# Macro 3

## Solving linear rational expectation models

Hernán D. Seoane

UC3M

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# Today's lecture

## Approximating DSGE models: Solving LREM

Determinacy and existence of equilibrium

Implementation: Schmitt-Grohe and Uribe (2004)

Implementation: Dynare

Implementation: Sims (2000)

This notes will explain the implementation in Schmitt-Grohe and Uribe (2004)

I follow Martín's notes and paper

# Solving LREM

Equilibrium conditions of DSGE: system of nonlinear stochastic difference equations

We cannot solve those systems exactly

We need to work with approximations

Approximations at a particular point: non-stochastic steady state.

Today: how to solve these systems

How to work with these systems (compute means, variances, correlations, autocorrelations and impulse response functions)

# Take again the baseline NGM

$$c_t^{-\gamma} = \beta \mathbb{E}_t \left[ c_{t+1}^{-\gamma} \left( \alpha A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right) \right]$$

$$c_t + k_{t+1} = A_t k_t^{\alpha} + (1-\delta) k_t$$

$$ln(A_{t+1}) = \rho ln(A_t) + \sigma \epsilon_{t+1}$$

for all $t \geq 0$, given $k_0$ and $A_0$.

# Solving LREM

The equilibrium conditions of our models can be written as

$$\mathbb{E}_t[f(y_{t+1}, y_t, x_{t+1}, x_t)] = 0,$$

where $\mathbb{E}_t$ is the expectation operator conditional on the information set available at time $t$

$$\mathbb{E}_t[f(y_{t+1}, y_t, x_{t+1}, x_t)] = \mathbb{E}_t \begin{bmatrix} y_t^{-\gamma} - \beta \mathbb{E}_t \left[ y_{t+1}^{-\gamma} \left( \alpha e^{x_{2t+1}} x_{1t+1}^{\alpha-1} + 1 - \delta \right) \right] \\ y_t + x_{1t+1} - e^{x_{2t}} x_{1t}^{\alpha} - (1-\delta) x_{1t} \\ x_{2t+1} - \rho x_{2t} \end{bmatrix}$$

$x_t = [k_t; \quad ln(A_t)]'$ is a vector of predetermined variables

$y_t = c_t$ is the vector of non-predetermined variables

Initial condition $x_0$ for the predetermined variables and also the no-Ponzi game constraint.

# Solving LREM

The vector of predetermined variables, $x_t$ can be partitioned as

$$x_t = [x_t^1; \quad x_t^2]'$$

$x_t^1$ are endogenous predetermined variables

$x_t^2$ are exogenous predetermined variables

Assume $x_t^2$ follows the following stochastic process

$$x_{t+1}^2 = \tilde{h}(x_t^2, \sigma) + \tilde{\eta}\sigma\epsilon_{t+1}$$

# Solving LREM

About the dimensions. $x_t$ is of size $n_x$

$y_t$ is of size $n_y$

Define $n = n_x + n_y$

Note that $f$ maps $R^{n_y} \times R^{n_y} \times R^{n_x} \times R^{n_x}$ into $R^n$

# Solving LREM

About $x_t^2$ and $\epsilon_t$

Both are of dimensions $n_\epsilon \times 1$

$\epsilon_t$ is i. i. d. with mean 0 and variance equal to 1

$\sigma$ is a positive scalar and $\tilde{\eta}$ is a matrix $n_\epsilon \times n_\epsilon$

# Solving LREM

A solution to this model is of the following form

$$y_t = \hat{g}(x_t)$$

and

$$x_{t+1} = \hat{h}(x_t) + \eta\sigma\epsilon_{t+1}$$

where $\eta = [0; \tilde{\eta}]'$

# Solving LREM

Now the shape of $\hat{g}$ and $\hat{h}$ depend on the degree of uncertainty of the economy

$$y_t = g(x_t, \sigma)$$

and

$$x_{t+1} = h(x_t, \sigma) + \eta\sigma\epsilon_{t+1}$$

where $\eta = [0; \tilde{\eta}]'$

# Solving LREM

The perturbation method will find a local approximation of $g$ and $h$ around $(\bar{x}, \bar{\sigma})$

Note, the solution will be valid only around the approximation point!

