Appendices to the paper "Detecting Big Structural Breaks in Large Factor Models" (2013) by Chen, Dolado and Gonzalo.

A.1: Proof of Propositions 1 and 2

The proof proceeds by showing that the errors, factors and loadings in model (5) satisfy Assumptions A to D of Bai and Ng (2002) (BN 2002 hereafter). Then, once these results are proven, Propositions 1 and 2 just follow immediately from application of Theorems 1 and 2 of BN (2002). Define $F_t^* = [F_t' \quad G_t^{1'}]'$, $\epsilon_t = HG_t^2 + e_t$, and $\Gamma = [A \quad \Lambda]$.

Lemma 1. *$E||F_t^*||^4 < \infty$ and $T^{-1}\sum_{t=1}^T F_t^* F_t^{*'} \xrightarrow{p} \Sigma_F^*$ as $T \to \infty$ for some positive matrix Σ_F^*.*

Proof. $E||F_t^*||^4 < \infty$ follows from $E||F_t||^4 < \infty$ by Assumption 2 and the definition of G_t^1.

To prove the second part, we partition the matrix $\Sigma_F (= \lim_{T\to\infty} T^{-1}\sum_{t=1}^T F_t F_t')$ into:

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix}$$

where $\Sigma_{11} = \lim_{T\to\infty} T^{-1}\sum_{t=1}^T F_t^1 F_t^{1'}$, $\Sigma_{22} = \lim_{T\to\infty} T^{-1}\sum_{t=1}^T F_t^2 F_t^{2'}$, $\Sigma_{12} = \lim_{T\to\infty} T^{-1}\sum_{t=1}^T F_t^1 F_t^{2'}$, and F_t^1 is the $k_1 \times 1$ subvector of F_t that has big breaks in their loadings, F_t^2 is the $k_2 \times 1$ subvector of F_t that doesn't have big breaks in their loadings. By the definition of F_t^* and G_t^1 we have:

$$T^{-1}\sum_{t=1}^T F_t^* F_t^{*'} = \begin{pmatrix} T^{-1}\sum_{t=1}^T F_t^1 F_t^{1'} & T^{-1}\sum_{t=1}^T F_t^1 F_t^{2'} & T^{-1}\sum_{t=\tau+1}^T F_t^1 F_t^{1'} \\ T^{-1}\sum_{t=1}^T F_t^2 F_t^{1'} & T^{-1}\sum_{t=1}^T F_t^2 F_t^{2'} & T^{-1}\sum_{t=\tau+1}^T F_t^2 F_t^{1'} \\ T^{-1}\sum_{t=\tau+1}^T F_t^1 F_t^{1'} & T^{-1}\sum_{t=\tau+1}^T F_t^1 F_t^{2'} & T^{-1}\sum_{t=\tau+1}^T F_t^1 F_t^{1'} \end{pmatrix}.$$

By Assumption 2, the above matrix converges to

$$\Sigma_F^* = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & (1-\pi^*)\Sigma_{11} \\ \Sigma_{12}' & \Sigma_{22} & (1-\pi^*)\Sigma_{12}' \\ (1-\pi^*)\Sigma_{11} & (1-\pi^*)\Sigma_{12} & (1-\pi^*)\Sigma_{11} \end{pmatrix}.$$

Moreover,

$$\det(\Sigma_F^*) = \det \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & 0 \\ \Sigma_{12}' & \Sigma_{22} & (1-\pi^*)\Sigma_{12}' \\ 0 & 0 & \pi^*(1-\pi^*)\Sigma_{11} \end{pmatrix} = \det(\Sigma_F)\det(\pi^*(1-\pi^*)\Sigma_{11}) > 0$$

because Σ_F is positive definite by assumption. This completes the proof. □

Lemma 2. *$||\Gamma_i|| < \infty$ for all i, and $N^{-1}\Gamma'\Gamma \to \Sigma_\Gamma$ as $N \to \infty$ for some positive definite matrix Σ_Γ.*

Proof. This follows directly from Assumptions 1.a and 3. □

The following lemmae involve the new errors ϵ_t. Let M and M^* denote some positive constants.

Lemma 3. *$E(\epsilon_{it}) = 0$, $E|\epsilon_{it}|^8 \le M^*$ for all i and t.*

Proof. For $t = 1, \dots, \tau$, $E|\epsilon_{it}|^8 = E|e_{it}|^8 < M$ by Assumption 4. For $t = \tau + 1, \dots, T$,

$$E|\epsilon_{it}|^8 = E|e_{it} + \eta_i' F_t^2|^8 \le 2^7 * \left(E|e_{it}|^8 + E|\eta_i' F_t^2|^8\right)$$

by Loève's inequality. Next, $E|\eta_i' F_t^2|^8 \le \|\eta_i\|^8 E\|F_t\|^8 < \infty$ by Assumptions 1.a and 2. Then the result follows. □

Lemma 4. *$E(\epsilon_s'\epsilon_t/N) = E(N^{-1}\sum_{i=1}^N \epsilon_{is}\epsilon_{it}) = \gamma_N^*(s,t)$, $|\gamma_N^*(s,s)| \le M^*$ for all s, and $\sum_{s=1}^T \gamma_N^*(s,t)^2 \le M^*$ for all t and T.*

Proof.

