end{aligned}$$

where C_t is consumption in period t, K_t is the capital stock in period t, N_t is the population size in period t, $\delta \in [0,1]$ is the rate of depreciation, $A > 0$ is total factor productivity, and $\eta > 0$ is the growth rate of the population. Note that the total time worked equals population size N_t.

(a) *In the problem it is assumed that each individual works its full time endowment. Why is that justified?*

(b) *Write variables in per capita form.*

(c) *Write down the Bellman equation.*

3. *Consider the following Production Economy with Labor Choice*

$$\sup_{\{c_t, \ell_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t),$$

subject to the dynamic constraints

$$\begin{aligned} c_t + i_t &= \widetilde{f}(k_t, \ell_t), \quad t \geq 0, \\ k_{t+1} &= (1-\delta)k_t + i_t, \quad t \geq 0, \\ k_t &\geq 0, \ c_t \geq 0, \ 0 \leq \ell_t \leq 1, \\ & k_0 \textit{ given}, \end{aligned}$$

where

c	*consumption*
ℓ	*labor supply*
k	*capital stock*
i	*investment*
$u(c,\ell)$	*utility function*
$\widetilde{f}(k,\ell)$	*production function*
δ	*depreciation rate.*

(a) *Describe the problem. Eliminating the investment variable from the formulation, identify the state and decision variables, the law of motion and the correspondence of feasible actions.*

(b) *Write down the Bellman equation.*

(c) *Assuming that the value function is differentiable and that an interior optimal policy exists, find the first order conditions. Use the Envelope Theorem of Benveniste and Scheinkman to find the Euler Equations.*

(d) *Suppose that $u(c,\ell) = \ln c + \ln(1-\ell)$, $\widetilde{f}(k,\ell) = Ak^{\alpha}\ell^{1-\alpha}$, $A > 0$, $0 < \alpha < 1$ and $\delta = 1$ (full depreciation). Assuming that consumption is proportional to output, $c_t = \theta \widetilde{f}(k_t, \ell_t)$, $\theta > 0$, find a solution of the Euler Equations.*

Solution:

(a) A representative agent maximizes discounted flow of utilities over an infinite horizon. Each period the agent gets utility from consumption c_t, and disutility from labor supply ℓ_t. The production technology is $\widetilde{f}(k_t, \ell_t)$, where k_t is the capital available in period t and ℓ_t represents labor input. Capital available at period $t+1$ is composed of the investment decision taken in t and on the non-depreciated capital inherited from period t, $k_{t+1} = (1-\delta)k_t + i_t$. Investment made in period t is output minus consumption, $i_t = \widetilde{f}(k_t, \ell_t) - c_t$. The state variable is capital, k, and the decision variables are consumption, c and labor supply, ℓ. Eliminating investment, the transition dynamics is $k_{t+1} = \tilde{f}(k_t, \ell_t) + (1-\delta)k_t - c_t$ thus, in our notation, $q(k, c, \ell) = \tilde{f}(k, \ell) + (1-\delta)k - c$. The feasible action correspondence is $D(k) = \{(c, \ell) : c \in [0, f(k, \ell)],\ \ell \in [0, 1]\}$, where we have normalized maximum labor supply to 1, and denoted $f = \widetilde{f} + (1-\delta)k$.

(b) Denoting tomorrow's capital with k' we have

$$v(k) = \sup_{\substack{k' = f(k,\ell) - c \\ c \in [0, f(k,\ell)],\ \ell \in [0,1]}} \left\{ u(c, \ell) + \beta v(k') \right\}.$$

(c) The first order conditions (FOC) for an interior maximum are:

$$u_c(c, \ell) - \beta v'(k') = 0, \tag{18}$$

$$u_\ell(c, \ell) + \beta v'(k') f_\ell(k, \ell) = 0, \tag{19}$$

where a subindex means partial derivative, v' is the derivative of the value function, that we assume exists, and $k' = f(k, \ell) - c$.

Note that they are not very useful, as they depend on the (unknown) value function. However, in concave problems, it is known that the value function is differentiable when the optimal policy is interior, and that the envelope formula of static optimization holds (the derivative of the value function is equal to the partial derivative of the objective function with respect to the parameter, evaluated at the optimal solution). This is the famous Theorem of Benveniste and Scheinkman (Econometrica, 1979). In this case

$$v'(k) = \beta v'(k') f_k(k, \ell). \tag{20}$$

This equality holds also one step ahead, hence denoting k'' the capital for next period following k' we have

$$v'(k') = \beta v'(k'') f_k(k', \ell'), \tag{21}$$

where ℓ' is labor supply one period ahead; in the same way, c' denotes one period ahead consumption. It is possible to manage the four identities obtained to eliminate the value function. This is done as follows: substituting $\beta v'(k') = u_c(c, \ell)$ from (18) into (19) we get

$$u_c(c, \ell) f_\ell(k, \ell) = -u_\ell(c, \ell). \tag{22}$$

Now, consider (18) one period ahead

$$u_c(c', \ell') - \beta v'(k'') = 0,$$

which substituted into (21) gives $v'(k') = u_c(c', \ell') f_k(k', \ell')$. But $\beta v'(k') = u_c(c, \ell)$, thus finally

$$u_c(c, \ell) = \beta u_c(c', \ell') f_k(k', \ell'). \tag{23}$$

Rewriting these equations in terms of t, $t+1$, we have

$$u_c(c_t, \ell_t) f_\ell(k_t, \ell_t) = -u_\ell(c_t, \ell_t), \tag{24}$$

that says that marginal utility from increasing labor supply must be equal to marginal disutility to increasing labor supply, and

$$u_c(c_t, \ell_t) = \beta u_c(c_{t+1}, \ell_{t+1}) f_k(k_{t+1}, \ell_{t+1}), \tag{25}$$

which means that marginal decrease in utility from increasing investment equals discounted future marginal increase in utility from increasing investment.

Equations (24) and (25) are called the Euler Equations of the problem. Along with the law of motion they constitute optimality conditions of optimality.

$$\begin{aligned} u_c(c_t, \ell_t) f_\ell(k_t, \ell_t) &= -u_\ell(c_t, \ell_t), \\ u_c(c_t, \ell_t) &= \beta u_c(c_{t+1}, \ell_{t+1}) f_k(k_{t+1}, \ell_{t+1}), \\ k_{t+1} &= f(k_t, \ell_t) - c_t. \end{aligned}$$

These three equations form a system of difference equations for the paths $\{k_t\}$, $\{c_t\}$ and $\{\ell_t\}$, but note that we only have one initial condition, k_0. We need two more conditions to fix the optimal paths. These conditions are given by a transversality condition as $t \to \infty$.

(d) In this particular case the Euler Equations become (note that $f = \widetilde{f}$, since $\delta = 1$)

$$\frac{1}{c_t} A k_t^\alpha \ell_t^{-\alpha} = \frac{1}{1-\ell_t}, \tag{26}$$

$$\frac{1}{c_t} = \frac{1}{c_{t+1}} \beta A \alpha k_{t+1}^{\alpha-1} \ell_{t+1}^{1-\alpha}, \tag{27}$$

$$k_{t+1} = A k_t^\alpha \ell_t^{1-\alpha} - c_t. \tag{28}$$

Plugging $c_t = \theta A k_t^\alpha \ell_t^{1-\alpha}$ into (26) we have

$$\frac{\ell_t}{1-\ell_t} = \frac{1-\alpha}{\theta},$$

so that ℓ_t is constant, equal to $\overline{\ell} = \frac{1-\alpha}{1-\alpha+\theta}$. Now, from (27) we have

$$\frac{c_{t+1}}{c_t} = \frac{k_{t+1}^\alpha}{k_t^\alpha} = \beta A \alpha k_{t+1}^{\alpha-1} \overline{\ell}^{1-\alpha},$$

hence

$$k_{t+1} = \beta A \alpha \overline{\ell}^{1-\alpha} k_t^\alpha. \tag{29}$$

On the other hand, from the law of motion of capital (28),

$$k_{t+1} = (1-\theta) A k_t^\alpha \overline{\ell}^{1-\alpha},$$

and both expressions coincide if

$$\theta = 1 - \beta\alpha.$$

Iterating in (29) we get

$$k_{t+1} = (\beta A \alpha \overline{\ell}^{1-\alpha})^{1+\alpha} k_{t-1}^{\alpha^2} = \cdots = (\beta A \alpha \overline{\ell}^{1-\alpha})^{\sum_{s=0}^t \alpha^s} k_0^{\alpha^{t+1}}.$$

From this it is easy to obtain c_t.

4. *Consider the following model of Capacity Expansion. A monopolist has the following production technology. Given current capacity, Q, he can produce any amount of output, q, up to Q units at zero cost, but he cannot produce more than Q in the current period. Capacity can be increased over time but cannot be sold. Any nonnegative amount a of capacity can be added in any period at cost $c(a) = a^2$, but the new capacity cannot be used until the next period. The monopolist faces the same demand for his product each period given by $q = 1 - p$, where p is the price of output. The monopolist seeks to maximize the present value over an infinite horizon of the flow of profits.*

   (a) *Formulate the problem as a dynamic programming problem, identifying the state an decision variables, the feasible choice correspondence, the law of motion and the one-period return function.*

   (b) *Write down the Bellman equation.*

   (c) *Justify why once $Q \geq 1/2$ it does not pay to build more capacity in the future. Consequently, identify a candidate for optimal capacity investment and optimal output when $Q \geq 1/2$. Which should be the value function for $Q \geq 1/2$?*

   (d) *For $Q < 1/2$ guess that the value function is quadratic, $v(Q) = \gamma Q(1 - Q) + \delta$, and check that it satisfies the Bellman equation for suitable constants γ and δ.*

   (e) *Find the optimal policies.*

Solution:

(a) Decision variables are output q and investment in capacity a. The state variable is total capacity Q. The state space is $X = \mathbb{R}_+$ and the decision space is formed by pairs (q, a) in $\mathbb{R}_+ \times \mathbb{R}_+$. The feasible correspondence is $D(Q) = \{(q, a) : 0 \leq q \leq Q, a \geq 0\}$, the transition law is given by $q(Q, a) = Q + a$ and the reward function is $U(Q, q, a) = q(1 - q) - a^2$.

(b) The Bellman equation is

$$v(Q) = \sup_{\substack{Q'=Q+a \\ 0\leq q\leq Q, a\geq 0}} \left\{q(1 - q) - a^2 + \beta v(Q')\right\}.$$

(c) Output q does not influence the value of Q', hence the monopolist choose q to maximize one-shot profits, thus $q^* = 1/2$ if $Q \geq 1/2$ and $q^* = Q$ if $Q < 1/2$ is the optimal output. When $Q \geq 1/2$ the monopolist do not want to build more capacity, so $a^* = 0$ and $Q' = Q$ Plugging this information into the Bellman equation we find

$$v(Q) = \frac{1}{4} + \beta v(Q) \Rightarrow v(Q) = \frac{1}{4(1 - \beta)}.$$

(d) When $Q < 1/2$, we have $q^* = Q$ and the Bellman equations reads

$$v(Q) = Q(1 - Q) + \sup_{a\geq 0} \left\{-a^2 + \beta v(Q + a)\right\}.$$

We guess that in this region the value function is quadratic, $v(Q) = \gamma Q(1 - Q) + \delta$, with $\gamma > 0$. Plugging this expression into the Bellman equation we get

$$\gamma Q(1 - Q) + \delta = Q(1 - Q) + \sup_{a\geq 0} \left\{-a^2 + \beta\gamma(Q + a)(1 - Q - a) + \beta\delta\right\}$$

Maximizing with respect to a we have the FOC $-2a + \beta\gamma(1 - 2Q - 2a) = 0$ or $a(Q) = \beta\gamma(1/2 - Q)/(1 + \beta\gamma) > 0$, which is a maximizer since the function is concave. Moreover, note that $Q' = Q + a(Q) < 1/2$, since $a(Q) < 1/2 - Q$ because $\beta\gamma/(1 + \beta\gamma) < 1$.

