Macroeconomics 3

Approximating nonlinear models with Heterogeneous agents and aggregate shocks

Hernán D. Seoane

UC3M

November 14, 2018

Today's lecture

- The Krusell and Smith (1998) solution
- The local approximation approach: Winberry (2018)

Heterogeneity in macro

- What is the problem of heterogeneous agents in macro?

- Suppose you have a model with many firms that receive an idiosyncratic shock? Would this be hard to solve? (hint: NO... well, a bit, but not as much as the next one)

- What if firms are also subject to aggregate shocks? (hint: YES!)

- Why? If aggregate shocks, the firms distribution is a state variable and you need to keep track of it... but this is an infinitely dimensional object

Heterogeneity in macro

- One of the first (and widely used) approaches is the one by Krusell and Smith (1998)
- This paper develops a global solution method
- I present an exposition of this paper here, but we will focus in Winberry (2018), that presents a local solution method based on perturbation

Krusell and Smith (1998)

- Do changes in income/wealth distribution affect macro?
  - In principle, they find that the behavior of macro aggregates can be be very well described by using the mean wealth distribution
- How to deal with heterogeneity in macro?
- Here: brief discussion on the method

- Fixed measure 1 of agents $i \in [0,1]$ with preferences $\sum_{t=0}^{\infty} \beta^t \ln(c_{it})$
- Labor endowment follows Markov process $\epsilon_{i,t} \in \{0, \bar{l}\}$.
- That is, they can be employed or unemployed and if employed they work all their time endowment (leisure does not enter in the utility function)
- We can also assume that agents are taxed when employed and get a transfer when unemployed, for now we skip this
- Agents can save in capital (or we call it "assets"... later I will change the notation to make it in line with Winberry)

- The budget constraint of agent "i" is

$$c_{it} + k_{it+1} = l_{it} w_t + (1 - \delta + r_t) k_{it}$$

- Assume no borrowing $k_{it} \geq 0$
- Representative firm: $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$
- Assume aggregate TFP follows MS process (later we just assume AR(1) process, to avoid dealing with MS in a Perturbation environment)

- In equilibrium we will have that

$$r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - \delta$$

$$w_t = (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}$$

- Let g be the measure (distribution) of agents over (k, ϵ), then
- (A_t, g) is the aggregate state vector of the economy, the problem is that this object is infinite-dimensional
- The distribution changes over time: $g_{t+1} = h(g_t, A_t; A_{t+1})$
- A potential solution: work with some moments of the distribution instead of the whole distribution

- we can write the households problem in recursive way

$$V(k,\epsilon;g,A) = \max_{c,k'} \left\{ u(c) + \beta \mathbb{E}\left[V(k',\epsilon';g',A') | g,A \right] \right\}$$

subject to

$$c + k' = r(K,L,A)k + w(K,L,A)l + (1-\delta)k$$

$$k' \geq 0$$

$$g' = h(g,A;A')$$

- Definition: A recursive competitive equilibrium is a law of motion h, a value and policy (capital accumulation) functions (V, y) and pricing functions (r, w) such that
  1. (V, y) solve the household's problem given g, r and w.
  2. (r, w) are competitive; and
  3. h is generated by y.

- Interpretation: we look for a fixed point in h.
- Agents have a perceived law of motion of the states h_p, compute their policy function and gives rise to the actual law of motion h_a.
- If $h_p = h_a$ we have a rational expectations equilibrium.
- Method: given a distribution, iterates until perceived and actual law of motions coincide

Krusell and Smith (1998)

Solution method

- Guess a law of motion for aggregate capital. Given log utility, log linear law of motion is not a bad guess

$$\ln(K') = a_0 + b_0 \ln(K); \qquad A = A_l$$

$$\ln(K') = a_1 + b_1 \ln(K); \qquad A = A_h$$

- g_1 is characterized by a_0, a_1, b_0, b_1
- find optimal policy given g_1, compute the actual law of motion and stop if they are closed enough