$$\begin{aligned}
\gamma_N^*(s,t) &= N^{-1}\sum_{i=1}^N E(\epsilon_{is}\epsilon_{it}) \\
&= N^{-1}\sum_{i=1}^N E(e_{is} + \eta_i' G_s^2) E(e_{it} + \eta_i' G_t^2) \\
&= N^{-1}\sum_{i=1}^N \left[E(e_{is}e_{it}) + E(\eta_i' G_s^2 \eta_i' G_t^2)\right] \\
&\le N^{-1}\sum_{i=1}^N E(e_{is}e_{it}) + N^{-1}\sum_{i=1}^N \sqrt{E\left(\eta_i' G_s^2\right)^2 E\left(\eta_i' G_t^2\right)^2}.
\end{aligned}$$

Since $N^{-1}\sum_{i=1}^N E(e_{is}e_{it}) = \gamma_N(s,t)$ by Assumption 4, and $E\left(\eta_i' G_t^2\right)^2 \le \|\eta_i\|^2 E\|F_t\|^2 = O(\frac{1}{NT})$ for all t by Assumptions 1.b and 2, we have $\gamma_N^*(s,t) \le \gamma_N(s,t) + O(\frac{1}{NT})$. Then

$$|\gamma_N^*(s,s)| \le |\gamma_N(s,s)| + O(\frac{1}{NT}) \le M^*$$

by Assumption 4. Moreover,

$$\begin{aligned}\sum_{s=1}^{T}\gamma_N^*(s,t)^2 &\leq \sum_{s=1}^{T}\left(\gamma_N(s,t)+O(\frac{1}{NT})\right)^2\\ &= \sum_{s=1}^{T}\gamma_N(s,t)^2+O(\frac{1}{N})\\ &\leq M+O(\frac{1}{N})\leq M^*\end{aligned}$$

by Assumption 4. Thus, the proof is complete. □

Lemma 5. *$E(\epsilon_{it}\epsilon_{jt})=\tau_{ij,t}^*$ with $|\tau_{ij,t}^*|\leq|\tau_{ij}^*|$ for some τ_{ij}^* and for all t; and $N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}|\tau_{ij}^*|\leq M^*$.*

Proof. By Assumption 4, $|\tau_{ij,t}|\leq|\tau_{ij}|$ for some τ_{ij} and all t, where $\tau_{ij,t}=E(e_{it}e_{jt})$. Then:

$$\begin{aligned}|\tau_{ij,t}^*| &= |E(\epsilon_{it}\epsilon_{jt})|\\ &= |E(e_{it}+\eta_i'G_t^2)(e_{jt}+\eta_j'G_t^2)|\\ &\leq |E(e_{it}e_{jt})|+\sqrt{E(\eta_i'G_s^2)^2E(\eta_i'G_t^2)^2}\\ &\leq |\tau_{ij}|+O(\frac{1}{NT})\end{aligned}$$

for all t. Therefore

$$\begin{aligned}N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}|\tau_{ij}^*| &\leq N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(|\tau_{ij}|+O(\frac{1}{NT})\right)\\ &\leq M+O(\frac{1}{T})\\ &\leq M^*\end{aligned}$$

by Assumption 4. □

Lemma 6. *$E(\epsilon_{it}\epsilon_{js})=\tau_{ij,ts}^*$ and $(NT)^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}|\tau_{ij,ts}^*|\leq M^*$.*

Proof. By Assumption 4, $(NT)^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}|\tau_{ij,ts}|\leq M$, where $E(e_{it}e_{js})=\tau_{ij,ts}$. Then:

$$E(\epsilon_{it}\epsilon_{js})=\tau_{ij,ts}^*=E(e_{it}e_{js})+E(\eta_i'G_t^2\eta_j'G_s^2)=\tau_{ij,ts}+E(\eta_i'G_t^2\eta_j'G_s^2)$$

and we have

$$\begin{aligned}(NT)^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{t=1}^{T}|\tau^*_{ij,ts}| &\leq (NT)^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{t=1}^{T}|\tau_{ij,ts}| + (NT)^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{t=1}^{T}|E(\eta_i' G_t^2 \eta_j' G_s^2)| \\ &\leq M + O(1) \\ &\leq M^*\end{aligned}$$

following the same arguments as above.

□

Lemma 7. *For every* (t,s), $E|N^{-1/2}\sum_{i=1}^{N}[\epsilon_{is}\epsilon_{it} - E(\epsilon_{is}\epsilon_{it})]|^4 \leq M^*$.

Proof. Since $\epsilon_{it} = e_{it} + \eta_i' G_t^2$, we have:

$$\epsilon_{it}\epsilon_{is} - E(\epsilon_{it}\epsilon_{is}) = e_{it}e_{is} - E(e_{it}e_{is}) + e_{it}\eta_i' G_s^2 + e_{is}\eta_i' G_t^2 + \eta_i' G_t^2 \eta_i' G_s^2 - E(\eta_i' G_t^2 \eta_i' G_s^2).$$

Since $E|e_{it}\eta_i G_s^2|^4 \leq \|\eta_i\|^4 E|e_{it}|^4 E\|F_t\|^4 = O_p(N^{-2}T^{-2})$, and $E|\eta_i' G_t^2 \eta_i' G_s^2|^4 \leq \|\eta_i\|^8 E\|F_t\|^4 = O_p(N^{-4}T^{-4})$, the result follows from Loève's inequality and that

$$E\Big|N^{-1/2}\sum_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]\Big|^4 \leq M.$$

□

Lemma 8. $E\Big(\frac{1}{N}\sum_{i=1}^{N}\Big\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t^*\epsilon_{it}\Big\|^2\Big) \leq M^*$.