Substituting again into the Bellman equations and rearranging terms, we check the identity

$$\gamma Q(1-Q) + \delta = Q(1-Q) - \frac{(\beta\gamma)^2}{(1+\beta\gamma)^2}(1/2 - Q)^2 + \frac{\beta\gamma}{(1+\beta\gamma)^2}(Q(1-Q) + \beta\gamma/2) + \beta\delta.$$

Equating coefficients we find

$$\gamma = \frac{2\beta - 1 + \sqrt{1 + 4\beta^2}}{2\beta} > 0,$$

and an expression for δ can be also found. So the guess is correct.

(e) We have guessed

$$v(Q) = \begin{cases} \gamma Q(1-Q) + \delta, & \text{if } Q < \frac{1}{2}; \\ \frac{1}{4(1-\beta)}, & \text{otherwise} \end{cases}$$

and

$$q(Q) = \begin{cases} Q, & \text{if } Q < \frac{1}{2}; \\ \frac{1}{2}, & \text{otherwise} \end{cases}, \qquad a(Q) = \begin{cases} \frac{\beta\gamma}{1+\beta\gamma}(\frac{1}{2} - Q), & \text{if } Q < \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that we can restrict the state space to $[0,1]$. Since v is bounded in $[0,1]$, $\beta^t v(Q_t)$ tends to 0 for any admissible path, thus v is the value function and $(q(Q), a(Q))$ is the optimal policy.

5. *A model of Gambling. In each play of a game, a gambler can bet any non-negative amount up to his current fortune and he will either win or lose that amount with probabilities p and $q = 1 - p$, respectively. He is allowed to make T bets in succession, and her/his objective is to maximize the expectation of the utility B of the final fortune (no discount is involved here). Suppose that utility is increasing in wealth.*

   (a) *Formulate the problem as a dynamic programming problem, identifying the state an decision variables, the feasible choice correspondence, the law of motion and the one-period return function.*
   (b) *Write down the Bellman equation.*
   (c) *Solve the problem assuming that the gambler has logarithm utility over final wealth, distinguishing the cases $p \leq q$ and $p > q$.*

Solution:

(a) Let x be wealth and a be the amount bet. The law of motion is as follows: $x' = x + a$ with probability p and $x' = x - a$ with probability q. Hence in fact now $q(x, a)$ is a conditional probability. Feasible choices satisfy $0 \leq a \leq x$. There is only a final payoff, $U(x)$.

(b) Let $v_n(x)$ the optimal value of the game when the gambler has wealth x and last n bets. The Bellman equation is

$$\begin{aligned} v_0(x) &= U(x), \\ v_n(x) &= \sup_{0 \leq a \leq x} \left\{ p v_{n-1}(x+a) + q v_{n-1}(x-a) \right\}, \qquad n = 1, \ldots, T. \end{aligned}$$

(c) The gambler is risk averse, thus when $p \le q$ (the game is unfavorable to the gambler) the optimal strategy is never gamble. Check this. Suppose now that $p > q$. We have $v_0(x) = \ln x$ and

$$v_1(x) = \sup_{0 \le a \le x} \{p \ln(x+a) + q \ln(x-a)\}. \tag{30}$$

We find the FOC

$$\frac{p}{x+a} = \frac{q}{x-a} \Rightarrow a = (p-q)x.$$

When this is substituted into (30), it leads to

$$v_1(x) = \ln x + p \ln p + q \ln q + \ln 2.$$

In general, it can be shown that

$$v_n(x) = \ln x + n\alpha,$$

where $\alpha = p \ln p + q \ln q + \ln 2$. Note that the optimal stake does not depend on n and it prescribes bets which are proportional to the wealth.

6. *A decision agent has capital $k_0 > 0$ which can be consumed or invested over T years. At the beginning of each year the agent must decide how much of the current capital k to consume. The utility of consuming c, $0 < c \le k$, is given by $\ln c$. The value of the capital at the beginning of the following year is $R(k-c)$ where $R = 1 + r$, $r > 0$. Future utilities are discounted at the rate β per year, $0 < \beta < 1$. Let $v_n(k)$ be the maximum total discounted utility when it remains n years to the end of the planned horizon, starting with capital k.*

   (a) *Obtain the optimality equation for $v_n(k)$.*

   (b) *Show that $v_n(k) = b_n \ln k + a_n$, where b_n and a_n are constants, $n \ge 1$.*

   (c) *Evaluate b_n for each n,* but not *a_n, and deduce that the optimal policy is to consume a proportion $(1-\beta)/(1-\beta^n)$ of the remaining capital when there are n years left.*

   Solution:

   (a) $v_n(k) = \max_{0 \le c \le k}\{\ln c + \beta v_{n-1}(R(k-c))\}$, $n \ge 1$, $v_0(k) = 0$.

   (b) Plugging $v_{n-1} = b_{n-1}k + a_{n-1}$ in the functional equation one finds

   $$c_n(k) = \frac{k}{1 + \beta b_{n-1}}, \quad v_n(k) = b_n \ln k + a_n,$$

   where

   $$b_n = 1 + \beta b_{n-1}, n \ge 1, \quad b_0 = 0.$$

   (c) Obviously $b_n = (1-\beta^n)/(1-\beta)$, and $c_n(k) = k/b_n$.

7. *Check that the value function of the problem*

$$\sup_{\substack{x_{t+1}=x_t+a_t \\ a_t \in \mathbb{R}}} \sum_{t=0}^{\infty} \beta^t \left(-\frac{2}{3}x_t^2 - a_t^2\right), \qquad x_0 \textit{ given,}$$

is $v(x) = -\alpha x^2$. Find a quadratic equation for α and find the optimal policy function, $a = \phi(x)$.

Solution:
The Bellman equation is

$$v(x) = \max_{a\in\mathbb{R}}\{-\frac{2}{3}x^2 - a^2 + \beta v(x+a)\}.$$

Plugging $v(x) = -\alpha x^2$ into the r.h.s. we get

$$\max_{a\in\mathbb{R}}\{-\frac{2}{3}x^2 - a^2 - \alpha\beta(x+a)^2\} = -\frac{2}{3}x^2 - \frac{\alpha\beta}{1+\alpha\beta}x^2,$$

since $a = -\alpha\beta x/(1+\alpha\beta)$. Returning to the functional equation we get

$$-\alpha x^2 = -\frac{2}{3}x^2 - \frac{\alpha\beta}{1+\alpha\beta}x^2,$$

which gives the quadratic equation for α

$$\alpha = \frac{2}{3} + \frac{\alpha\beta}{1+\alpha\beta}.$$

We consider only the positive solution.
To confirm that v is the value function we check the condition

$$\lim_{t\to\infty} \beta^t v(x_t) = 0.$$

From the law of motion we have $x_{t+1} = x_t + a_t = x_t/(1+\alpha\beta)$, so

$$x_t = \left(\frac{1}{1+\alpha\beta}\right)^t x_0, \quad t \geq 1.$$

Then

$$\beta^t v(x_t) = -\alpha\beta^t \left(\frac{1}{1+\alpha\beta}\right)^{2t} x_0^2 \to 0$$

as $t \to \infty$, since $\beta < 1 < (1+\alpha\beta)^2$.

8. *Consider the following planner's problem:*

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

$$s.t. \quad c_t + \eta k_{t+1} - (1-\delta)k_t = Ak_t^\alpha,$$

$$c_t, k_t \geq 0, \quad k_0 \textit{ given}, \quad 0 < \alpha < 1,$$

which is a Ramsey model in per-capita variables when the growth rate of the population is $\eta > 0$.

(a) *Find the Euler equation.*
(b) *Find the transversality condition.*
(c) *Find the steady state of the problem, $k_t = \overline{k}$, $c_t = \overline{c}$, using the Euler equation and the dynamics constraint.*

9. *(Exogenous growth). Consider the following planner's problem:*

$$\sup_{\{C_t,L_t,K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}-1}{1-\sigma}$$

$$\begin{aligned} s.t. \quad & C_t + K_{t+1} - (1-\delta)K_t = AK_t^{\alpha}L_t^{1-\alpha}, \\ & L_{t+1} = (1+g)L_t, \\ & C_t, K_{t+1} \geq 0, \quad K_0, L_0 \textit{ given; } \sigma, g > 0. \end{aligned}$$

The notation is as usual. Suppose that $\beta(1+g)^{1-\sigma} < 1$.

(a) Explain the model in words.

(b) Rewrite the model in terms of new variables $c_t = C_t/(1+g)^t$, $k_t = K_t/(1+g)^t$ *and new discount factor* $\tilde{\beta}$. *Write down the Bellman equation.*

(c) Derive the Euler Equation and the transversality condition and show that they are necessary and sufficient for optimality.

(d) Find the unique steady state and deduce that the original model has a unique balanced growth path, deriving the growth rates of the different variables.

Solution:

(a)

(b) Let $\tilde{\beta} = \beta(1+g)^{1-\sigma}$.

$$\sup_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{c_t^{1-\sigma}-1}{1-\sigma}$$

$$\begin{aligned} \text{s.t.} \quad & c_t + (1+g)k_{t+1} - (1-\delta)k_t = Ak_t^{\alpha}, \\ & c_t, k_{t+1} \geq 0, \quad k_0 \text{ given; } \sigma, g > 0. \end{aligned}$$

The Bellman equation is straightforward to obtain.

(c) We can proceed as follows: substitute the constraint into the utility function to get

$$W(k,k') = \frac{(Ak^{\alpha} + (1-\delta)k - (1+g)k')^{1-\sigma}}{1-\sigma}.$$

The Euler equation is $0 = W_y(k_t,k_{t+1}) + \tilde{\beta}W_x(k_{t+1},k_{t+2})$, so that

$$0 = -(1+g)(Ak_t^{\alpha}+(1-\delta)k_t-(1+g)k_{t+1})^{-\sigma}+\tilde{\beta}(A\alpha k_t^{\alpha-1}+1-\delta)(Ak_{t+1}^{\alpha}+(1-\delta)k_{t+1}-(1+g)k_{t+2})^{-\sigma}),$$

or

$$0 = -(1+g)c_t^{-\sigma} + \tilde{\beta}(A\alpha k_t^{\alpha-1} + 1 - \delta)c_{t+1}^{-\sigma},$$

that is,

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta(1+g)^{-\sigma}(A\alpha k_t^{\alpha-1} + 1 - \delta).$$

The model is concave and the solution interior, thus the Euler equation is necessary and sufficient if the transversality condition holds. It is straightforward to write the transversality condition.

(d) Let again the Euler equation

$$\left(\frac{c_{t+1}}{c_t}\right)^\sigma = \beta(1+g)^{-\sigma}(A\alpha k_t^{\alpha-1} + 1 - \delta).$$

In a balanced growth path $\frac{c_{t+1}}{c_t}$ is constant, so k_t is also constant. But the accumulation equation implies that c_t is also constant over time. Thus K_t and C_t grow at rate $g > 0$.

10. Consider the following cake–eating problem

$$\sup_{\{c_t, k_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$
$$\text{s.t. } c_t + k_{t+1} = k_t,$$
$$c_t, k_{t+1} \geq 0, \quad k_0 \text{ given},$$

where $\sigma > 0$. The notation is as usual.

(a) Show that the period utility function converges to the log as $\sigma \to 1$.