# Solving LREM

Set $n_x = n_y = 1$

By Taylor's theorem

$$g(x,\sigma) = g(\bar{x},\bar{\sigma}) + g_x(\bar{x},\bar{\sigma})(x-\bar{x}) + g_\sigma(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma}) +$$

$$+\frac{1}{2}g_{xx}(\bar{x},\bar{\sigma})(x-\bar{x})^2 + \frac{1}{2}g_{x\sigma}(\bar{x},\bar{\sigma})(x-\bar{x})(\sigma-\bar{\sigma}) +$$

$$+\frac{1}{2}g_{\sigma\sigma}(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma})^2 + \ldots$$

and

# Solving LREM

$$h(x,\sigma) = h(\bar{x},\bar{\sigma}) + h_x(\bar{x},\bar{\sigma})(x-\bar{x}) + h_\sigma(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma}) +$$

$$+\frac{1}{2}h_{xx}(\bar{x},\bar{\sigma})(x-\bar{x})^2 + \frac{1}{2}h_{x\sigma}(\bar{x},\bar{\sigma})(x-\bar{x})(\sigma-\bar{\sigma}) +$$

$$+\frac{1}{2}h_{\sigma\sigma}(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma})^2 + \ldots$$

# Solving LREM

The unknowns of an nth order expansion are the n order derivatives of $g$ and $h$, always evaluated at the point $(\bar{x}, \bar{\sigma})$

# Solving LREM

To find these unknowns, note that

$$F(x, \sigma) = E_t[f(g(h(x, \sigma) + \eta\sigma\epsilon', \sigma), g(x, \sigma), h(x, \sigma) + \eta\sigma\epsilon', x)],$$

$$F(x, \sigma) = 0,$$

As $F(x, \sigma) = 0$ for any $(x, \sigma)$, all its derivatives are equal to zero.

$$F_{x^k, \sigma^j}(x, \sigma) = 0,$$

# Solving LREM

Approximate around the non-stochastic steady state, $x = \bar{x}$ and $\sigma = 0$

This implies $y = \bar{y} = g(\bar{x}, 0)$ and $x = \bar{x} = h(\bar{x}, 0)$

Here note that: $f(\bar{y}, \bar{y}, \bar{x}, \bar{x}) = 0$ and $E_t f = f$

Good news is that we can solve for the non-stochastic steady state in most models (algebraically or numerically)

# Solving LREM

Around the non-stochastic steady state, the first order approximations are

$$g(x,\sigma) = g(\bar{x},0) + g_x(\bar{x},0)(x-\bar{x}) + g_\sigma(\bar{x},0)\sigma$$

$$h(x,\sigma) = h(\bar{x},0) + h_x(\bar{x},0)(x-\bar{x}) + h_\sigma(\bar{x},0)\sigma$$

We know that $\bar{y} = g(\bar{x},0)$ and $\bar{x} = h(\bar{x},0)$. Now we need to find the remaining coefficients

# Solving LREM

Let's find first $h_\sigma$ and $g_\sigma$ Note

$$F_\sigma(\bar{x},0) = E_t f_{y'}[g_x(h_\sigma + \eta\epsilon') + g_\sigma] + f_y g_\sigma + f_{x'}(h_\sigma + \eta\epsilon')$$

$$F_\sigma(\bar{x},0) = f_{y'}[g_x h_\sigma + g_\sigma] + f_y g_\sigma + f_{x'} h_\sigma$$

But $F_\sigma(\bar{x},0) = 0$ This is a set of $n$ equations in $n$ unknowns

# Solving LREM

We can write the system as Note

$$\begin{bmatrix} f_{y'} g_x + f_{x'} & f_{y'} + f_y \end{bmatrix} \begin{bmatrix} h_\sigma \\ g_\sigma \end{bmatrix} = 0$$

Is linear, and homogeneous in $h_\sigma$ and $g_\sigma$ If a unique solution

exists, $h_\sigma = 0$ and $g_\sigma = 0$

# Solving LREM

What does this result say?