Proof. By the definition of ϵ_{it} we have:

$$E\Big(\frac{1}{N}\sum_{i=1}^{N}\Big\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t^*\epsilon_{it}\Big\|^2\Big) \leq E\Big(\frac{1}{N}\sum_{i=1}^{N}\Big\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t^* e_{it}\Big\|^2\Big) + E\Big(\frac{1}{N}\sum_{i=1}^{N}\Big\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t^*\eta_i' G_t^2\Big\|^2\Big)$$

then by the definition of F_t^* and G_t^2,

$$\Big\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t^* e_{it}\Big\|^2 = \Big\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t e_{it}\Big\|^2 + \Big\|\frac{1}{\sqrt{T}}\sum_{t=\tau+1}^{T}F_t^1 e_{it}\Big\|^2,$$

$$\Big\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t^*\eta_i' G_t^2\Big\|^2 = \Big\|\frac{1}{\sqrt{T}}\sum_{t=\tau+1}^{T}F_t\eta_i' F_t^2\Big\|^2 + \Big\|\frac{1}{\sqrt{T}}\sum_{t=\tau+1}^{T}F_t^1\eta_i' F_t^2\Big\|^2.$$

First, by Assumption 4 we have

$$E\Big(\frac{1}{N}\sum_{i=1}^{N}\Big\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t e_{it}\Big\|^2\Big) \leq M.$$

Second,

$$
\begin{aligned}
& E\Big\|\frac{1}{\sqrt{T}}\sum_{t=\tau+1}^{T}F_t\eta_i'F_t^2\Big\|^2 \\
= \ & \frac{1}{T}\sum_{k=1}^{r}E\Big(\sum_{t=\tau+1}^{T}F_{kt}\eta_i'F_t^2\Big)^2 \\
= \ & \frac{1}{T}\sum_{p=1}^{r}\sum_{t=\tau+1}^{T}\sum_{s=\tau+1}^{T}E\Big(F_{kt}F_{ks}(\eta_i'F_t)(\eta_i'F_s)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
& E\Big(F_{kt}F_{ks}(\eta_i'F_t^2)(\eta_i'F_s^2)\Big) \\
\leq \ & \|\eta_i\|^2 E\|F_t\|^4 = O(\frac{1}{NT}),
\end{aligned}
$$

so we have $E\Big\|\frac{1}{\sqrt{T}}\sum_{t=\tau+1}^{T}F_t\eta_i'F_t^2\Big\|^2 = O(1/N)$. The result then follows by noting that $\Big\|\frac{1}{\sqrt{T}}\sum_{t=\tau+1}^{T}F_t^1 e_{it}\Big\|^2 \leq \Big\|\frac{1}{\sqrt{T}}\sum_{t=\tau+1}^{T}F_t e_{it}\Big\|^2$ and $\Big\|\frac{1}{\sqrt{T}}\sum_{t=\tau+1}^{T}F_t^1\eta_i'F_t^2\Big\|^2 \leq \Big\|\frac{1}{\sqrt{T}}\sum_{t=\tau+1}^{T}F_t\eta_i'F_t^2\Big\|^2$. $\square$

As mentioned before, once it has been shown that the new factors: F_t^*, the new loadings: Γ and the new errors: ϵ_t all satisfy the necessary conditions of BN (2002), Propositions 1 and 2 just follow directly from their Theorems 1 and 2, with r replaced by $r + k_1$ and F_t replaced by F_t^*.

A.2: Proof of Theorem 1

We only derive the limiting distributions for the two versions of the LM test, since the proof for the Wald tests is very similar. Let $\hat{F}_t$ define the $r \times 1$ vector of estimated factors. Under the null: $k_1 = 0$, when $\bar{r} = r$ we have

$$\hat{F}_t = DF_t + o_p(1).$$

Let $D_{(i\cdot)}$ denote the ith row of D, and $D_{(\cdot j)}$ denote the jth column of D. Define $\hat{\mathcal{F}}_t = DF_t$, and $\hat{\mathcal{F}}_{kt} = D_{(k\cdot)} \times F_t$ as the kth element of $\hat{\mathcal{F}}_t$. Let $\hat{F}_{1t}$ be the first element of $\hat{F}_t$, and $\hat{F}_{-1t} = [\hat{F}_{2t}, \ldots, \hat{F}_{rt}]'$, while $\hat{\mathcal{F}}_{1t}$ and $\hat{\mathcal{F}}_{-1t}$ can be defined in the same way. Note that $\hat{\mathcal{F}}_t$ depends on N and T. For simplicity, let $T\pi$ denote $[T\pi]$.

Note that under H_0, we allow for the existence of small breaks, so that the model can be written as $X_{it} = \alpha_i F_t + e_{it} + \eta_i G_t^2$. However, since $\eta_i G_t^2$ is $O_p(1/\sqrt{NT})$ by Assumption 1, we can use similar methods as in Appendix A.1 to show that an error term of this order can be ignored and that the asymptotic properties of $\hat{F}_t$ will not be affected (See Remark 5 of Bai, 2009). Therefore, for simplicity in the presentation below, we eliminate the last term and consider instead the model $X_{it} = \alpha_i F_t + e_{it}$ in the following lemmae (9 to 13) required to prove Lemma 14 which is the key one in the proof of Theorem 1.