- The problem now can be written as

$$V(k, \epsilon; K, A) = \max_{c,k'} \left\{ u(c) + \beta \mathbb{E} \left[V(k', \epsilon'; K', A') | g, A \right] \right\}$$

subject to

$$c + k' = r(K, L, A)k + w(K, L, A)l + (1 - \delta)k$$

$$\ln(K') = a_0 + b_0 \ln(K); \quad A = A_l$$

$$\ln(K') = a_1 + b_1 \ln(K); \quad A = A_h$$

$$k' \geq 0$$

Krusell and Smith (1998)

The method

- Obtain a nonlinear decision rule $k' = y_1(k, \epsilon; K, A)$
- Simulate it, and compare the aggregate behavior of the moments with H_1.
- We want to find a fixed point for H_1 in the form of a vector a_0, a_1, b_0, b_1.
- Once we do it, stop. Otherwise, we may need to reconsider the functionla form of the distribution or add more moments.

- The method may become slow when dealing with larger models
- Also methods are not general or easy to apply
- We explore now another extension of the perturbation approach

Winberry (2018)

The method

- The method combines some local and global solution methods
- First, instead of working with infinitely large objects, it approximates the cross-sectional distribution of heterogeneous agents using a parametric family
- Following Algan, Allais, and Den Haan (2008) to approximate the stationary cross-section distribution
- combine this with perturbation

- The method is a mix between global and local methods
- 3 steps
  1. approximate equilibrium objects using finite-dimensional global methods wrt individual state variables
  2. compute stationary equilibrium of the finite dimensional model without aggregate shocks but with idiosyncratic shocks
  3. compute aggregate dynamics using Taylor expansions around the stationary steady state

- First we want to get a simple expression for the distribution of agents
- In this setting there are 2 problems: (1) first the infinite-dimensional feature, but also (2) given the borrowing constraint, we may have mass in a point
- we first deal with the latest issue

Winberry (2018)

The method: step 1

- $\hat{m}_{\epsilon,t}$ is the fraction of households with labor productivity ϵ at the borrowing constraint
- start by characterizing the mass at the constraint

$$\hat{m}_{\epsilon,t+1} = \frac{1}{\pi(\epsilon)} \Bigg[\sum_{\tilde{\epsilon}} (1 - \hat{m}_{\tilde{\epsilon},t}) \pi(\tilde{\epsilon}) \pi(\epsilon|\tilde{\epsilon}) \int 1\{a'_t(\tilde{\epsilon}, a) = \underline{a}\} g_{\tilde{\epsilon},t}(a) da$$

$$+ \sum_{\tilde{\epsilon}} \hat{m}_{\tilde{\epsilon},t} \pi(\tilde{\epsilon}) \pi(\epsilon|\tilde{\epsilon}) 1\{a'_t(\tilde{\epsilon}, \underline{a}) = \underline{a}\} \Bigg]$$

- This eq says how this mass evolves, we are just counting individuals at the constraint

- We now turn to find the distribution of agents that are not in the constraint

- approximate it using finite-dimensional object: use Algan, Allais, and Den Haan (2008)

$$g_{\epsilon,t}(k) \approx g^0_{\epsilon,t} \exp\left\{ g^1_{\epsilon,t}(k - m^1_{\epsilon,t}) + \sum_{i=1}^{n_g} g^i_{\epsilon,t} \left[(k - m^1_{\epsilon,t})^i - m^i_{\epsilon,t} \right] \right\}$$

- n_g is the degree of approximation, $\{g^i_{\epsilon,t}\}_{i=0}^{n_g}$ are parameters and $\{m^i_{\epsilon,t}\}_{i=0}^{n_g}$ are centralized moments of the distribution

- What are these moments?
- (to make notation in line with Winberry, I use assets to refer to capital, switch k by a) and write the nonnegativity constraint in a more general way as $a' \geq \underline{a}$
- parameters and moments need to be consistent. In particular,

$$m^1_{\epsilon,t} = \int a g_{\epsilon,t}(a) da$$

$$m^i_{\epsilon,t} = \int (a - m^1_{\epsilon,t})^i g_{\epsilon,t}(a) da, \quad \forall i = 2, ..., n_g$$