Up to first order, there is no need to correct the constant term of the approximation for the size of the variance shock

In other words, the ergodic mean of the linear approximation is the same as the steady state value

In other words, the linear approximation is certainty equivalent. Changes in the variance will not affect the policy functions up to the linear term!

# Solving LREM

Let's find first $h_x$ and $g_x$

Note

$$F_x(\bar{x},0) = f_{y'}g_xh_x + f_yg_x + f_{x'}h_x + f_x$$

But $F_x(\bar{x},0) = 0$

This system is a set of $n \times n_x$ quadratic equations in $n \times n_x$ unknowns

We will find two solution, keep the stable one

# Solving LREM

Note

$$[f_{x'} \quad f_{y'}] \begin{bmatrix} I \\ g_x \end{bmatrix} h_x = -[f_x \quad f_y] \begin{bmatrix} I \\ g_x \end{bmatrix}$$

Denote $A = [f_{x'} \quad f_{y'}]$ and $B = -[f_x \quad f_y]$ Define $\hat{x} \equiv x_t - \bar{x}$

# Solving LREM

Note

$$A \begin{bmatrix} I \\ g_x \end{bmatrix} h_x \hat{x}_t = B \begin{bmatrix} I \\ g_x \end{bmatrix} \hat{x}_t$$

and

$$A \begin{bmatrix} I \\ g_x \end{bmatrix} \hat{x}_{t+1} = B \begin{bmatrix} I \\ g_x \end{bmatrix} \hat{x}_t$$

we want a stable solution, that is

$$\lim_{t \to \infty} |\hat{x}_t| < \infty$$

# Solving LREM

We use Schur decomposition to find $g_x$ from the previous equation

# Solving LREM

Generalized Schur Decomposition

For matrix $A$ and $B$, $n \times n$ matrices the decomposition gives

1. $QAZ = a$ is upper triangular
2. $QBZ = b$ is upper triangular
3. For each $i$, $a_{ii}$ and $b_{ii}$ are not both zero
4. $\lambda(A, B) = \left\{ \frac{b_{ii}}{a_{ii}} : a_{ii} \neq 0 \right\}$
5. The pairs $(a_{ii}, b_{ii})$ can be rearranged in any order for all $i$

Here the $\lambda$ are the generalized eigenvalues such that $|\lambda_i| > 1$ will be called unstable $a$ and $b$ are upper triangular and $Q$ and $Z$ are orthonormal

# Solving LREM

We use Schur decomposition to find $g_x$ from the previous equation

Schur decomposition is given by upper triangular matrices a and b and orthonormal matrices $Q$ and $Z$ such that

$$QAZ = a$$

$$QBZ = b$$

Define $s_t \equiv Z' \begin{bmatrix} I \\ g_x \end{bmatrix} \hat{x}_t$

# Solving LREM

Hence, we can write our system as

$$as_{t+1} = bs_t$$

We can partition the matrices as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

$$b = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}$$

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

$$s = \begin{bmatrix} s^1 \\ s^2 \end{bmatrix}$$

# Solving LREM

Here $a_{22}$ and $b_{22}$ have dimensions $n_y \times n_y$, $z_{12}$ is $n_x \times n_y$ and $s_t^2$ is $n_y \times 1$

Hence

$$b_{22}^{-1} a_{22} s_{t+1}^2 = s_t^2$$

Notice this is linear difference equation in $s^2$

Assume $abs(a_{ii}/b_{ii})$ are decreasing in $i$ and that the number of ratios less than 1 is $n_y$ and the number of ratios larger than 1 is $n_x$

Then, the eigenvalues of $b_{22}^{-1} a_{22}$ are less than unity.