Lemma 9.

$$\sup_{\pi\in[0,1]} \Big\| \frac{1}{T} \sum_{t=1}^{T\pi} (\hat{F}_t - \hat{\mathcal{F}}_t) F_t' \Big\| = O_p(\delta_{N,T}^{-2}).$$

Proof. Following Bai (2003) we have:

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T\pi}(\hat{F}_t - \hat{\mathcal{F}}_t)F_t' &= T^{-2}\sum_{t=1}^{T\pi}\sum_{s=1}^{T}\hat{F}_s F_t' \gamma_N(s,t) + T^{-2}\sum_{t=1}^{T\pi}\sum_{s=1}^{T}\hat{F}_s F_t' \zeta_{st} + T^{-2}\sum_{t=1}^{T\pi}\sum_{s=1}^{T}\hat{F}_s F_t' \kappa_{st} \\ &\quad + T^{-2}\sum_{t=1}^{T\pi}\sum_{s=1}^{T}\hat{F}_s F_t' \xi_{st} \\ &= I + II + III + IV\end{aligned}$$

where

$$\begin{aligned}\zeta_{st} &= \frac{e_s' e_t}{N} - \gamma_N(s,t). \\ \kappa_{st} &= F_s' A' e_t / N. \\ \xi_{st} &= F_t' A' e_s / N.\end{aligned}$$

First, note that:

$$I = T^{-2}\sum_{t=1}^{T\pi}\sum_{s=1}^{T}(\hat{F}_s - DF_s)F_t'\gamma_N(s,t) + T^{-2}D\sum_{t=1}^{T\pi}\sum_{s=1}^{T}F_s F_t'\gamma_N(s,t).$$

Consider the first part of the right hand side, we have

$$\begin{aligned}&\Big\|T^{-2}\sum_{t=1}^{T\pi}\sum_{s=1}^{T}(\hat{F}_s - DF_s)F_t'\gamma_N(s,t)\Big\| \\ =\ &\Big\|T^{-2}\sum_{s=1}^{T}\Big((\hat{F}_s - DF_s)\sum_{t=1}^{T\pi}F_t'\gamma_N(s,t)\Big)\Big\| \\ \le\ &T^{-1/2}\sqrt{\frac{1}{T}\sum_{s=1}^{T}\left\|\hat{F}_s - DF_s\right\|^2}\sqrt{\frac{1}{T}\sum_{t=1}^{T\pi}\left\|F_t\right\|^2}\sqrt{\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T\pi}\gamma_N(s,t)^2}.\end{aligned}$$

$\frac{1}{T}\sum_{s=1}^{T}\left\|\hat{F}_s - DF_s\right\|^2$ is $O_p(\delta_{N,T}^{-2})$ by Theorem 1 of BN (2002), $\sup_{\pi\in[0,1]}\frac{1}{T}\sum_{t=1}^{T\pi}\left\|F_t\right\|^2 \le \frac{1}{T}\sum_{t=1}^{T}\left\|F_t\right\|^2 = O_p(1)$ by Assumption 2, and $\sup_{\pi\in[0,1]}\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T\pi}\gamma_N(s,t)^2 \le \frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\gamma_N(s,t)^2 =$

$O_p(1)$ by Lemma 1(i) of BN (2002). Therefore:

$$\sup_{\pi\in[0,1]} \Big\| T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^{T} (\hat{F}_s - DF_s) F_t' \gamma_N(s,t) \Big\| = O_p(\delta_{N,T}^{-1} T^{-1/2}).$$

For the second part, note that:

$$\begin{aligned} & \sup_{\pi\in[0,1]} \Big\| T^{-2} D \sum_{t=1}^{T\pi} \sum_{s=1}^{T} F_s F_t' \gamma_N(s,t) \Big\| \\ \leq \;& T^{-2} \|D\| \sum_{t=1}^{T} \sum_{s=1}^{T} \|F_s F_t'\| |\gamma_N(s,t)| \end{aligned}$$

and

$$\begin{aligned} & E\bigg(\frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} \|F_s F_t'\| |\gamma_N(s,t)| \bigg) \\ \leq \;& \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} E||F_t||^2 |\gamma_N(s,t)| = E||F_t||^2 \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} |\gamma_N(s,t)| \leq M \end{aligned}$$

by Assumptions 2 and 4, so the second part is $O_p(T^{-1})$ given that $||D||$ is $O_p(1)$. Therefore, we have

$$\sup_{\pi\in[0,1]} \|I\| = O_p\Big(\frac{1}{\delta_{N,T}\sqrt{T}} \Big). \tag{A.1}$$

Next, II can be written as:

$$II = T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^{T} (\hat{F}_s - DF_s) F_t' \zeta_{st} + T^{-2} D \sum_{t=1}^{T\pi} \sum_{s=1}^{T} F_s F_t' \zeta_{st}.$$