- The distribution is characterized by its moments

- If no borrowing limit is imposed, the evolution of its moments is given by

$$m_\epsilon^{1\prime} = \frac{1}{\pi(\epsilon)} \sum_{\tilde{\epsilon}} \pi(\tilde{\epsilon})\pi(\epsilon|\tilde{\epsilon}) \int a'(\tilde{\epsilon}, a) g_{\tilde{\epsilon}}(a) da$$

$$m_\epsilon^{i\prime} = \frac{1}{\pi(\epsilon)} \sum_{\tilde{\epsilon}} \pi(\tilde{\epsilon})\pi(\epsilon|\tilde{\epsilon}) \int \left[a'(\tilde{\epsilon}, a) - m_\epsilon^{1\prime}\right] g_{\tilde{\epsilon}}(a) da$$

- The law of motion of the distribution can be approximated by solving for m moments... that is, (A, g) by (A, m)

- With borrowing limit, the evolution of its moments is given by

$$m^1_{\epsilon,t+1} = \frac{1}{\pi(\epsilon)}\left[\sum_{\tilde{\epsilon}}(1-\hat{m}_{\tilde{\epsilon},t})\pi(\tilde{\epsilon})\pi(\epsilon|\tilde{\epsilon})\int a'_t(\tilde{\epsilon},a)g_{\tilde{\epsilon},t}(a)da + \sum_{\tilde{\epsilon}}\hat{m}_{\tilde{\epsilon},t}\pi(\tilde{\epsilon})\pi(\epsilon|\tilde{\epsilon})a'_t(\tilde{\epsilon},\underline{a})\right]$$

$$m^i_{\epsilon,t+1} = \frac{1}{\pi(\epsilon)}\left[\sum_{\tilde{\epsilon}}(1-\hat{m}_{\tilde{\epsilon},t})\pi(\tilde{\epsilon})\pi(\epsilon|\tilde{\epsilon})\int\left[a'(\tilde{\epsilon},a)-m^{1\prime}_{\epsilon}\right]^i g_{\tilde{\epsilon},t}(a)da + \sum_{\tilde{\epsilon}}\hat{m}_{\tilde{\epsilon},t}\pi(\tilde{\epsilon})\pi(\epsilon|\tilde{\epsilon})\left[a'(\tilde{\epsilon},\underline{a})-m^{1\prime}_{\epsilon}\right]^i\right]$$

- The law of motion of the distribution can be approximated by solving for m moments... that is, (A,g) by (A,m)

- The integrals in the previous expression are solved using Gauss-Legendre Quadrature (i.e. define nodes and weights and substitute the integral by summations)
- Use the same quadrature to approximate aggregate capital

$$K_t = \sum_{\epsilon} \pi(\epsilon) \sum_{j=1}^{m_g} \omega_j a_j g_{\epsilon,t}(a_j)$$

Winberry (2018)

The method: step 1

- Next, we approximate the households decision rules. Why? the borrowing constraint
- If this were not the case, we would just go straight to build the system F of first order conditions
- So, here we have another way of dealing with the OBC
- Approximate Conditional Expectation with Chebyshev Polynomials

$$\psi_t(\epsilon, a) = \mathbb{E}\left[\beta(1 + r_{t+1})c_{t+1}(\epsilon', a'_t(\epsilon, a))^{-\sigma}\right]$$

- the savings and consumption policies that can be derived from the conditional expectations using

$$a'_t(\epsilon,a) = \max\left\{\underline{a}, w_t l + (1+r_t)a - \psi_t(\epsilon,a)^{\frac{-1}{\sigma}}\right\}$$

$$c_t(\epsilon,a) = w_t l + (1+r_t)a - a'_t(\epsilon,a)$$

- We approximate the conditional expectation with Chebyshev polinomials

$$\psi_t(\hat{\epsilon},a) \sim \exp\left\{\sum_{i=1}^{n_\psi} \theta_{\epsilon i,t} T_i(\xi(a))\right\}$$

- Here, $\xi(a)$ is the order of approximation
- T_i is the polinomial of order i
- $\xi(a) = 2\frac{a-\underline{a}}{\bar{a}-\underline{a}} - 1$ transforms the asset interval to the interval -1 to 1 (Cheb pols are defined in this interval)
- θ are coefficients

