Macroeconomics 3

Extensions : time iteration, Markov Switching and occasionally binding constraints

Hernán D. Seoane

UC3M

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Today's lecture

- The Rendahl (2017) solution method
- Pretty much in line with standard log-linearization methods
- Intuition: suppose a 2 state Markov chain: High and low average output growth
- You have one economy (model/set of log-linearized conditions) conditional on high and other one conditional on low
- You can think of the full model as a combination between these 2 worlds, where agents internalize the probability of moving between them

Today's lecture

- These slides follow Rendahl (2017)
- and the presentation in Wouter's class-notes: `http://www.wouterdenhaan.com/teach/Occasional.pdf`

Model

- As we did so far, we start with the set of FOCs that accomodate to most of our models

$$\mathbb{E}_t\left[x_{t-1}, x_t, x_{t+1}\right] = 0$$

- where x_t is a vector of endogenous and exogenous variables (keep together predetermined and jump variables)

- Forget about MS for a while

- We know we can solve the FOCs around the non-stochastic steady state, for the vector at x^*

Model

- If we log-linearize the conditions, the log-lin system becomes

$$J_{\hat{x}_{t-1}}\hat{x}_{t-1} + J_{\hat{x}_t}\hat{x}_t + J_{\hat{x}_{t-1}}\mathbb{E}_t\hat{x}_{t+1} = 0$$

- or simplifying notation

$$Au_{t-1} + Bu_t + Cu_{t+1} = 0$$

- We solve this now with time-iteration, a very intuitive method that can be used in linearized as well as non-linear settings

Model

- We are looking for a solution as: $u_t = Fu_{t-1}$
- plugging in
$$Au_{t-1} + Bu_t + CFu_t = 0$$
- We can then start guessing $F = F_0$
$$Au_{t-1} + Bu_t + CF_0u_t = 0$$
- working out the expression we get
$$u_t = \underbrace{-(B + CF_0)^{-1}A}_{F_1} u_{t-1}$$
- we just did one step update of F

Model

- We can repeat until convergence
- We can have a high degree of accuracy (conditional on that we already log-linearized the system) since this is super-fast
- It is usually faster that the QZ because you don't solve a quadratic system. Instead you need to compute an inverse.

Markov Switching

- If we loglinearize around another point that is not the steady state

$$\mathbb{E}_t\left[x_{t-1}, x_t, x_{t+1}\right] = 0$$

- The solution to this system would be of

$$F(\bar{x}, \bar{x}, \bar{x}) = D$$

- Now

$$D + J_{\hat{x}_{t-1}} \hat{x}_{t-1} + J_{\hat{x}_t} \hat{x}_t + J_{\hat{x}_{t-1}} \mathbb{E}_t \hat{x}_{t+1} = 0$$

Markov Switching

$$D + Au_{t-1} + Bu_t + Cu_{t+1} = 0$$

- Now the solution is not centered in 0 but

$$u_t = E + Fu_{t-1}$$

- the rest of the method looks alike

Markov Switching

- Guess

$$u_t = E_0 + F_0 u_{t-1}$$

- plug

$$D + Au_{t-1} + Bu_t + C[E_0 + F_0 u_t] = 0$$

$$u_t = \underbrace{(B + CF_0)^{-1}(-(D + CE_0))}_{E_1} + \underbrace{-(B + CF_0)^{-1}A}_{F_1} u_{t-1}$$

- we got an update for F and E
- we keep repeating until convergence

Markov Switching

- Lets move to a MS setting

$$\mathbb{E}_t \left[x_{t-1}, x_t, x_{t+1}; z_t, z_{t+1} \right] = 0$$

- where z are discrete stochastic variables with a given transition probability

Markov Switching

- We linearize the system wrt the x_t variables, but we can't do it with respect to z_t, because this last one is discrete
- For each regime, we linearize a different set of FOCs around an arbitrary point (i.e. the steady state in a particular regime)

$$E_j[F(\bar{x}, z^i, \bar{x}, z^j, \bar{x})] = D^i$$

- Then for each regime, we have a set of linearized FOCs

$$D^i + J^i_{x_{t-1}}(x_{t-1} - \bar{x}) + J^i_{x_t}(x_t - \bar{x}) + E_j[J^j_{x_{t+1}}(x_{t+1}(j) - \bar{x})|i] = 0$$

Markov Switching

- If we have 2 regimes, using $\tilde{x} = x - \bar{x}$

$$A^i \tilde{x}_{t-1} + B^i \tilde{x}_t + \sum_j C^j \tilde{x}_{t+1}(j) + D^i = 0$$

$$A^1 \tilde{x}_{t-1} + B^1 \tilde{x}_t + C^{1,1} \tilde{x}_{t+1}(1) + C^{1,2} \tilde{x}_{t+1}(2) + D^1 = 0$$

$$A^2 \tilde{x}_{t-1} + B^2 \tilde{x}_t + C^{2,1} \tilde{x}_{t+1}(1) + C^{2,2} \tilde{x}_{t+1}(2) + D^2 = 0$$

- Solution has the form: $\tilde{x}_t = E^i + F^i \tilde{x}_{t-1}$, for $i = 1, 2$