Similarly, we have

$$\begin{aligned} & \sup_{\pi\in[0,1]} \Big\| T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^{T} (\hat{F}_s - DF_s) F_t' \zeta_{st} \Big\| \\ \leq \;& \sup_{\pi\in[0,1]} \sqrt{\frac{1}{T} \sum_{s=1}^{T} \|\hat{F}_s - DF_s\|^2} \sqrt{\frac{1}{T} \sum_{t=1}^{T\pi} \|F_t\|^2} \sqrt{\frac{1}{T^2} \sum_{s=1}^{T} \sum_{t=1}^{T\pi} \zeta_{st}^2} \\ \leq \;& \sqrt{\frac{1}{T} \sum_{s=1}^{T} \|\hat{F}_s - DF_s\|^2} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \|F_t\|^2} \sqrt{\frac{1}{T^2} \sum_{s=1}^{T} \sum_{t=1}^{T} \zeta_{st}^2} \\ = \;& O_p\Big(\frac{1}{\delta_{N,T}\sqrt{N}} \Big) \end{aligned}$$

because

$$E|\zeta_{st}|^2 = N^{-1}E\Big|N^{-1/2}\sum_{i=1}^{N}[e_{it}e_{is} - E(e_{it}e_{is})]\Big|^2 = O(N^{-1})$$

by Assumption 4. As for the second term of II, we have:

$$T^{-2}D\sum_{t=1}^{T\pi}\sum_{s=1}^{T}F_sF_t'\zeta_{st} = \frac{1}{\sqrt{NT}}\frac{1}{T}D\sum_{t=1}^{T\pi}q_tF_t'$$

where

$$q_t = \frac{1}{\sqrt{NT}}\sum_{s=1}^{T}\sum_{i=1}^{N}[e_{it}e_{is} - E(e_{it}e_{is})]F_s.$$

Since $E||q_t||^2 \leq M$ by Assumption 4, we have

$$\begin{aligned}
&\sup_{\pi\in[0,1]}\Big\|T^{-2}D\sum_{t=1}^{T\pi}\sum_{s=1}^{T}F_sF_t'\zeta_{st}\Big\| \\
=\;& \frac{1}{\sqrt{NT}}\sup_{\pi\in[0,1]}\Big\|T^{-1}D\sum_{t=1}^{T\pi}q_tF_t'\Big\| \\
\leq\;& \frac{1}{\sqrt{NT}}\|D\|\sup_{\pi\in[0,1]}\Big\|\sqrt{\frac{1}{T}\sum_{t=1}^{T\pi}\|q_t\|^2}\sqrt{\frac{1}{T}\sum_{t=1}^{T\pi}\|F_t\|^2} \\
\leq\;& O_p(1)\frac{1}{\sqrt{NT}}\Big\|\sqrt{\frac{1}{T}\sum_{t=1}^{T}\|q_t\|^2}\sqrt{\frac{1}{T}\sum_{t=1}^{T}\|F_t\|^2} \\
=\;& O_p\Big(\frac{1}{\sqrt{NT}}\Big).
\end{aligned}$$

Then it follows that

$$\sup_{\pi\in[0,1]}\|II\| = O_p\Big(\frac{1}{\delta_{N,T}\sqrt{N}}\Big). \tag{A.2}$$

Regarding III, it can be written as:

$$III = T^{-2}\sum_{t=1}^{T\pi}\sum_{s=1}^{T}(\hat{F}_s - DF_s)F_t'\kappa_{st} + T^{-2}D\sum_{t=1}^{T\pi}\sum_{s=1}^{T}F_sF_t'\kappa_{st}$$

and the second part on the right hand side can be written as

$$D\Big(\frac{1}{T}\sum_{s=1}^{T}F_sF_s'\Big)\frac{1}{NT}\sum_{t=1}^{T\pi}\sum_{i=1}^{N}\alpha_iF_t'e_{it}.$$

Therefore:

$$\begin{aligned}
& \sup_{\pi\in[0,1]} \Big\| T^{-2} D \sum_{t=1}^{T\pi}\sum_{s=1}^{T} F_s F_t' \kappa_{st} \Big\| \\
\leq\ & \frac{1}{\sqrt{NT}} \|D\| \Big\| \frac{1}{T}\sum_{s=1}^{T} F_s F_s' \Big\| \sup_{\pi\in[0,1]} \Big\| \frac{1}{\sqrt{NT}} \sum_{t=1}^{T\pi}\sum_{i=1}^{N} \alpha_i F_t' e_{it} \Big\| \\
=\ & O_p\Big(\frac{1}{\sqrt{NT}}\Big)
\end{aligned}$$

by Assumption 8.

As for the first part on the right hand side of III, we have

$$\begin{aligned}
& \sup_{\pi\in[0,1]} \Big\| T^{-2} \sum_{t=1}^{T\pi}\sum_{s=1}^{T} (\hat{F}_s - DF_s) F_t' \kappa_{st} \Big\| \\
\leq\ & \sqrt{\frac{1}{T}\sum_{s=1}^{T} \|\hat{F}_s - DF_s\|^2} \sup_{\pi\in[0,1]} \sqrt{\frac{1}{T}\sum_{s=1}^{T} \Big\| \frac{1}{T}\sum_{t=1}^{T\pi} F_t' \kappa_{st} \Big\|^2} \\
=\ & O_p(\delta_{N,T}^{-1}) \frac{1}{\sqrt{NT}} \sup_{\pi\in[0,1]} \sqrt{\frac{1}{T}\sum_{s=1}^{T} \Big\| F_s' \frac{1}{\sqrt{NT}} \sum_{t=1}^{T\pi}\sum_{i=1}^{N} \alpha_i F_t' e_{it} \Big\|^2} \\
\leq\ & O_p(\delta_{N,T}^{-1}) \frac{1}{\sqrt{NT}} \sqrt{\frac{1}{T}\sum_{s=1}^{T} \Big\| F_s \Big\|^2} \sup_{\pi\in[0,1]} \sqrt{\Big\| \frac{1}{\sqrt{NT}} \sum_{t=1}^{T\pi}\sum_{i=1}^{N} \alpha_i F_t' e_{it} \Big\|^2} \\
=\ & O_p\Big(\frac{1}{\delta_{N,T}} \frac{1}{\sqrt{NT}}\Big)
\end{aligned}$$