(b) Discuss whether the problem satisfies the assumptions in Blackwell's Theorem.

(c) Starting with $v_0(k) = 0$, carry out analytically two iterations of the value function iteration algorithm.

(d) Deduce a guess from (c) for the value function and compute it analytically.

(e) Derive the Euler equation and find that $c_t = \beta^{t/\sigma} c_0$.

(f) Derive the transversality condition and suppose that it is a necessary condition for optimality. Find $\lim_{t\to\infty} k_t$. From this find c_0 and give the optimal consumption rule.

Solution:

(a) Simply use the L'hospital rule get

$$\lim_{\sigma\to 1} \frac{c^{1-\sigma} - 1}{1-\sigma} = \lim_{\sigma\to 1} \frac{-c^{1-\sigma}\ln c}{-1} = \ln c.$$

(b) Note that if the size of the cake is $k_0 > 0$, the state space is $[0, k_0]$ when $\sigma < 1$ and $(0, k_0]$ if $\sigma \geq 1$. In the first case the utility function is bounded, but in the second case is not, as the utility is $-\infty$ at zero. Thus, Blackwell's Theorem is not applicable in the second case.

(c) The Bellman equation is

$$v(k) = \max_{k' \in \Gamma(k)} \left\{ \frac{(k-k')^{1-\sigma} - 1}{1-\sigma} + \beta v(k') \right\},$$

where $\Gamma(k) = [0, k]$ if $\sigma < 1$ and $= (0, k)$ if $\sigma \geq 1$. In both cases we have, starting with $v_0 = 0$

$$v_1(k) = \max_{k' \in \Gamma(k)} \frac{(k-k')^{1-\sigma} - 1}{1-\sigma} = \frac{k^{1-\sigma} - 1}{1-\sigma},$$

$$v_2(k) = v(k) = \max_{k' \in \Gamma(k)} \left\{ \frac{(k-k')^{1-\sigma} - 1}{1-\sigma} + \beta \frac{(k')^{1-\sigma} - 1}{1-\sigma} \right\}.$$

Applying the FOC we get

$$-(k-k')^{-\sigma}+\beta(k')^{-\sigma}=0,$$

hence

$$k'=\frac{1}{1+\beta^{-1/\sigma}}k.$$

Plugging this into the Bellman equation for v_2 we find

$$v_2(k)=\frac{\left(1-\frac{1}{1+\beta^{-1/\sigma}}\right)^{1-\sigma}k^{1-\sigma}+\beta\left(\frac{1}{1+\beta^{-1/\sigma}}\right)^{1-\sigma}k^{1-\sigma}-(1+\beta)}{1-\sigma}.$$

(d) Guess that

$$v(k)=\frac{Ak^{1-\sigma}-B}{1-\sigma}$$

with $A>0$ (so that v is concave) and substitute into the Bellman equation for v. We get

$$\frac{Ak^{1-\sigma}-B}{1-\sigma}=\max_{k'\in\Gamma(k)}\left\{\frac{(k-k')^{1-\sigma}-1}{1-\sigma}+\beta\frac{A(k')^{1-\sigma}-B}{1-\sigma}\right\}.$$

The FOC gives

$$k'=\frac{1}{1+(\beta A)^{-1/\sigma}}k.$$

Back into Bellman equation obtains

$$\frac{Ak^{1-\sigma}-B}{1-\sigma}=\frac{\left(1-\frac{1}{1+(\beta A)^{-1/\sigma}}\right)^{1-\sigma}k^{1-\sigma}-1+\beta A\left(\frac{1}{1+(\beta A)^{-1/\sigma}}\right)^{1-\sigma}k^{1-\sigma}-\beta B}{1-\sigma}.$$

Comparison of coefficients gives

$$-B=-1-\beta B\Rightarrow B=\frac{1}{1-\beta};$$

$$A=\frac{(\beta A)^{-(1-\sigma)/\sigma}+\beta A}{\left(1+(\beta A)^{-1/\sigma}\right)^{1-\sigma}}\Rightarrow A=\left(\frac{1}{1-\beta^{1/\sigma}}\right)^{\sigma}.$$

(e) The Euler equation is $c_{t+1}=\beta^{1/\sigma}c_t$, hence one can solve for $c_t=\beta^{t/\sigma}c_0$. The transversality condition is

$$\lim_{t\to\infty}\beta^t W_x(k_t,k_{t+1})k_{t+1}=\lim_{t\to\infty}\beta^t(\beta^{t/\sigma})^{-\sigma}k_{t+1}=\lim_{t\to\infty}k_{t+1}=0. \tag{31}$$

To find k_t, substitute c_t into the resource constraint to get $k_t=k_0-c_0\sum_{s=0}^{t}\beta^{s/\sigma}$. Hence (31) holds if and only if

$$\lim_{t\to\infty}k_t=k_0-\frac{c_0}{1-\beta^{1/\sigma}}=0.$$

Hence it must be $c_0=(1-\beta^{1/\sigma})k_0$ and, in general, the optimal consumption policy is $c(k)=(1-\beta^{1/\sigma})k$.

11. Consider the following Ramsey model with quadratic preferences and linear production function:

$$\sup_{\{C_t,K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t(-0.5C_t^2 + BC_t)$$
$$\text{s.t. } C_t + K_{t+1} = (A+1-\delta)K_t,$$
$$C_t, K_{t+1} \geq 0, \quad K_0 \text{ given},$$

where A, B are positive constants. The rest of notation is as usual. Assume that $A+1-\delta > 1/\beta$. Note that the one period utility function has a maximum at $C = B$, which for this reason is called the bliss point.

(a) Show that there is a unique steady state (K^*, C^*). What is the behavior of the utility function to the right of C^*?

(b) Characterize the solution to the planner problem in the following steps:

i. Rewrite the Euler Equation into a first-order system and compute the corresponding eigenvalues.

ii. Guess that *when interior*, the policy function $K_{t+1} = h(K_t)$ is affine, i.e. $h(K) = aK+b$ for suitable constants a, b and use the fact that the derivative of the policy function equals the stable eigenvalue to determine a and b. Deduce that there is an interval of capital stocks $[0, \tilde{K}]$ such that it is optimal to consume nothing, $C^* = 0$ and so $h(K) = (A+1-\delta)K$ for $K \in [0, \tilde{K}]$ (you do not need to prove this rigorously). Find $\tilde{K}$.

Solution:

(a) Let $\alpha = A+1-\delta$. The Euler equation is

$$0 = \alpha\beta K_{t+2} - (1+\alpha^2\beta)K_{t+1} + \alpha K_t + B(\alpha\beta - 1),$$

thus the steady state K^0 satisfies

$$\begin{aligned} 0 &= \alpha\beta K^0 - (1+\alpha^2\beta)K^0 + \alpha K^0 + B(\alpha\beta-1) \\ &= K^0(1-\alpha)(\alpha\beta-1) + B(\alpha\beta-1), \end{aligned}$$

and given that $\alpha\beta - 1 \neq 0$, we have

$$K^0 = \frac{B}{\alpha-1} > 0.$$

From this we find

$$C^0 = \alpha K^0 - K^0 = B,$$

that is, the bliss point is the steady state. The utility function is decreasing at the right of C^0, since C^0 is the maximum.

(b) i. We construct the stacked system as follows (note the that the system is linear in this model. This is always the case in linear-quadratic problems).

$$\begin{pmatrix} \hat{k}_{t+2} \\ \hat{k}_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1+\alpha^2\beta}{\alpha\beta} & -\frac{1}{\beta} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{k}_{t+1} \\ \hat{k}_t \end{pmatrix}$$

The eigenvalues solve $f(\lambda) \equiv \lambda^2 - \frac{1+\alpha^2\beta}{\alpha\beta}\lambda + \frac{1}{\beta} = 0$. We want to identify whether one of them is smaller than one in absolute value. The function f is a convex parabola, with $f(0) = 1/\beta$ ¿0 and $f(1) = 1 - \frac{1+\alpha^2\beta}{a\beta}\frac{1}{\beta} = \frac{(\alpha\beta-1)(1-\alpha)}{\alpha\beta} < 0$. Hence f has a zero in the interval $(0, 1)$, which is the stable eigenvalue, λ_1. The other eigenvalue must be $\lambda_2 = \frac{1}{\lambda_1\beta} > 1$, as we now.

ii. Suppose that the policy function is $h(K) = aK + b$ *when it is interior*. We know that the slope of h at the steady state is λ_1, thus $a = \lambda_1$. To find b, note that $K^0 = h(K^0) = \lambda_1 K^0 + b$, thus

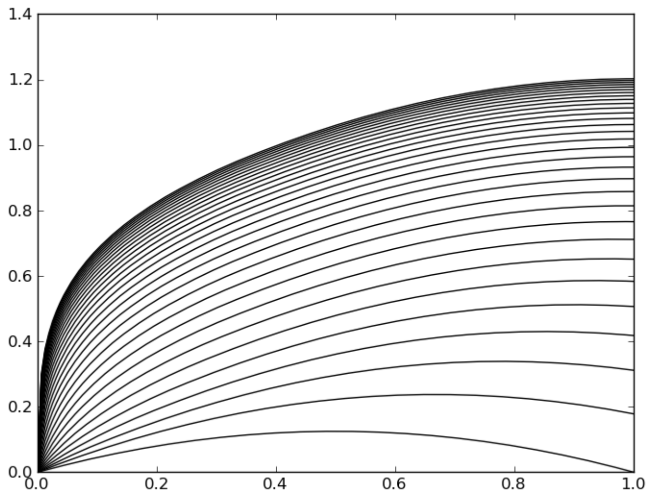
$$b = (1 - \lambda_1)K^0.$$

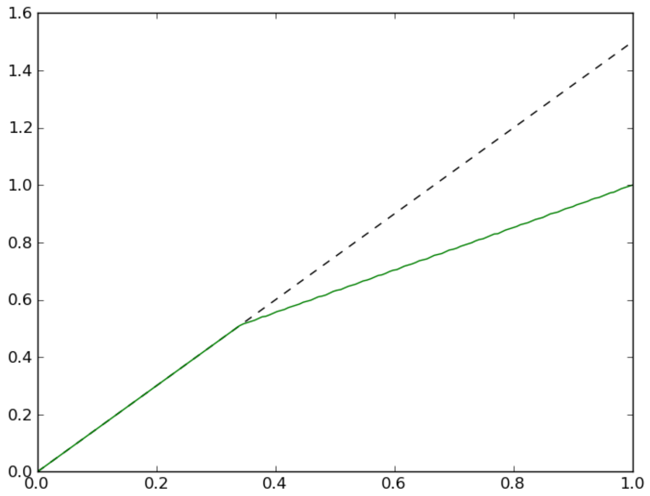
thus, $K' = h(K) = \lambda_1 K + (1 - \lambda_1)K^0$ and

$$C^*(K) = \alpha K - h(K) = (\alpha - \lambda_1)K - (1 - \lambda_1)K^0 > 0$$

only if $K \geq \tilde{K} = \frac{1-\lambda_1}{\alpha-\lambda_1}K^0$. Notice that $\tilde{K} < K^0$ since $\alpha > 1$. Obviously, $C^*(K) = 0$ in the interval $[0, \tilde{K}]$. For these values of capital, the optimal policy is $h'(K) = \alpha K$: all capital is devoted to production and nothing is consumed.

The figures below show the value function and the optimal policy. They have been found with the value function iteration algorithm, following the numerical routines shown in the book of J. Stachurski "Economic Dynamics: Theory and Computation" (The MIT Press, 2009). The parameters values are: $B = 0.5$, $A = 1$, $\delta = 0.5$ and $\beta = 0.9$.