Markov Switching

- Plugging the solution

$$A^i \tilde{x}_{t-1} + B^i \tilde{x}_t + \sum_j C^j \left[E_n^j + F_n^j \tilde{x}_t \right] + D^i = 0$$

- Using time Iteration we will be able to update the E_n^j and F_n^j in the same way as before, using all the regime systems
- Policy function coefficients of different regimes are interconnected

Markov Switching: alternative 1

- Another alternative is to implement an eigenvalue decomposition as in Foerster (2015)
- It is implemented in a very similar way, start with a guess of the solution and iterate until convergence
- This is also an iterative procedure that extends Schmitt-Grohé and Uribe (2004)
- It is a perturbation approach

Markov Switching: alternative 2

- Another alternative is to implement an eigenvalue decomposition as in Foerster, Rubio-Ramírez, Waggoner, and Zha (2016)
- Rely on the use of Groebner Basis
- Allows to recover all solutions from a quadratic system
- Problem of the method: it may be unfeasible even for medium scale models

Markov Switching: alternative 3

- Older method Farmer, Waggoner, and Zha (2011)
- Problem: the underlaying model that the agents solve do not have MS

Occasionally binding constraints: a perturbation methods approach

- A feature that makes the MS environment easy is that the switch is exogenously driven
- Models with occasionally binding constraints are problematic in particular because the timing of the switch between regimes depend on endogenous variables
- Dealing with this using perturbation methods is possible, but so far there is no "elegant" local method that solves the right underlaying model

Guerrieri and Iacoviello (2015)

- Guerrieri and Iacoviello (2015) presents a method that, in their examples, works well
- Examples: ZLB, Borrowing constraints, capital irreversibility, to name a few
- In any of these, you can define a regime where the constraint is slack (let's say, model M_1)
- an other regime where it is binding, M_2
- As before, we can derive a set of FOCs for each model

Guerrieri and Iacoviello (2015)

- For M_1, we call it base regime

$$A_t \mathbb{E}_t X_{t+1} + BX_t + CX_{t-1} + E\epsilon_t = 0$$

- set of log-linearized conditions
- In regime 2, the conditions will be different as the obc becomes relevant

$$D^* + A_t^* \mathbb{E}_t X_{t+1} + B^* X_t + C^* X_{t-1} + E^* \epsilon_t = 0$$

- where u in this regime may not share the same SS as regime 1

Guerrieri and Iacoviello (2015)

- **Definition 1 (in Guerrieri and Iacoviello (2015))** A solution for a model with an occasionally binding constraint is a function $f : X_{t1} \times \epsilon_t \to X_t$ such that the conditions under system $(M1)$ or the system $(M2)$ hold, depending on the evaluation of the occasionally binding constraint, governed by g and h.

- Here g and h are the OBC when slack or binding, respectively.

Guerrieri and Iacoviello (2015)

- We need two assumptions for this method to work
- BK conditions hold for M_1
- If a shock moves the economy from M_1 to M_2, the economy returns to M_1 for $t < \infty$ while agents do not expect future shocks to occur
- In other words, the method pastes a linear solution to a linear perfect foresight solution (piecewise linear solution in the end)

Guerrieri and Iacoviello (2015)

- The solution we look for is like

$$X_t = \mathcal{P}_t X_{t-1} + \mathcal{R}_t + \mathcal{Q}_t \epsilon_t, \text{ for t=1}$$

and

$$X_t = \mathcal{P}_t X_{t-1} + \mathcal{R}_t, \text{ for all t>2}$$

- note that the solution is non-linear, the matrices are time-varying
- the algorithm to find these matrices depend on the initial condition X_0 and the shock ϵ_1

Guerrieri and Iacoviello (2015)

- How do we solve this setting? Easy, solve for regime 1, ignoring the existence of regime 2... only check the constraint is slack
- Regime 2 is a temporary regime
- When in regime 2, you assume that the economy stays there for T periods and after that, the economy returns (forever) to regime 1
- Iterate until finding the right T

Guerrieri and Iacoviello (2015)

The algorithm

- We know that at period T the economy is back to M_1, forever. And we can compute that part of the solution rigth-away... for any $t \geq T$

$$X_t = PX_{t-1} + Q\epsilon_t$$

- Then we go backwards, we can plug $X_T = PX_{T-1}$ and

$$D^* + A_t^* PX_{T-1} + B^* X_{T-1} + C^* X_{T-2} = 0$$

together with the assumption of no further shocks

Guerrieri and Iacoviello (2015)

The algorithm

- Given a state X_{T-2} we can solve for X_{T-1}, we repeat this until getting to X_0, using M_1 or M_2 depending on the states
- Using the guesses for the solution, compute the path for X to verify if the current guess of regimes. If the guess was right,s top, if not update the guess and return to step 1

Guerrieri and Iacoviello (2015)

- Problems with the method
- The solution you obtained is not for the original problem... notice that agents in M_1 operate as if M_2 does not exist
- The solution is piecewise linear (you basically paste 2 solutions of different models)
- Being in regime 2 is like perfect foresight, you know when you leave it and you solve it assuming you will never go back
- All these may be irrelevant from a quantitative point of view, in the examples the authors show, the solution seems fine

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