by Assumption 8. Thus,

$$\sup_{\pi\in[0,1]} \|III\| = O_p\Big(\frac{1}{\sqrt{NT}}\Big). \tag{A.3}$$

It can also be proved in the similar way that

$$\sup_{\pi\in[0,1]} \|IV\| = O_p\Big(\frac{1}{\sqrt{NT}}\Big). \tag{A.4}$$

Finally we have:

$$\begin{aligned}
& \sup_{\pi\in[0,1]} \Big\| \frac{1}{T}\sum_{t=1}^{T\pi} (\hat{F}_t - \hat{\mathcal{F}}_t) F_t' \Big\| \leq \sup_{\pi\in[0,1]} \|I\| + \sup_{\pi\in[0,1]} \|II\| + \sup_{\pi\in[0,1]} \|III\| + \sup_{\pi\in[0,1]} \|IV\| \\
=\ & O_p\Big(\frac{1}{\sqrt{T}\delta_{N,T}}\Big) + O_p\Big(\frac{1}{\sqrt{N}\delta_{N,T}}\Big) + O_p\Big(\frac{1}{\sqrt{NT}}\Big) = O_p\Big(\frac{1}{\delta_{N,T}^2}\Big).
\end{aligned}$$

□

Lemma 10.

$$\sup_{\pi\in[0,1]}\left\|\frac{1}{T}\sum_{t=1}^{T\pi}\hat{F}_t\hat{F}_t' - \frac{1}{T}\sum_{t=1}^{T\pi}\hat{\mathcal{F}}_t\hat{\mathcal{F}}_t'\right\| = O_p(\delta_{N,T}^{-2}).$$

Proof. Note that:

$$\begin{aligned}
& \frac{1}{T}\sum_{t=1}^{T\pi}\hat{F}_t\hat{F}_t' - \frac{1}{T}\sum_{t=1}^{T\pi}\hat{\mathcal{F}}_t\hat{\mathcal{F}}_t' \\
= & \frac{1}{T}\sum_{t=1}^{T\pi}\hat{F}_t\hat{F}_t' - \frac{1}{T}\sum_{t=1}^{T\pi}(DF_t)(F_t'D') \\
= & \frac{1}{T}\sum_{t=1}^{T\pi}\hat{F}_t(\hat{F}_t' - F_t'D') + \frac{1}{T}\sum_{t=1}^{T\pi}(\hat{F}_t - DF_t)(F_t'D') \\
= & \frac{1}{T}\sum_{t=1}^{T\pi}(\hat{F}_t - DF_t)(\hat{F}_t - DF_t)' + \frac{1}{T}D\sum_{t=1}^{T\pi}F_t(\hat{F}_t - DF_t)' + \frac{1}{T}\sum_{t=1}^{T\pi}(\hat{F}_t - DF_t)(F_t'D').
\end{aligned}$$

Thus,

$$\begin{aligned}
& \sup_{\pi\in[0,1]}\left\|\frac{1}{T}\sum_{t=1}^{T\pi}\hat{F}_t\hat{F}_t' - \frac{1}{T}\sum_{t=1}^{T\pi}\hat{\mathcal{F}}_t\hat{\mathcal{F}}_t'\right\| \\
\leq & \sup_{\pi\in[0,1]}\left\|\frac{1}{T}\sum_{t=1}^{T\pi}(\hat{F}_t - DF_t)(\hat{F}_t - DF_t)'\right\| + 2\|D\|\sup_{\pi\in[0,1]}\left\|\frac{1}{T}\sum_{t=1}^{T\pi}(\hat{F}_t - DF_t)F_t'\right\| \\
\leq & \frac{1}{T}\sum_{t=1}^{T}\left\|\hat{F}_t - DF_t\right\|^2 + 2\|D\|\sup_{\pi\in[0,1]}\left\|\frac{1}{T}\sum_{t=1}^{T\pi}(\hat{F}_t - DF_t)F_t'\right\|
\end{aligned}$$

since $\frac{1}{T}\sum_{t=1}^{T}\left\|\hat{F}_t - DF_t\right\|^2 = O_p(\delta_{N,T}^{-2})$ and $\sup_{\pi\in[0,1]}\left\|\frac{1}{T}\sum_{t=1}^{T\pi}(\hat{F}_t - DF_t)F_t'\right\|$ is $O_p(\delta_{N,T}^{-2})$ by Lemma 9, the proof is complete.