1.6
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2 Stochastic Stationary Discounted Dynamic Programming

2.1 Motivation

Consider the Ramsey model with technology shocks

$$c_t + k_{t+1} = f(k_t, z_t) + (1-\delta)k_t,$$

where z_t is a stochastic technological shock. Suppose that $z \in \{z^1, \ldots, z^n, \ldots\}$ and $z = z_i$ with probability π_i where $\sum_{i=1}^{\infty} \pi_i = 1$. The Bellman equation is

$$v(k,z) = \sup_{0 \le k' \le f(k,z)+(1-\delta)k} \{U(f(k,z) + (1-\delta)k - k') + \beta \sum_{i=1}^{\infty} \pi_i v(k', z_i)\},$$

since the value starting from capital k and shock z is the result of choosing an optimal action today given that the process terminates tomorrow with the receipt of the expected optimal value as a function of tomorrow's capital k'.

Basically, this case can be handled with the techniques developed for deterministic models. However, more interesting cases need other tools, as when the state space for z is continuous and/or if the shocks z are not iid. For these cases the Bellman equation should be

$$v(k,z) = \sup_{0 \le k' \le f(k,z)+(1-\delta)k} \{U(f(k,z) + (1-\delta)k - k') + \beta \mathbb{E}_{k,z} v(k', z')\},$$

where $\mathbb{E}_{k,z}$ denotes conditional expectation. We briefly develop here the tools needed to give a meaning to the above functional equation and to to understand the methods of stochastic dynamic programming.

2.2 Events and probability

Let S be a non-empty set. We interpret $s \in S$ as a elementary event or state of the world, and a subset of S as an event.

Measurable sets. A σ-field $\mathcal{S}$ on S is a family of subsets of S such that

1. $\emptyset \in \mathcal{S}$;
2. $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$;
3. $A_1, A_2, \ldots \in \mathcal{S} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$.

A σ-field $\mathcal{S}$ represents what we know about the state of the world. Given a collection $\mathcal{A}$ of events, the σ-field generated by $\mathcal{A}$, $\sigma(\mathcal{A})$, is the smallest σ-field that contains $\mathcal{A}$.

Example 2.1. *Let (S,d) be a metric space, and let $\mathcal{U}$ the collection of all open subsets of S. Then $B(S) = \sigma(\mathcal{U})$ is called the family of Borel sets, and includes all open sets, closed sets, and countable union and intersections of theses sets.*

Example 2.2. *Let $S = \{1,2,3\}$ and let $\mathcal{A}_1 = \{\{1\},\{2\},\{3\}\}$ be a partition of Ω (the sets forming $\mathcal{A}_1$ are pairwise disjoint and their union is S). Note that $S \in \sigma(\mathcal{A}_1)$. If the information is represented by $\sigma(\mathcal{A}_1)$, then we know exactly which state of the world occurred. Note that σ-field generated by $\mathcal{A}_2 = \{\{1,2\},\{3\}\}$ also contains S.*

We say that the pair $(S, \mathcal{S})$ is a measurable space, and any $A \in \mathcal{S}$ a measurable set.

Measurable functions Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. An extended real valued function $f : \mathcal{S} \longrightarrow \overline{\mathbb{R}}$ is measurable (or $\mathcal{S}$-measurable) if for every Borel set $B \in B(\mathbb{R})$

$$f^{-1}(B) = \{s \in S \,:\, f(s) \in B\} \in \mathcal{S}.$$

For instance, let $S = \{1, 2\}$ and let the σ-fields $\mathcal{S}_1 = \{\emptyset, S, \{1\}, \{2\}\}$, $\mathcal{S}_2 = \{\emptyset, S\}$. All functions on S are $\mathcal{S}_1$-measurable, but only constant functions are $\mathcal{S}_2$-measurable.

We will denote $f^{-1}(B)$ as $\{f \in B\}$.

The σ-field generated by a measurable function $f : S \longrightarrow \overline{\mathbb{R}}$ consists of all sets of the form $f^{-1}(B)$, where B ia a Borel set in $\mathbb{R}$. The σ-field generated by a family of measurable functions $f_i : S \longrightarrow \mathbb{R}$, $i \in I$, is the smallest σ-field containing all events of the form $f_i^{-1}(B)$, where B is a Borel set in $\mathbb{R}$ and $i \in I$.

Let $\{A_i\}_{i=1}^n$ a finite family of measurable sets and $\{a_i\}_{i=1}^n$ a finite family of real numbers. Let χ_{A_i} be the indicator function of set A_i. The function $\varphi(s) = \sum_{i=1}^n a_i\chi_{A_i}(s)$ is called a simple function. If $\{A_i\}_{i=1}^n$ is a partition of S and if all a_i's are distinct, then it is a standard representation.

Theorem 2.3. *Let $(S, \mathcal{S})$ be a measurable space and let $f : S \longrightarrow \overline{\mathbb{R}}$. then f is measurable if and only if it is the pointwise limit of simple functions.*

Probability measure. A probability measure is a set function $P : \mathcal{S} \longrightarrow [0, 1]$ such that

- $P(S) = 1$;
- If $A_1, A_2, \ldots$ are pairwise disjoint sets belonging to $\mathcal{S}$, then

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

The triplet $(S, \mathcal{S}, P)$ is a probability space. An event A occurs almost surely (a.s.) if $P(A) = 1$.

Example 2.4. *Let the unit interval $S = [0, 1]$ with the Borel σ-field, and let the Lebesgue measure $P = \lambda$ on $[0, 1]$. Then $(S, B(S), P)$ is a probability space. Lebesgue measure is the unique measure defined on Borel sets such that $\lambda[a, b] = b - a$ for any interval $[a, b]$.*

Conditional probability. For any events $A, B \in \mathcal{S}$ such that $P(B) \neq 0$ the conditional probability of A given B is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

The total probability formula says that for any event $A \in \mathcal{S}$

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots ,$$

for any sequence of pairwise disjoint events $B_1, B_2, \ldots \in \mathcal{S}$ such that $B_1 \cup B_2 \cup \cdots = S$ and $P(B_i) \neq 0$ for any i.

Random variable. A random variable is a function $\xi : S \longrightarrow \mathbb{R}$, where $(S, \mathcal{S}, P)$ is a probability space (in fact P plays no role here). Every random variable $\xi : S \longrightarrow \mathbb{R}$ defines a probability measure

$$P_\xi(B) = P(\{f \in B\})$$

on $\mathbb{R}$, where $B \in B(\mathbb{R})$. We call P_ξ the distribution of ξ and the function $F_\xi : \mathbb{R} \longrightarrow [0, 1]$ defined by

$$F_\xi(x) = P\{s \in S \,:\, \xi(s) \leq x\}.$$

is called the distribution function of the random variable ξ. Note that F_ξ is non-decreasing, right-continuous, and

$$\lim_{x \to -\infty} F_\xi(x) = 0, \qquad \lim_{x \to \infty} F_\xi(x) = 1.$$

Given the random variable ξ, the σ-field generated by ξ is denoted $\sigma(\xi)$.

Lebesgue integral. Given $(S, \mathcal{S}, P)$ a probability space let $M(S, \mathcal{S})$ be the space of measurable, extended real-valued functions and $M^+(S, \mathcal{S})$ the non-negative cone of $M(S, \mathcal{S})$.

1. $\varphi \in M^+(S, \mathcal{S})$. The integral of φ with respect to P is

$$\int_S \varphi(s) P(ds) = \sum_{i=1}^{n} a_i P(A_i).$$

2. $\xi \in M^+(S, \mathcal{S})$. The integral of ξ with respect to P is

$$\int_S \xi(s) P(ds) = \sup \left\{ \int_S \varphi(s) P(ds) \,:\, 0 \leq \varphi \leq \xi, \varphi \in M^+(S, \mathcal{S}) \text{ simple function} \right\}$$

3. $\xi \in M(S, \mathcal{S})$. The integral of ξ with respect to P is

$$\int_S \xi(s) P(ds) = \int_S \xi^+(s) P(ds) - \int_S \xi^-(s) P(ds)$$

   if both $\int_S \xi^+(s) P(ds) < \infty$ and $\int_S \xi^-(s) P(ds) < \infty$, where

$$\xi^+(s) = \begin{cases} \xi(s), & \text{if } \xi(s) \geq 0; \\ 0, & \text{otherwise.} \end{cases} \qquad \xi^-(s) = \begin{cases} -\xi(s), & \text{if } \xi(s) \leq 0; \\ 0, & \text{otherwise.} \end{cases}$$

   If either ξ^+ or ξ^- have an infinite integral, then integral of ξ is not defined and we say that ξ is not integrable.

Let $L^1(S, \mathcal{S}, P)$ be the set of integrable random variables, i.e., $\xi \in L^1 = L^1(S, \mathcal{S}, P)$ if

$$\int_S |\xi(s)| P(ds) < \infty.$$

Expectation. The expectation of $\xi \in L^1$ is

$$\mathbb{E}(\xi) = \int_S \xi(s) P(ds).$$

Expectation is linear: $\mathbb{E}(a\xi_1 + b\xi_2) = a\mathbb{E}(\xi_1) + b\mathbb{E}(\xi_2)$, $a, b \in \mathbb{R}$.

Theorem 2.5. *For any Borel function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f \circ \xi \in L^1$*

$$E(g(\xi)) = \int_{\mathbb{R}} g(x) P_\xi(dx).$$

If there is a Borel function $f_\xi : \mathbb{R} \longrightarrow \mathbb{R}$ such that for any Borel set B in $\mathbb{R}$

$$P(\{\xi \in B\}) = \int_B f_\xi(x) \lambda(dx),$$

then ξ is a random variable with an absolutely continuous distribution and f_ξ is the density of ξ. The expectation of ξ is, according to Theorem 2.5

$$E(\xi) = \int_S x f_\xi(x) \lambda(dx).$$

More generally,

$$E(g(\xi)) = \int_S g(x) f_\xi(x) \lambda(dx).$$

If $x_1, x_2, \ldots$ is a finite or infinite sequence of pairwise distinct real numbers such that for all $B \in B(\mathbb{R})$

$$P(\{\xi \in B\}) = \sum_{x_i \in B} P\{\xi = x_i\},$$

then ξ has a discrete distribution with mass $P\{\xi = x_i\}$ at x_i.

Conditional expectation

- Conditioning on an event. For any random variable $\xi \in L^1$ and any event $A \in \mathcal{S}$ such that $P(A) \neq 0$, the conditional expectation of ξ given A is defined by

  $$\mathbb{E}(\xi | A) = \frac{1}{P(A)} \int_A \xi dP.$$

  Note that $\mathbb{E}(\xi | S) = \mathbb{E}(\xi)$.

- Conditioning on a random variable. Let $\xi \in L^1$ and let η be an arbitrary random variable. The conditional expectation of ξ given η is defined as the random variable denoted $E(\xi|\eta)$ such that

  - $\mathbb{E}(\xi|\eta)$ is $\sigma(\eta)$-measurable ($\sigma(\eta)$ is the σ-field generated by η);
  - For any $A \in \sigma(\eta)$

    $$\int_A \mathbb{E}(\xi|\eta) dP = \int_A \xi dP.$$

- Conditioning on a σ-field. Let $\xi \in L^1(S, \mathcal{S}, P)$ and let $\mathcal{G}$ be a σ-field contained in $\mathcal{S}$. The conditional expectation of ξ given $\mathcal{G}$ is defined to be the random variable $\mathbb{E}(\xi|\mathcal{G})$ such that

  - $\mathbb{E}(\xi|\mathcal{G})$ is $\mathcal{G}$-measurable;
  - For any $A \in \mathcal{G}$

    $$\int_A \mathbb{E}(\xi|\mathcal{G}) dP = \int_A \xi dP.$$

  Conditional expectation with respect to a σ-field extends conditioning on a random variable η in the sense that

  $$\mathbb{E}(\xi|\sigma(\eta)) = \mathbb{E}(\xi|\eta).$$

The conditional probability of an event $A \in \mathcal{S}$ given a σ-field $\mathcal{G}$ is defined as

$$P(A|\mathcal{G}) = \mathbb{E}(1_A|\mathcal{G}),$$

where 1_A is the indicator function of A.