Winberry (2018)

$$f\left(\mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t+1}; \chi\right) = \begin{bmatrix} m^{1\prime}_{\varepsilon,t} - \frac{1}{\pi(\varepsilon)}[\sum_{\tilde{\varepsilon}} \left(1 - \widehat{m}_{\tilde{\varepsilon},t}\right) \pi\left(\tilde{\varepsilon}\right) \pi\left(\varepsilon|\tilde{\varepsilon}\right) \sum_j \omega_j a'_t\left(\tilde{\varepsilon}, a_j\right) g_{\tilde{\varepsilon},t}\left(a_j\right) \\ + \sum_{\tilde{\varepsilon}} \widehat{m}_{\tilde{\varepsilon},t} \pi\left(\tilde{\varepsilon}\right) \pi\left(\varepsilon|\tilde{\varepsilon}\right) a'_t\left(\tilde{\varepsilon}, \underline{a}\right)] \\ m^{i\prime}_{\varepsilon,t} - \frac{1}{\pi(\varepsilon)}[\sum_{\tilde{\varepsilon}} \left(1 - \widehat{m}_{\tilde{\varepsilon},t}\right) \pi\left(\tilde{\varepsilon}\right) \pi\left(\varepsilon|\tilde{\varepsilon}\right) \sum_j \left[a'_t\left(\tilde{\varepsilon}, a_j\right) - m^{1\prime}_{\varepsilon,t}\right]^i g_{\tilde{\varepsilon},t}\left(a_j\right) \\ + \sum_{\tilde{\varepsilon}} \widehat{m}_{\tilde{\varepsilon},t} \pi\left(\tilde{\varepsilon}\right) \pi\left(\varepsilon|\tilde{\varepsilon}\right) \left[a'_t\left(\tilde{\varepsilon}, \underline{a}\right) - m^{1\prime}_{\varepsilon}\right]^i] \\ \exp\left\{\sum_{i=1}^{n_\psi} \theta_{\varepsilon i,t} T_i\left(\xi\left(a_j\right)\right)\right\} - \beta\left(1 + r_{t+1}\right) \sum_{\varepsilon'} \pi\left(\varepsilon'|\varepsilon\right) \widehat{c}_t\left(\varepsilon', \widehat{a}'_t\left(\varepsilon, a_j\right)\right)^{-\sigma} \\ m^1_{\varepsilon,t} - \sum_j \omega_j a_j g_{\varepsilon,t}\left(a_j\right) \\ m^i_{\varepsilon,t} - \sum_j \omega_j \left(a_j - m^1_{\varepsilon,t}\right)^i g_{\varepsilon,t}\left(a_j\right) \\ r_t - \left(\alpha e^{z_t} K_t^{\alpha-1} L^{1-\alpha} - \delta\right) \\ w_t - \left(1 - \alpha\right) e^{z_t} K_t^{\alpha} L^{-\alpha} \\ z_{t+1} - \rho_z z_t - \chi \times \sigma_z \omega_{t+1} \end{bmatrix}$$

such that

$$E_t\left[f\left(\mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t+1}; \chi\right)\right] = \mathbf{0}, \tag{5}$$

Figure: Winberry (2016)

- Here the $\mathbf{x}_t = [z_t, \mathbf{m}]$
- $\mathbf{y}_t = [\theta_t, \mathbf{g}_t, r_t, w_t]$
- Now the rest of the steps are the standard, first solve for the steady state and then the dynamics using our standard Perturbation methods

ALGAN, Y., O. ALLAIS, AND W. J. DEN HAAN (2008): "Solving heterogeneous-agent models with parameterized cross-sectional distributions," *Journal of Economic Dynamics and Control*, 32(3), 875–908.

KRUSELL, P., AND A. A. SMITH, JR (1998): "Income and wealth heterogeneity in the macroeconomy," *Journal of political Economy*, 106(5), 867–896.

WINBERRY, T. (2018): "A toolbox for solving and estimating heterogeneous agent macro models," *Forthcoming Quantitative Economics*.