□

The next two lemmae follow from Lemma 10 and Assumption 6:

Lemma 11.

$$\sup_{\pi\in[0,1]}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T\pi}\hat{F}_{-1t}\hat{F}_{1t} - \frac{1}{\sqrt{T}}\sum_{t=1}^{T\pi}\hat{\mathcal{F}}_{-1t}\hat{\mathcal{F}}_{1t}\right\| = o_p(1).$$

Proof. See Lemma 10 and Assumption 6. □

Lemma 12.

$$\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\hat{\mathcal{F}}_{-1t}\hat{\mathcal{F}}_{1t}'\right\| = o_p(1).$$

Proof. By construction we have $\frac{1}{T}\sum_{t=1}^{T}\hat{F}_{-1t}\hat{F}_{1t}' = 0$, and then the result follows from Lemma 11. □

Let $\Rightarrow$ denote *weak convergence*. D^*, $\mathcal{F}_t$, $\mathcal{F}_{1t}$, $\mathcal{F}_{-1t}$ and S are defined as in the paper (see Page 12). Similarly, let $D^*_{(i\cdot)}$ denote the ith row of D^*, and $D^*_{(\cdot j)}$ denote the jth column of D^*. Then:

Lemma 13.

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T\pi}\left(\mathcal{F}_{-1t}\mathcal{F}_{1t}-E(\mathcal{F}_{-1t}\mathcal{F}_{1t})\right)\Rightarrow S^{1/2}\mathcal{W}_{r-1}(\pi)$$

for $\pi\in[0,1]$*, where* $\mathcal{W}_{r-1}(\cdot)$ *is a* $r-1$ *vector of independent Brownian motions on* $[0,1]$*.*

Proof. $\mathcal{F}_{-1t}\mathcal{F}_{1t}$ is stationary and ergodic because F_t is stationary and ergodic by Assumption 7. First, we show that $\{\mathcal{F}_{kt}\mathcal{F}_{1t}-E(\mathcal{F}_{kt}\mathcal{F}_{1t}),\Omega_t\}$ is an adapted mixingale of size -1 for $k=2,\ldots,r$. By definition, we have $\mathcal{F}_{kt}\mathcal{F}_{1t}=(D^*_{(k\cdot)}F_t)(D^*_{(1\cdot)}F_t)=\big(\sum_{p=1}^r D^*_{kp}F_{pt}\big)\big(\sum_{p=1}^r D^*_{1p}F_{pt}\big)=\sum_{h=1}^r\sum_{p=1}^r D^*_{kp}D^*_{1h}F_{pt}F_{ht}$, and $\mathcal{F}_{kt}\mathcal{F}_{1t}-E(\mathcal{F}_{kt}\mathcal{F}_{1t})=\sum_{h=1}^r\sum_{p=1}^r D^*_{kp}D^*_{1h}(F_{pt}F_{ht}-E(F_{pt}F_{ht}))=\sum_{h=1}^r\sum_{p=1}^r D^*_{kp}D^*_{1h}Y_{hp,t}$. Thus:

$$\begin{aligned}
&\sqrt{E\Big(E(\mathcal{F}_{kt}\mathcal{F}_{1t}-E(\mathcal{F}_{kt}\mathcal{F}_{1t})|\Omega_{t-m})\Big)^2}\\
=\;&\sqrt{E\Big(\sum_{h=1}^r\sum_{p=1}^r D^*_{kp}D^*_{1h}E(Y_{hp,t}|\Omega_{t-m})\Big)^2}\\
\le\;&\sum_{h=1}^r\sum_{p=1}^r|D^*_{kp}D^*_{1h}|\sqrt{E\big(E(Y_{hp,t}|\Omega_{t-m})\big)^2}\\
\le\;&\Delta\sum_{h=1}^r\sum_{p=1}^r c_t^{hp}\gamma_m^{hp}\\
\le\;&\Delta r^2\max(c_t^{hp})\max(\gamma_m^{hp})
\end{aligned}$$

since $\max(\gamma_m^{hp})$ is $O(m^{-1-\delta})$ for some $\delta>0$ by Assumption 7, we conclude that $\{\mathcal{F}_{kt}\mathcal{F}_{1t}-E(\mathcal{F}_{kt}\mathcal{F}_{1t}),\Omega_t\}$ is an adapted mixingale of size -1 for $k=2,\ldots,r$.

Next, we prove the weak convergence using the Crame-Rao device. Define

$$z_t=a'S^{-1/2}(\mathcal{F}_{-1t}\mathcal{F}_{1t}-E(\mathcal{F}_{-1t}\mathcal{F}_{1t}))$$

where $a\in\mathbb{R}^{r-1}$, and $a'a=1$. Note that

$$z_t=\sum_{k=2}^r\tilde{a}_k[\mathcal{F}_{kt}\mathcal{F}_{1t}-E(\mathcal{F}_{kt}\mathcal{F}_{1t})]$$

where $\tilde{a}_k$ is the $k-1$th element of $a'S^{-1/2}$.