Variance. A random variable $\xi : S \longrightarrow \mathbb{R}$ is called square integrable if

$$\int_S |\xi(s)|^2 P(ds) < \infty.$$

The family of square integrable random variables will be denoted $L^2 = L^2(S, \mathcal{S}, P)$. It holds that $L^2 \subseteq L^1$. The variance of ξ is

$$\text{var}(\xi) = \int_S (\xi - \mathbb{E}(\xi))^2 dP.$$

Note that $\text{var}(a + b\xi) = b^2\text{var}(\xi)$ for any $a, b \in \mathbb{R}$ and that $\text{var}(\xi) = \mathbb{E}(\xi^2) - \mathbb{E}^2(\xi)$.

Covariance. Given two random variables $\xi, \chi \in L^2$ the covariance of ξ and χ is

$$\text{cov}(\xi, \chi) = \mathbb{E}\big((\xi - \mathbb{E}\xi)(\chi - \mathbb{E}\chi)\big).$$

If $\text{cov}(\xi, \chi) = 0$, then $\text{var}(\xi + \chi) = \text{var}(\xi) + \text{var}(\chi)$.

Information. The evolution of information over time can be modeled using σ-fields. As time increases, our knowledge of what happened on the past also increases.

A sequence of σ-fields $\mathcal{S}_1, \mathcal{S}_2, \ldots$ on S such that

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots \subseteq \mathcal{S}_t \subseteq \cdots$$

is called a filtration of S. Each $\mathcal{S}_t$ contains all events A such that at time t one can tell whether A has occurred or not. If $\xi = \{\xi_t\}$ is a sequence of random variables $\xi_t : S \longrightarrow \mathbb{R}$ such that ξ_t is $\mathcal{S}_t$-measurable for every t, then we say that ξ is adapted to the filtration $\mathcal{F} = \{\mathcal{S}_t : t = 1, 2, \ldots\}$.

Example 2.6. *Let $\xi_1, \xi_2, \ldots$ be a sequence of coin tosses and let $\mathcal{S}_n$ be the σ-field generated by $\xi_1, \xi_2, \ldots, \xi_n$. For the events*
$A = \{$*the first occurrence of tails is preceded by no more than 5 heads*$\}$,
$B = \{$*there is at least 1 head*$\}$,
$C = \{$*the first 100 tosses produce the same outcome*$\}$,
$D = \{$*the first 5 tosses produces at least 3 heads*$\}$,
find the smallest n such that the event belongs to $\mathcal{S}_n$.

A belongs to $\mathcal{S}_6$ but not to $\mathcal{S}_5$.
B does not belong to $\mathcal{S}_n$ for any n.
C belongs to $\mathcal{S}_{100}$ but not to $\mathcal{S}_{99}$.
D belongs to $\mathcal{S}_5$ but not to $\mathcal{S}_4$.

2.3 General Development

The setting of the stochastic dynamic programming is quite similar to the deterministic one, but incorporating events, random variables, probabilities and expectations.

The general setup is as follows.

1. t denotes time and it is supposed to be discrete, $t = 0, 1, 2, \ldots$.

2. S, the set of possible states, is a subset of a Euclidean space. Most often $S = X \times Z$, where $X \subseteq \mathbb{R}^m$ is the endogenous state space and $Z \subseteq \mathbb{R}^l$ is the space of stochastic shocks, so that in this case $s = (x, z) \in S$.

3. $D : S \rightrightarrows A \subseteq \mathbb{R}^p$, a correspondence that associates with state s a nonempty set $D(s)$ of feasible decisions $a \in D(s)$. We denote $A = \bigcup_{s \in S} D(s)$.

4. $q : S \times A \longrightarrow S$, the law of motion. Given $s \in S$ and $a \in D(s)$ (we say that the pair (s, a) is admissible), $q(\cdot|s, a)$ is the conditional probability on S given (s, a), that is, for any Borel set $B \in S$, $q(B|s, a)$ is the probability that next state of the system $s' \in B$ if the current state is s and the action taken is a. Note that we assume that the law of motion is a first-order Markov process, and that q could be degenerated (covering deterministic models).

5. $U : S \times A \longrightarrow \mathbb{R}$, the one-period return function. For (s, a) admissible, $U(s, a)$ is the current return, utility or income if the current state is s and the current action taken is a.

6. β, the discount factor, $0 < \beta < 1$.

A Markov policy or decision rule is a sequence $\pi = \{\pi_t, t = 0, 1, \ldots\}$ such that $\pi_t(s) \in D(s)$ for all t and for all $s \in S$. Let Π the set of Markov policies. A policy is stationary if there exists ϕ such that $\pi_t = \phi$ for all t. Any policy π, along with the law of motion q defines a distribution on all possible evolutions of the system, $(a_0, s_1, a_1, \ldots)$, conditional on a given s_0. Let E_π be the associated conditional expectation. Define the value

$$I(s, \pi) = E_\pi \left\{ \sum_{t=0}^{\infty} \beta^t U(s_t, a_t) | s_0 = s \right\},$$

that is, the expected total discounted return from policy π starting from s.

The problem is then to find a policy $\pi \in \Pi$ such that for any $s_0 \in S$, $I(s, \pi) \geq I(s, \pi')$ for every $\pi' \in \Pi$. We shall say that such π is an optimal policy. It is possible to show in this framework that it suffices to look for stationary policies, thus we reduce our exposition to this type. However, if the time period is finite, then usually the optimal policy depends on time.

The value function $v : S \longrightarrow \mathbb{R}$ is defined as

$$v(s) = \sup_{\pi \in \Pi} I(s, \pi).$$

2.4 The Bellman Equation

The value function satisfies a Bellman Equation, analogous to the deterministic case. The proof is more delicate now and require more hypotheses. See Assumptions 9.1, 9.2 and 9.3 in Stokey, Lucas and Prescott (1989). We take for granted that those assumptions hold in what follows.

Lemma 2.7. *For any $s_0 \in S$ and $\pi \in \Pi(s_0)$*

$$I(s_0, \pi) = U(s_0, \pi_0) + \beta \overbrace{\int_S I(s_1, \pi^1) dq(s_1|s_0, \pi_0)}^{\mathbb{E}_{s_0,\pi_0} I(s_1, \pi^1)},$$

where π^1 is the continuation policy of π at period 1, contingent to next period state, s_1.

Theorem 2.8. *The value function satisfies the Bellman equation: for any $s \in S$*

$$v(s) = \sup_{a \in D(s)} \left\{ U(s, a) + \beta \int_S v(s') dq(s'|s, a) \right\}. \tag{32}$$

Particular cases.

- q is degenerated: The notation $\int_S v(s') dq(s'|s, a)$ means $v(s')$, where $s' = q(s, a)$ is a deterministic transition law. Hence we recover the deterministic Bellman Equation.

- The shock sequence $\{z_t\}$ is i.i.d. with common distribution function q: then (32) is

$$v(s) = \sup_{a \in D(s)} \left\{ U(s,a) + \beta \int_S v(s') dq(s'|a) \right\}.$$

If q is also independent of action,

$$v(s) = \sup_{a \in D(s)} \left\{ U(s,a) + \beta \int_S v(s') dq(s') \right\}$$

and if q has a density function f then

$$v(s) = \sup_{a \in D(s)} \left\{ U(s,a) + \beta \int_S v(s') f(s') ds' \right\}.$$

- q is discrete: $\int_S v(s') dq(s'|s,a) = \sum_j P(s' = s_j | s = s_i, a) v(s_j)$ so that (32) is

$$v(s) = \sup_{a \in D(s)} \left\{ U(s,a) + \beta \sum_j P(s' = s_j | s = s_i, a) v(s_j) \right\}.$$

- Assume $S = X \times Z$, where X is the endogenous state space and Z is the space of shocks. Then (32) is

$$v(x,z) = \sup_{a \in D(x,z)} \left\{ U(x,z,a) + \beta \int_S v(x',z') dq(x',z'|x,z,a) \right\}.$$

Example 2.9. *(Harris, 1987) Consider the following problem.*

An agent possesses a share stock and a put option on this share (the right to sell the share at any time for an exercise price of X). The stock pays a dividend $\tilde{d}_t$ in period $t = 1, 2, \ldots$, where $\{\tilde{d}_t\}$ is a Markov process with transition function $F(d'|d) = P(\tilde{d}_{t+1} \le d' | \tilde{d}_t = d)$. The stock market is efficient, so that there is no difference between selling the share of the stock and keeping it. The agent is risk neutral and discounts the future using discount factor β.

Obviously, the objective is to find the optimal exercise date. Actions variable is $a \in \{0,1\}$, so that $a = 0$ if the agent do not exercise, and $a = 1$ if agent do. State variable is $s = (m,d)$, where $m \in \{0,1\}$, $m = 0$ if the option is not exercised, whereas $m = 1$ if it is. The other component d is simply the dividend. The return function reflects that once the option is exercised, the agent receives X but not the current dividend d. The law of motion is: $q((0,d')|(0,d), a = 0) = F(d',d)$, $q((0,d)|(0,d), a = 1) = 0$, $q((1,d)|(0,d), a = 1) = 1$. There is no transition from $m = 1$.

The return function is

$$U(0,d,a) = \begin{cases} d, & \text{if } a = 0; \\ X, & \text{if } a = 1. \end{cases}$$

Moreover, $U(1,d,a) = 0$, so that $v(1,d) = 0$ for any d. The Bellman equation is

$$\begin{aligned} v(0,d) &= \max \left\{ U(0,d,0) + \beta \int_{\mathbb{R}_+} v(0,d') dF(d',d),\ U(0,d,1) + \beta v(1,d') \right\} \\ &= \max \left\{ d + \beta \int_{\mathbb{R}_+} v(0,d') dF(d',d), X \right\} \end{aligned}$$

We will write $v(0,d) = v(d)$.