The sequence is non explosive only if $s_t^2 = 0$

# Solving LREM

By definition of $s_t^2$

Hence

$$(z_{12}^{'} + z_{22}^{'} g_x)\hat{x}_t = 0$$

From where we can get $g_x$!

$$g_x = -(z_{22}^{'})^{-1} z_{12}^{'}$$

# Solving LREM

$s_t^2 = 0$ implies that

$$a_{11} s_{t+1}^1 = b_{11} s_t^1$$

But $s_t^1 = (z'_{11} + z'_{21} gx)\hat{x}_t = (z'_{11} - z'_{21}(z'_{22})^{-1} z'_{12})\hat{x}_t$

Hence

$$h_x = [z'_{11} - z'_{21}(z'_{22})^{-1} z'_{12}]^{-1} a_{11}^{-1} b_{11} [z'_{11} - z'_{21}(z'_{22})^{-1} z'_{12}]$$

# Solving LREM

And we found $g_x$ and $h_x$

We knew $g_\sigma = h_\sigma = 0$

We knew $g(\bar{x}, 0) = \bar{y}$ and $h(\bar{x}, 0) = \bar{x}$

Hence, we solved the model up to first order!

# Solving LREM

## Local existence and uniqueness of equilibrium

We made a big assumption. We assumed the number of eigenvalues computed from the Schur decomposition larger than 1 is equal to $n_y$ and the number of eigenvalues smaller than 1 is equal to $n_x$

This implies the existence of unique local equilibrium

# Solving LREM

## Local existence and uniqueness of equilibrium

Suppose the number of eigenvalues larger than 1 is $m < n_y$

Hence, to study the solutions for which $\lim_{t\to\infty} E_t|\hat{x}_{t+j}| < \infty$ we are now imposing $m$ constraints, instead of $n_y$

We don´t have enough constraints, we could construct sequences of $\hat{x}_t$ that are convergent for $n_y - m$ arbitrarily chosen number of $y_0$

Equilibrium is indeterminate. We have more than one saddle path!

# Solving LREM

## Local existence and uniqueness of equilibrium

Suppose the number of eigenvalues larger than 1 is $m > n_y$

Hence, to study the solutions for which $\lim_{t\to\infty} E_t|\hat{x}_{t+j}| < \infty$ we are now imposing $m$ constraints, instead of $n_y$

We impose too many constraints. We should fix $y_0$ values to be over a saddle path. But this is not enough, we also need to determine $m - n_y$, $x_0$ values. But this is impossible, given that $x$ are predetermined

Equilibrium does not exist

# Solving LREM

## Unconditional moments

We solved a model, and so what?

We need to analyze a model

Unconditional moments: standard errors, correlations autocorrelations

# Solving LREM

## Covariance matrix

Recall

$$\hat{x}_{t+1} = h_x \hat{x}_t + \sigma \eta \epsilon_{t+1}$$

Covariance matrix

$$\Sigma_x \equiv E \hat{x}_t \hat{x}_t'$$

$$\Sigma_\epsilon \equiv E \sigma^2 \eta \eta'$$

Then

$$\Sigma_x = h_x \Sigma_x h_x' + \Sigma_\epsilon$$

Which is a Lyapunov equation and can be solved with doubling algorithm

# Solving LREM

## Other second order moments

Other moments

$$E_t\hat{x}_t\hat{x}'_{t-j} = E\left[h_x^j\hat{x}_{t-j} + \sum_{k=0}^{j-1} h_x^k\mu_{t-k}\right]\hat{x}'_{t-j}$$

Where $\mu = \sigma\eta\epsilon_t$, hence

$$E_t\hat{x}_t\hat{x}'_{t-j} = h_x^j E\left[\hat{x}_{t-j}\hat{x}'_{t-j}\right]$$

$$E_t\hat{x}_t\hat{x}'_{t-j} = h_x^j\Sigma_x$$