$$\begin{aligned} E(z_t^2) &\le \left(\sum_{k=2}^{r}\sqrt{E\Big(\tilde{a}_k[\mathcal{F}_{kt}\mathcal{F}_{1t}-E(\mathcal{F}_{kt}\mathcal{F}_{1t})]\Big)^2}\right)^2 \\ &\le \Delta\left(\sum_{k=2}^{r}\sqrt{E\Big(\mathcal{F}_{kt}\mathcal{F}_{1t}\Big)^2-\Big(E(\mathcal{F}_{kt}\mathcal{F}_{1t})\Big)^2}\right)^2 \le M \end{aligned}$$

because $E\|F_t\|^4<\infty$ and $\mathcal{F}_{kt}=D_{k\cdot}^{*}F_t$. Moreover, z_t is stationary and ergodic, and we can show $\{z_t,\Omega_t\}$ is an adapted mixingale sequence of size -1 because:

$$\begin{aligned} \sqrt{E\Big(E(z_t|\Omega_{t-m})\Big)^2} &= \sqrt{E\Big(\sum_{k=2}^{r}\tilde{a}_k E\Big(\mathcal{F}_{kt}\mathcal{F}_{1t}-E(\mathcal{F}_{kt}\mathcal{F}_{1t})|\Omega_{t-m}\Big)\Big)^2} \\ &\le \sum_{k=2}^{r}|\tilde{a}_k|\sqrt{E\Big(E\Big(\mathcal{F}_{kt}\mathcal{F}_{1t}-E(\mathcal{F}_{kt}\mathcal{F}_{1t})|\Omega_{t-m}\Big)^2\Big)} \\ &\le \max(|\tilde{a}_k|)\sum_{k=2}^{r}\tilde{c}_t^k\tilde{\gamma}_m^k. \end{aligned}$$

By the results above we know that $\tilde{\gamma}_m^k$ is $O(m^{-1-\delta})$ for $k=2,\ldots,r$. Hence it follows that $\{z_t,\Omega_t\}$ is an adapted mixingale sequence of size -1. Then it follows from Theorem 7.17 of White (2001) that:

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T\pi}z_t = a'S^{-1/2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T\pi}\left(\mathcal{F}_{-1t}\mathcal{F}_{1t}-E(\mathcal{F}_{-1t}\mathcal{F}_{1t})\right)\Rightarrow\mathcal{W}(\pi).$$

Moreover, it can be shown that:

$$a_1'\frac{1}{\sqrt{T}}\sum_{t=T\pi_1}^{T\pi_2}\left(\mathcal{F}_{-1t}\mathcal{F}_{1t}-E(\mathcal{F}_{-1t}\mathcal{F}_{1t})\right)+a_2'\frac{1}{\sqrt{T}}\sum_{t=1}^{T\pi_0}\left(\mathcal{F}_{-1t}\mathcal{F}_{1t}-E(\mathcal{F}_{-1t}\mathcal{F}_{1t})\right)\xrightarrow{d}N(0,(\pi_2-\pi_1)a_1'Sa_1+\pi_0a_2'Sa_2)$$

by using Corollary 3.1 of Woodridge and White (1988). The proof is completed by using Lemma A.4 of Andrews (1993).

□

A.3: More discussions on Remark 9

In Remark 9 of the paper we mention that, although our tests are designed for single break, they should also have power against multiple breaks. To see this, consider the simple example of a FM with one factor and two big breaks:

$$X_t = Af_t\cdot 1(t\le\tau_1)+Bf_t\cdot 1(\tau_1<t<\tau_2)+Df_t\cdot 1(t\ge\tau_2)+e_t$$

$$= Ag_t + Bh_t + Ds_t + e_t$$

where $g_t = f_t \cdot 1(t \leq \tau_1)$, $h_t = f_t \cdot 1(\tau_1 < t < \tau_2)$, and $s_t = f_t \cdot 1(t \geq \tau_2)$. In view of Proposition 2, Bai and Ng's (2002) IC will lead to the choice of 3 factors which, when estimated by PCA, implies the following result:

$$\begin{pmatrix} \hat{f}_{1t} \\ \hat{f}_{2t} \\ \hat{f}_{3t} \end{pmatrix} = \begin{pmatrix} d_1 & d_4 & d_7 \\ d_2 & d_5 & d_8 \\ d_3 & d_6 & d_9 \end{pmatrix} \begin{pmatrix} g_t \\ h_t \\ s_t \end{pmatrix} + o_p(1).$$

Then, by the definition of g_t, h_t and s_t we have:

$$\begin{pmatrix} \hat{f}_{1t} \\ \hat{f}_{2t} \\ \hat{f}_{3t} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} f_t + o_p(1) \text{ for } t = 1, \ldots, \tau_1,$$

$$\begin{pmatrix} \hat{f}_{1t} \\ \hat{f}_{2t} \\ \hat{f}_{3t} \end{pmatrix} = \begin{pmatrix} d_4 \\ d_5 \\ d_6 \end{pmatrix} f_t + o_p(1) \text{ for } t = \tau_1, \ldots, \tau_2,$$

$$\begin{pmatrix} \hat{f}_{1t} \\ \hat{f}_{2t} \\ \hat{f}_{3t} \end{pmatrix} = \begin{pmatrix} d_7 \\ d_8 \\ d_9 \end{pmatrix} f_t + o_p(1) \text{ for } t = \tau_2, \ldots, T.$$

Hence, we can find one vector $[p_1, p_2, p_3]'$ which is orthogonal to $[d_1, d_2, d_3]'$ and $[d_4, d_5, d_6]'$, plus another vector $[p_4, p_5, p_6]'$ which is orthogonal to $[d_7, d_8, d_9]'$. It is easy to see that $[p_1, p_2, p_3] \neq a[p_4, p_5, p_6]$ for any $a \neq 0$ (otherwise the D matrix will be singular), and thus we can find a breaking relationship between the estimated factors and even use Bai and Perron's (1998, 2003) to detect a second break. The simulation results about the power of our tests against multiple breaks are available upon request.

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