To solve the problem, let us assume that F is monotone decreasing in d for all d', meaning that if today's dividend increases, then tomorrow's dividend increases in the first order stochastic dominance *sense. Then it is easy to show that v is increasing. Using this fact we will prove that the optimal policy involves a reservation value d^*, such that it is optimal to exercise if and only if $d \leq d^*$. To see this, note that it is optimal to exercise if*

$$d + \beta \int_{\mathbb{R}_+} v(d')dF(d', d) \leq X.$$

Since that $\beta \int_{\mathbb{R}_+} v(d')dF(d', d)$ increases with d, it is optimal to exercise for all $d \leq d^*$ where

$$d^* + \beta \int_{\mathbb{R}_+} v(d')dF(d', d^*) = X. \tag{33}$$

To get further insights into the solution, suppose that $\{\tilde{d}_t\}$ is i.i.d., that is, F is independent of d. Then, let $\xi = \int_0^\infty v(d')dF(d')$. Using the Bellman equation we get

$$\xi = \int_0^\infty \max\left\{d' + \beta\xi, X\right\} dF(d')$$

Now, d^* above satisfies $X = d^* + \beta\xi$, so that

$$\begin{aligned}\xi &= \int_0^{d^*} X dF(d') + \int_{d^*}^\infty (d' + \beta\xi)dF(d') \\ &= XF(d^*) + \mathbb{E}(d|d \geq d^*)(1 - F(d^*)) + \beta\xi(1 - F(d^*)),\end{aligned}$$

where we have used the definition of conditional expectation

$$\mathbb{E}(d|d \geq d^*) = \frac{1}{P(d \geq d^*)} \int_{[d^*,\infty)} d'dF(d').$$

Solving for ξ we find

$$\xi = \frac{XF(d^*) + \mathbb{E}(d|d \geq d^*)(1 - F(d^*))}{1 - \beta(1 - F(d^*))}.$$

To find an equation for d^*, we substitute this value of ξ into (33), to get

$$X = d^* + \frac{\beta(XF(d^*) + \mathbb{E}(d|d \geq d^*)(1 - F(d^*))}{1 - \beta(1 - F(d^*))}.$$

Letting $x = X(1 - \beta)$ the flow equivalent of the exercise price the equation can be rewritten

$$x = d^*\big(1 - \beta(1 - F(d^*))\big) + \beta\mathbb{E}(d|d \geq d^*)(1 - F(d^*)).$$

Assuming a specific distribution of dividends one can get explicit expressions. For instance, assume that the $\{\tilde{d}_t\}$ is i.i.d. and $\tilde{d}_t$ is uniform in $[0, 1]$. Then $F(d^*) = d^*$ and

$$\mathbb{E}(d|d \geq d^*)(1 - F(d^*)) = \int_{[d^*,\infty)} d'dF(d') = \frac{1}{2}(1 - (d^*)^2).$$

Hence the quadratic equation

$$0 = -x + d^*\big(1 - \beta(1 - d^*)\big) + \frac{\beta}{2}(1 - (d^*)^2)$$

gives the threshold d^*. Let $f(d)$ be the r.h.s. of the above equation (a convex parabola). The vertex is obtained from $f'(\cdot) = 0$, and it is $\beta - 1 < 0$. Since $f(0) = \beta/2 - x$, we have that the roots of the *equation are both negative (if any) when $\beta/2 - x > 0$, thus in this case we shall take $d^* = 0$. A positive root exists if $\beta/2 - x < 0$, that is*

$$d^* = \frac{1}{\beta}\big(- (1-\beta) + \sqrt{1 - 2\beta(1-x)}\big)$$

Thus we conclude that the threshold level of dividends is

$$d^* = \begin{cases} \frac{1}{\beta}\big(- (1-\beta) + \sqrt{1 - 2\beta(1-x)}\big), & \text{if } x > \frac{\beta}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, for $x \leq \beta/2$ one never exercises the put.

Markov chain. *Suppose now that dividends takes on finite values $d_1, \ldots, d_n$ and that $\{\tilde{d}_t\}$ is not i.i.d. anymore, so that the transition from d_i to d_j has probability $\pi_{ji} = P(d' = d_j | d = d_i)$. The matrix $\pi = (\pi_{ji})$ is called the transition matrix of the chain $\tilde{d}_t$. In this case the Bellman equation is*

$$v(d_i) = \max\left\{ d_i + \beta \sum_{j=1}^{n} \pi_{ji} v(d_j), X \right\}, \qquad i = 1, \ldots, n.$$

As shown above, the solution involves a threshold d^*, but this case is harder to solve. In the case $n = 2$ let H denotes a high dividend and L a low one, with probabilities $\pi_1 = P(\tilde{d}_{t+1} = H | \tilde{d}_t = H)$, $\pi_2 = P(\tilde{d}_{t+1} = L | \tilde{d}_t = L)$. The Bellman equation is the system of non-linear equations

$$\begin{aligned} v(H) &= \max\left\{ H + \beta\big(\pi_1 v(H) + (1-\pi_1) v(L)\big), X \right\}, \\ v(L) &= \max\left\{ L + \beta\big((1-\pi_2) v(H) + \pi_2 v(L)\big), X \right\}, \end{aligned}$$

that can be explicitly solved. For $n > 2$ it is better to use a numerical approach based in iteration of the Bellman operator until $\|v_{n+1} - v_n\|$ is smaller than a fixed tolerance. For instance, when $n = 3$ and dividends are Low ($d = 0$), Medium ($d = 1$) or High, ($d = 2$), transition matrix

$$\begin{array}{ccc} P(H|H) = 0.3 & P(H|M) = 0.2 & P(H|L) = 0.1 \\ P(M|H) = 0.4 & P(M|M) = 0.4 & P(M|L) = 0.3 \\ P(L|H) = 0.3 & P(L|M) = 0.4 & P(L|H) = 0.6 \end{array},$$

discount factor $\beta = 0.9$ and exercise price $X = 10$, one gets $v(L) = 10$, $v(M) = 10.4473$ and $v(H) = 11.5905$. Hence the optimal policy is to exercise if the dividend has been low, but not exercise if it has been medium or high.

The algorithm has been implemented in Excel©, which provides a very convenient framework for working with iterative techniques in simple problems. The following table is self explanatory. The iteration begins with $v(L) = v(M) = v(H) = 0$, computes the value of not exercising the option and then takes the maximum between this value and X. The drop and drag mechanism of Excel allows to obtain the successive iterations immediately.

29/03/2012 [**Put Option**]

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	N	V(H)	V(M)	V(L)	a=0, d=H	a=0, d=M	a=0, d=L		BETA	0,9	H	2		
2	0	0,0000	0,0000	0,0000	2,0000	1,0000	0,0000		X	10	M	1		
3	1	10,0000	10,0000	10,0000	11,0000	10,0000	9,0000				L	0		
4	2	11,0000	10,0000	10,0000	11,2700	10,1800	9,0900							
5	3	11,2700	10,1800	10,0000	11,4077	10,2934	9,1629		PHH	0,3	PHM	0,2	PHL	0,1
6	4	11,4077	10,2934	10,0000	11,4857	10,3590	9,2059		PMH	0,4	PMM	0,4	PML	0,3
7	5	11,4857	10,3590	10,0000	11,5304	10,3967	9,2306		PLH	0,3	PLM	0,4	PLH	0,6
8	6	11,5304	10,3967	10,0000	11,5560	10,4183	9,2448							
9	7	11,5560	10,4183	10,0000	11,5707	10,4307	9,2530							
10	8	11,5707	10,4307	10,0000	11,5791	10,4378	9,2576							
11	9	11,5791	10,4378	10,0000	11,5840	10,4418	9,2603							
12	10	11,5840	10,4418	10,0000	11,5867	10,4442	9,2619							
13	11	11,5867	10,4442	10,0000	11,5883	10,4455	9,2627							
14	12	11,5883	10,4455	10,0000	11,5892	10,4463	9,2632							
15	13	11,5892	10,4463	10,0000	11,5898	10,4467	9,2635							
16	14	11,5898	10,4467	10,0000	11,5901	10,4470	9,2637							
17	15	11,5901	10,4470	10,0000	11,5902	10,4471	9,2638							
18	16	11,5902	10,4471	10,0000	11,5903	10,4472	9,2638							
19	17	11,5903	10,4472	10,0000	11,5904	10,4473	9,2639							
20	18	11,5904	10,4473	10,0000	11,5904	10,4473	9,2639							
21	19	11,5904	10,4473	10,0000	11,5904	10,4473	9,2639							
22	20	11,5904	10,4473	10,0000	11,5904	10,4473	9,2639							
23	21	11,5904	10,4473	10,0000	11,5904	10,4473	9,2639							
24	22	11,5904	10,4473	10,0000	11,5905	10,4473	9,2639							
25	23	11,5905	10,4473	10,0000	11,5905	10,4473	9,2639							
26														
27														
28	Cell E2: $L$1+$J$1*($J$5*B2+$J$6*C2+$J$7*D2)													
29	Cell F2: $L$2+$J$1*($L$5*B2+$L$6*C2+$L$7*D2)													
30	Cell G2: $L$3+$J$1*($N$5*B2+$N$6*C2+$N$7*D2)													
31	Cell B3: MAX($J$2;E2)													
32	Cell C3: MAX($J$2;F2)													
33	Cell D3: MAX($J$2;G2)													

2.5 Finite horizon

Suppose that the problem ends at a fixed time T. Following the same notation as for the deterministic case, $v_n(s)$ will denote the value function when current state is s and it remains n periods to the end. The Bellman equation is

$$v_n(s) = \sup_{a \in D(s)} \left\{ U(s,a) + \beta \int_S v_{n-1}(s') dq(s'|s,a) \right\}, \qquad n = 1, 2, \dots$$

and $v_0(s) = 0$ if there is no bequest function, or $v_0(s) = b(s)$ if b is the non-null bequest function.

Example 2.10. *Find the optimal policy in the problem of control a sequence $\{x_0, x_1, \dots, x_T\}$ near to zero minimizing the expected sum of all control costs, where the cost of exerting effort a is $U(a) = (ca^2)$, $c > 0$. The transitions are driven by the stochastic process*

$$x_{t+1} = x_t + a_t + z_t,$$

where $\{z_t\}$ is a i.i.d. sequence of random shocks with mean μ and variance σ^2.

We can model the problem assuming that the penalization of deviating form 0 ant $t = T$ is quadratic, given by the function $b(x) = mx^2$, $m > 0$. We have

$$v_n(x) = \inf_{a\in\mathbb{R}}\{ca^2 + \mathbb{E}v_{n-1}(y)|x, a)\}, \qquad n \geq 1, \qquad v_0(x) = mx^2.$$

Let us find the conditional expectation

$$\begin{aligned}
\mathbb{E}(v_0(y)|x, a) &= \mathbb{E}(my^2|x, a) \\
&= m\mathbb{E}(x + a + z)^2 \\
&= m\mathbb{E}^2(x + a + z) + m\mathrm{var}(x + a + z)) \\
&= m((x + a + \mu)^2 + \sigma^2).
\end{aligned}$$

Hence

$$v_1(x) = \inf_{a\in\mathbb{R}}\{ca^2 + m((x + a + \mu)^2 + \sigma^2)\}.$$

The FOC gives

$$a_1 = -\frac{m}{c+m}(x + \mu).$$

Thus

$$v_1(x) = \frac{cm}{c+m}(\mu + x)^2 + m\sigma^2.$$

It is possible to show by an induction argument that for general n the optimal control is $a_n = -m(c+nm)^{-1}(x + m\mu)$, independent of the variance. It is the same policy as the one for the deterministic problem obtained with the dynamics $x_{t+1} = x_t + a_t + \mu$. This is the certainty equivalence principle, that holds in linear-quadratic models of this kind.

2.6 Reduced form models

Assuming that $S = X \times Z$, where $X \subseteq \mathbb{R}^m$, $Z \subseteq \mathbb{R}^r$ and that the law of motion q is of the form $q(z'|z)$, eliminating the action variable from the formulation as in the deterministic case the Bellman equation is

$$v(x,z) = \sup_{x' \in \Gamma(x,z)} \left\{ W(x,x',z) + \beta \int_Z v(x',z') dq(z'|z) \right\}.$$

The policy function $x' = h(x,z)$ gives tomorrow's state as a function of today's state and shock. Obviously

$$h(x,z) \in \operatorname{argmax}_{x' \in \Gamma(x,z)} \{W(x,x',z) + \beta \int_Z v(x',z') dq(z'|z)\}.$$

2.7 Euler Equations. Reduced form

Suppose that an optimal policy function exists and is interior, that is, for any $(x,z) \in X \times Z$ there exists $\delta > 0$ such that $B(h(x,z),\delta) \subseteq \Gamma(x,z))$. Then, under suitable assumptions on W (concavity, continuous differentiability and integrability of F and of F_x), the sequence $x_{t+1} = h(x_t, z_t)$ with x_0, z_0 given satisfies the following Euler equation

$$0 = W_y(x_t, x_{t+1}, z_t) + \beta \int_Z W_x(x_{t+1}, x_{t+2}, z_{t+1}) dq(z_{t+1}|z_t), \qquad t \geq 1. \tag{34}$$

The proof is similar to the deterministic case. Under the assumptions stated, if $X = \mathbb{R}^m_+$ and W_x is non decreasing in each variable and the Euler equation and the transversality condition

$$\lim_{t \to \infty} \beta^t \int_Z W_x(x_{t+1}, x_{t+2}, z_{t+1}) x_{t+1} dq(z_{t+1}|z_t) = 0$$

hold, then the sequence $\{x_t\}$ generated from x_0 and z_0 by means of h is optimal. Note that $\{x_t\}$ is a sequence of random variables.

Stochastic Euler equations are not so useful as the deterministic Euler equations are. In fact the steady state of a stochastic system is a probability measure, not a fixed point, thus a linear approximation of the stochastic system around the steady state lose its meaning. Thus it is difficult to study the stability properties of the system using this method.

2.8 Envelope Theorem. Reduced form

Suppose that X is convex and that U is concave and differentiable. Let x_0 an interior point of X such that for $z \in Z$ the optimal policy $h(x_0, z)$ is interior to the set $\Gamma(x_0, z)$. Then $v(\cdot, z)$ is differentiable at x_0 and the partial derivatives are

$$\frac{\partial v}{\partial x_i}(x_0, z) = \frac{\partial U}{\partial x_i}(x_0, h(x_0, z), z).$$

2.9 The stochastic Bellman equation

Consider the law of motion q. We say that q has the Feller property if the function

$$Mf(s,a) = \int_S f(s') dq(s'|s,a) = \mathbb{E}(f|s,a)$$

is continuous and bounded for any continuous and bounded function f.

As an example, suppose that $S = X \times Z$ and the transition law is given by $x_{t+1} = F(x_t, a_t, z_t)$, where $\{z_t\}$ is i.i.d., with common distribution G, and $F : X \times A \times Z \longrightarrow X$ is a measurable function. The law of motion is given by

$$\begin{aligned} q(B|x, a) &= P(x_{t+1} \in B | x_t = x, a_t = a) \\ &= \int_Z \chi_B(F(x, a, z)) dG(z), \end{aligned}$$

for every Borel set B, where χ_B is the indicator function of set B: $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ otherwise. Then the operator M is

$$Mf(x, a) = \int_Z f(x')\chi_B(F(x, a, z)) dG(z) = \int_Z f(F(x, a, z)) dG(z).$$

For a continuous function f, continuity of the function Mf is assured if Z is compact and F is continuous.

Consider the stochastic Bellman operator associated to the Bellman equation (32)

$$Tf(s) = \sup_{a \in D(s)} \left\{ U(s, a) + \beta \int_S f(s') dq(s'|s, a) \right\}.$$

We assume that U is continuous and bounded, D is continuous and compact-valued and the transition law q has the Feller property.

Lemma 2.11. *For any function $f \in C_b(S)$, $Tf \in C_b(S)$.*

Proof. Let $f \in C_b(S)$. By assumption, $Mf(s, a) = \int_S f(s') dq(s'|s, a)$ is continuous, thus by the Theorem of the Maximum, Tf is continuous because f is continuous. Moreover, Tf is bounded, since $|f| \leq K$ implies that $|Mf(s, a)| \leq C$ is bounded, again by the Feller property of q. It is easy then to show that $|Tf(s)| \leq D + \beta \int_S D dq(s'|s, a) = D + \beta D$, because q is a probability measure and hence $\int_S dq(s'|s, a) = 1$ for any s, a, and where $|U(s, a)| \leq D$. □

Theorem 2.12. *T has a unique fixed point $f \in C_b(S)$ and $T^n(f_0) \to f$ as $n \to \infty$ from any $f_0 \in C_b(S)$. Moreover, f is the value function.*

Proof. The proof is the same as for the deterministic case. □

2.10 Application. Asset pricing in an exchange economy or Lucas' tree model

- The economy is populated by a large number of identical consumers, identified with the interval $[0, 1]$ (measure 1 of consumers).
- An asset produces stochastic dividend stream $\{z_t\}$. In each period, after z_t is realized, the agents trade on consumption good, c_t, and asset, in a competitive spot market.
- Dividends $z \in Z$, $Z \subseteq \mathbb{R}_+$ compact, and they are Markov, that is, $\{z_t\}$ is follows the transition law $q(\cdot|z_t)$. This means that $P(z_{t+1} \leq z | z_t) = q(z|z_t)$. Moreover, it is assumed that q is Feller continuous.
- Preferences over consumption streams given by

$$\mathbb{E}\left(\sum_{t=0}^{\infty} \beta^t U(c_t) | x_0, z_0 \right).$$

- The consumer's problem at time $t = 0$ is

$$\max_{\{c_t, x_{t+1}\}_{t=0}^{\infty}} \mathbb{E}\left(\sum_{t=0}^{\infty} \beta^t U(c_t) | x_0, z_0\right)$$
$$\text{s.t. } c_t + p_t x_{t+1} \leq (z_t + p_t) x_t,$$
$$c_t \geq 0, x_t \geq 0, \quad x_0 = 1 \text{ given.}$$

where x_t is the asset demand at date t and p_t is asset's date t price.

- Asset price is given by a function $p : Z \longrightarrow \mathbb{R}_{++}$, $p_t = p(z_t)$.

- Equilibrium. The aim is to find an equilibrium for this model. An equilibrium is a sequence of prices $\{p_t^*\}$ such that: (i) given prices $\{p_t^*\}$, all agents optimize; (ii) markets clear.

  Since the agents are identical, (ii) means that prices are such that the representative agent wishes to neither buy nor sell the asset, $x_t = 1$ for all t: For, suppose that one agent wishes to buy one unit of the asset at price p_t; then, by the clearing market condition there must be another agent who is willing to sell him that unit of the asset at p_t. But given that all the agents are identical, all agents want to buy the asset.

  Condition (i) means that given a price function p, the agent solves the Bellman equation

$$v(x, z) = \max_{0 \leq p(z)x' \leq (z+p(z))x} \left\{ U((z + p(z))x - p(z)x') + \beta \int_Z v(x', z') dq(z', z) \right\}.$$

- Existence of equilibrium. Assume that U is bounded, concave and differentiable. Also, taking as state space for asset holdings $[0, \bar{x}]$ with $\bar{x} > 1$, we see that the equilibrium asset holdings $x' = 1$ is interior to the state space, since that at $x = 1$ the constraint is $x' = (z + p)/p > 1$ is never binding. Thus the solution is characterized by the stochastic Euler equation

$$p(z_t)U'(c_t) = \beta \int_Z (z_{t+1} + p(z_{t+1}))U'(c_{t+1}) dq(z_{t+1}, z_t), \qquad t = 0, 1, \ldots$$

  If $(p, x') = (p^*, 1)$ is an equilibrium, then

$$p^*(z_t)U'(z_t) = \beta \int_Z (z_{t+1} + p(z_{t+1}))U'(z_{t+1}) dq(z_{t+1}, z_t), \qquad t = 0, 1, \ldots$$

  Let us rewrite this equality dropping out time dependence

$$\overbrace{p^*(z)U'(z)}^{\phi(z)} = \beta \int_Z (z' + p(z))U'(z') dq(z', z),$$
$$= \overbrace{\beta \int_Z z'U'(z') dq(z', z)}^{h(z)} + \int_Z p(z)U'(z') dq(z', z), \qquad t = 0, 1, \ldots$$

  The issue is to prove that this functional equation admits a solution. The equation can be rewritten

$$\phi(z) = h(z) + \beta \int_Z \phi(z') dq(z', z),$$

  and needs to be solved for ϕ uniquely. This is a fixed point problem for which Banach's Theorem applies in the space of continuous bounded functions (if we are looking for bounded prices!). Hence a unique price function exists, that is given by $p^*(z) = \phi(z)/U'(z)$.

- Interpreting the Euler equation. Rewrite the Euler equation as

$$p_t^* U'(z_t) = \beta \mathbb{E}_t(U'(z_{t+1})(p_{t+1}^* + d_{t+1})).$$

The left hand side is the marginal utility to giving up a small amount of consumption (recall that in equilibrium $c_t = z_t$), using it to buy some amount of the asset at price p_t^*. The right hand side is the discounted expected marginal utility at date $t+1$ from increasing a small amount of the asset: part of the utility gain comes from the expected resale value of the asset and the other from the dividends.

Let again the Euler equation with current price isolated on the l.h.s

$$p_t^* = \beta \mathbb{E}_t \left(\frac{U'(z_{t+1})}{U'(z_t)} (p_{t+1}^* + d_{t+1}) \right).$$

Observe:

  - Assets' current price depends positively on future expected prices and dividends.
  - The more impatience the agent is (the lower the discount factor β), the less the agent is willing to pay to get any given expected resale value or dividend.
  - Current prices also depends on the ratio of marginal utilities $\frac{U'(z_{t+1})}{U'(z_t)}$. Since U is concave we have

    $$\mathbb{E}_t \left(\frac{U'(z_{t+1})}{U'(z_t)} \right) \text{ is higher } \Leftrightarrow \mathbb{E}_t \left(\frac{z_{t+1}}{z_t} \right) \text{ is lower.}$$

    Thus, if future consumption is expected to be lower than current consumption, then the price of the asset increases. This is known a the consumption smoothing effect, and it is due to the concavity of the utility function. If expected future consumption c_{t+1} is expected to be very low with respect to current consumption c_t, agents transfer consumption into the future, by buying more of the asset; the increasing demand drives the price up.

- Returns and asset pricing. Rewrite the Euler equation

$$1 = \mathbb{E}_t \left(\beta \frac{U'(z_{t+1})}{U'(z_t)} \frac{(p_{t+1}^* + d_{t+1})}{p_t^*} \right)$$

It is common to denote

$$\text{MRS}(t+1, t) = \beta \frac{U'(z_{t+1})}{U'(z_t)}$$

the marginal rate of substitution (the rate at which agents are willing to trade consumption tomorrow from consumption today) and

$$R_{t+1} = \frac{(p_{t+1}^* + d_{t+1})}{p_t^*}$$

the asset's return. Thus

$$1 = \mathbb{E}_t(\text{MRS}(t+1, t) R_{t+1}).$$

  - One-period real discount bond. A bond is an asset that pays no interest, it is a promise to pay out a certain amount in the future. At date t the agent pays m_t units of the consumption good for one unit of the bond. At day $t+1$, the bond-issuer promises to give you one unit of the consumption good. The expected return is

    $$R_{t+1} = \frac{1}{m_t},$$

    hence the price of the bond is

    $$m_t = \mathbb{E}_t(\text{MRS}(t+1, t)).$$

- One-period nominal discount bond. At date t the agent pays m_t units of currency for one unit of the bond. At time $t+1$ a corporation or the government give the agent one unit of currency. The expected return is

- Linear utility, $U(c) = c$. In this case, at any t the price is

$$p(z_t) = \sum_{s=1}^{\infty} \beta^s \mathbb{E}(z_{t+s}|z_t),$$

that is, the price of the share is simply the discounted sum of the expected future dividends. This can also be rewritten

$$p(z) = \beta \mathbb{E}(z_{t+1} + p(z_{t+1})|z).$$

3 Sample Final Exam

1. (40 points.) Consider the problem of choosing a consumption sequence $\{c_t\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t (\ln c_t + \gamma \ln c_{t-1}), \qquad 0 < \beta < 1,\ \gamma > 0,\ \beta\gamma < 1,$$

subject to

$$c_t + k_{t+1} = Ak_t^\alpha, \quad A > 0,\ 0 < \alpha < 1,\ (k_0, c_{-1}) > (0, 0) \text{ given.}$$

Here c_t is consumption at t and k_t is capital stock at the beginning of period t. The current utility function, $\ln c_t + \gamma \ln c_{t-1}$, represents habit formation in consumption.

(a) Considering as state variables the stock of capital k and lagged consumption c_{-1}, write down the dynamic programming equation, identifying the action variable, the choice correspondence and the law of motion.

(b) Write down the reduced form version of the dynamic programming equation.

(c) Prove that the value function is of the form $v(k, c_{-1}) = E + F \ln k + G \ln c_{-1}$, where E, F and G are constants.

(d) Prove that the optimal policy function is of the form $h(k, c_{-1}) = Hk^\alpha$ (thus, independent of lagged consumption), where H is a constant. Give explicit formulas for E, F, G and H.

Solution:

(a) Let $x_t = c_{t-1}$ be lagged consumption. State variables are k, x, decision variable is c. State space is $X = \mathbb{R}_+ \times \mathbb{R}_+$, action space is $A = \mathbb{R}_+$ and law of motion is $k_{t+1} = q(k_t, c_t) = Ak_t^\alpha - c_t$. The choice correspondence is

$$D(k) = \{c \in \mathbb{R}_+ \, : \, 0 \le c \le Ak_t^\alpha\}.$$

(b) Let v be the value function and let (k', x') denote tomorrow's capital stock and lagged consumption (thus, $x' = c$). The Bellman equation is

$$v(k, x) = \max_{c \in D(k), k' = q(k,c)} \{\ln c + \gamma \ln x + \beta v(k', x')\} = \max_{0 \le k' \le Ak^\alpha} \{\ln (Ak^\alpha - k') + \gamma \ln x + \beta v(k', Ak^\alpha - k')\}.$$

(c) Substitute the suggested form for v into the Bellman equation and look for an interior policy. Let $V(k, x, k', x')$ denotes the r.h.s. in the Bellman equation. The FOC gives

$$0 = \frac{\partial V}{\partial k'} = -\frac{1}{Ak^\alpha - k'} + \beta \left(\frac{F}{k'} - \frac{G}{Ak^\alpha - k'} \right),$$

thereby, solving for k'

$$k' = \frac{\beta F}{1 + \beta(F + G)} Ak^\alpha.$$

Observe that $k' \in (0, Ak^\alpha)$ since we assume that both $F, G > 0$. To prove that k' is indeed a maximum,observe that $V(k, x, 0, x') = V(k, x, Ak^\alpha, x') = -\infty$. Since the function $k' \mapsto V(k, x, k', x')$ is continuous and differentiable in the interior of the technological correspondence and there is only one interior critical point, it is the global maximum. Plugging this candidate for optimal policy into the Bellman equation we get

$$
\begin{aligned}
E + F\ln k + G\ln x = \ln\left(\frac{A(1+\beta G)}{1+\beta(F+G)}k^\alpha\right) + \gamma\ln x \\
+\beta E + \beta\ln\left(\frac{\beta AF}{1+\beta(F+G)}k^\alpha\right) + \beta G\ln\left(\frac{A(1+\beta G)}{1+\beta(F+G)}k^\alpha\right)
\end{aligned}
$$

Equating coefficients we get

$$
\begin{aligned}
E &= (1+\beta G)\ln\left(\frac{A(1+\beta G)}{1+\beta(F+G)}\right) + \beta E + \beta\ln\left(\frac{\beta AF}{1+\beta(F+G)}\right). \\
F &= \frac{\alpha(1+\beta G)}{1-\alpha\beta}, \\
G &= \gamma.
\end{aligned}
$$

The important thing is that $\beta < 1$ and $\alpha\beta < 1$ imply that the above equations are solvable for unique E, F, G, with both $F > 0$ and $G > 0$.

(d) The policy function is

$$
k' = h(k) = \frac{\beta F}{1+\beta(F+\gamma)}Ak^\alpha = \alpha\beta Ak^\alpha,
$$

independent of lagged consumption. To show it is optimal, let's prove that for the optimal path $\{(k_t, x_t)\}_{t=0}^\infty$

$$
\lim_{t\to\infty}\beta^t v(k_t, x_t) = \lim_{t\to\infty}\beta^t v(k_t, c_{t-1}) = 0. \tag{35}
$$

Note that $k_{t+1} = \alpha\beta Ak_t^\alpha$. Recursively one finds

$$
k_t = (\alpha\beta A)^{1+\alpha+\cdots+\alpha^t}k_0^{\alpha^t} \Rightarrow \lim_{t\to\infty} k_t = (\alpha\beta A)^{1/(1-\alpha)} > 0,
$$

and $c_t = Ak_t^\alpha - k_{t+1}$, hence

$$
\lim_{t\to\infty} c_t = A(\alpha\beta A)^{\alpha/(1-\alpha)} - (\alpha\beta A)^{1/(1-\alpha)} > 0,
$$

the inequality because $\alpha\beta < 1$. In consequence, (35) holds, and $h(k, x)$ is the optimal policy function.

2. (40 points.) Consider the following stochastic one-sector growth model. An infinitely-lived representative agent has preferences

$$
\mathrm{E}_0\sum_{t=0}^\infty \beta^t(c_t - \theta c_t^2),
$$

where $\theta > 0$. The production function is linear, so that the output is

$$
y_t = Ak_t + z_t, \quad A > 0
$$

where z_t is a stochastic shock satisfying

$$
z_t = \rho z_{t-1} + \epsilon_t, \quad 0 < \rho < 1
$$

and $\{\epsilon_t\}$ is i.i.d. with zero mean. The resource constraint is

$$
k_{t+1} + c_t = y_t + k_t
$$

(no capital depreciation). Along the exercise, assume that the optimal policy function is interior.

(a) Write down the dynamic programming equation of the representative consumer.

(b) Find the Euler equation associated to the problem in terms of the optimal consumption.

(c) Iterating forward in the Euler equation, express current consumption c_t as the sum of two summands: a fundamental part and a bubble component. State conditions over the parameters such that a solution for c_t exists with no bubble component, and find c_t.

(d) Let $A = \frac{1}{\beta} - 1$. Using the guess for the consumption function

$$c(k, z) = a + bk + dz, \qquad a, b, d \geq 0,$$

find a, b, d that solves the Euler equation. Check out that the consumption function is feasible.

Solution:

(a) For a general utility function U, the DP equation is

$$v(k_t, z_t) = \max_{\substack{0 \leq c_t \leq (1+A)k_t + z_t, \\ k_{t+1} = (1+A)k_t + z_t - c_t}} \{U(c_t) + \beta E_t v(k_{t+1}, z_{t+1}\}.$$

(b) FOC

$$0 = U'(c_t) + \beta E_t \left(v'_k(k_{t+1}, z_{t+1}) \frac{\partial k_{t+1}}{c_t} \right) = U'(c_t) - \beta E_t v'_k(k_{t+1}, z_{t+1}).$$

and envelope

$$v'_k(k_t, z_t) = \beta E_t \left(v'_k(k_{t+1}, z_{t+1}) \frac{\partial k_{t+1}}{k_t} \right) = \beta(1 + A) E_t v'_k(k_{t+1}, z_{t+1})$$

give $v'_k(k_t, z_t) = (1 + A)U'(c_t)$, and plugging this identity into the envelope formula yield

$$U'(c_t) = \beta(1 + A)E_t(U'(c_{t+1})).$$

In the particular case we are considering, it becomes

$$1 - 2\theta c_t = \beta(1 + A)E_t(1 - 2\theta c_{t+1}).$$

(c) Solving for c_t in the Euler equation above one gets

$$c_t = \frac{1 - \beta(1 + A)}{2\theta} + \beta(1 + A)E_t(c_{t+1}).$$

Iterating T periods we have

$$c_t = \left(\frac{1 - \beta(1 + A)}{2\theta} \right) \sum_{t=0}^{T-1} \beta^t (1 + A)^t + \beta^T (1 + A)^T E_t(c_{t+T}).$$

Taking limits (assuming that $\beta(1 + A) < 1$) we find c_t expressed as the sum of the fundamental part and the bubble

$$c_t = \frac{2}{\theta} + \lim_{T \to \infty} \beta^T (1 + A)^T E_t(c_{t+T}).$$

Assuming that the bubble component vanishes, $c_t = \frac{2}{\theta}$. Note that we are using the Euler equation here, so this equality hold only when the consumption is interior.

(d) In this case the Euler equation is

$$c_t = E_t(c_{t+1}),$$

thus consumption is a martingale. Let us rewrite in terms of the policy consumption rule

$$c(k, z) = E(c(k', z')|k, z) \Rightarrow a + bk + dz = E(a + bk' + dz')|k, z).$$

Let us compute $E(a + bk' + dz')|k, z)$ using the transition law and the properties of conditional expectation. We have

$$\begin{aligned} E(a + bk' + dz')|k, z) &= a + bE(k'|k, z) + dE(z'|k, z), \\ &= a + b(1 + A)k - bc(k, z) + bz + d\rho z + dE(\epsilon), \\ &= a + b(1 + A)k - b(a + bk + dz) + bz \\ &= a - ab + b((1 + A) - b)k + (b(1 - d) + d\rho)z. \end{aligned}$$

Equating coefficients we find

$$\begin{aligned} a &= a - ab, \\ b &= b((1 + A) - b), \\ d &= (b(1 - d) + d\rho). \end{aligned}$$

Take $a = 0$, $b = 1 + A = \frac{1}{\beta}$ and $d = \frac{1-\beta}{1-\beta\rho}$, which provides a feasible consumption function, as one can check easily. Other selections ($a > 0$) lead to infeasible consumption functions.

3. (10 points.) Let $(C(X), \|\cdot\|_\infty)$ be the space of continuous functions over the compact set X, $f : X \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}$. Let $\mathcal{B}$ be the Borel σ-field of X and let $F : \mathcal{B} \times \mathbb{R}^m \longrightarrow [0, 1]$ be such that for any $x \in X$, $F(\cdot, x)$ is a probability measure. Let the operator

$$Tf(x) = \alpha(x) + \beta(x) \int_X f(x')\, dF(x', x).$$

(a) Under what conditions on F, α and β, $T : C(X) \longrightarrow C(X)$?

(b) Find conditions Show that T is a contraction mapping.

Solution:

(a) Assuming that both α and β are continuous, if F is Feller continuous then Tf is also continuous, since any $f \in C(X)$ is bounded because X is compact.

(b) Note that for $f, g \in C(X)$

$$\begin{aligned} |\beta(x)| \int_X |f(x') - g(x')| dF(x', x) &\le |\beta(x)| \int_X \|f - g\|_\infty dF(x', x) \\ &= \|f - h\|_\infty |\beta(x)| \int_X dF(x', x) \\ &\le \|\beta\|_\infty \|f - h\|_\infty, \end{aligned}$$

since $dF(x', x)$ is a probability measure for all $x \in X$, so that $\int_X dF(x', x) = 1$ for all $x \in X$. Hence, for any $x \in X$

$$|Tf(x) - Tg(x)| = \beta(x)| \int_X f(x') - g(x') dF(x', x)| \le \|\beta\|_\infty \int_X |f(x') - g(x')| dF(x', x) \le \|\beta\|_\infty \|f - h\|_\infty.$$

Taking the supremum we have

$$\|Tf - Th\|_\infty \leq \|\beta\|_\infty \|f - h)\|_\infty.$$

For T to be a contraction $\|\beta\|_\infty < 1$ is required.