Macroeconomics 3 (MAE/PhD)

Perturbation methods

Hernán D. Seoane

UC3M

October 31, 2018

So far

So far we studied (log-)linear approximation methods

We solved for the dynamics and we know how the solution looks like

Up to first order in simple models we can work algebraically

No good for welfare analysis nor cases where non-linearities matter

Today's lecture

Perturbation method

Non-linear perturbation

We will see how higher order solutions look like

We need 2 theorems for perturbation theory

Taylor's theorem (again!)

Implicit function theorem for R^n

Basic idea of perturbation

Consider this function $f(x, \epsilon) = 0$

we want to solve this function for x

ϵ is a parameter

Assume for each value of ϵ the equation has solution(s) for x

This means there is a collection of equations in x parametrized by ϵ

Let $x = x(\epsilon)$ smooth function. Then $f(x(\epsilon), \epsilon) = 0$

The key to apply perturbation method (as in linearization) is that we can solve the equation for x, for a particular value of ϵ

Basic idea of approximation

Suppose we know $x(0)$ Implicit differentiation defines $x'(\epsilon)$

$$f_x(x(\epsilon),\epsilon)x'(\epsilon) + f_\epsilon(x(\epsilon),\epsilon) = 0$$

This is useless, it depends on $x(\epsilon)$, which is unknown. But we know it for $\epsilon = 0$

Then,

$$x'(0) = -\frac{f_\epsilon(x(0),0)}{f_x(x(0),0)}$$

The linear approximation around $\epsilon = 0$ is

$$x(\epsilon) \equiv x(0) - \frac{f_\epsilon(x(0),0)}{f_x(x(0),0)}\epsilon$$

Basic idea of approximation

We can go on for a 2nd order

$$(f_{xx}x' + f_{x\epsilon})x' + f_x x'' + f_{\epsilon x}x' + f_{\epsilon\epsilon} = 0$$

$$f_{xx}(x')^2 + f_{x\epsilon}x' + f_x x'' + f_{\epsilon x}x' + f_{\epsilon\epsilon} = 0$$

$$f_{xx}(x')^2 + 2f_{x\epsilon}x' + f_x x'' + f_{\epsilon\epsilon} = 0$$

At $\epsilon = 0$

$$x''(0) = -\frac{f_{xx}(x(0),0)(x'(0))^2 + 2f_{x\epsilon}(x(0),0)x'(0) + f_{\epsilon\epsilon}(x(0),0)}{f_x(x(0),0)}$$

$$x(\epsilon) \equiv x(0) + x'(0)\epsilon + \frac{1}{2}x''(0)\epsilon^2$$

A simple example

This example is from Jesus Fernandez-Villaverde notes

Solve the following equation (find the roots)

$$x^3 - 4.1x + 0.2 = 0$$

such that $x < 0$

Perturbation approach will be useful implemented in 3 steps

- Rewrite the problem in terms of a perturbation parameter
- Solve the problem for a given value of the parameter
- Approximate the solution of the problem around that point

A simple example

The first step is to introduce a perturbation parameter in order to rewrite the problem

There are many ways of doing it (even in the true models we are going to use)

For instance here

$$x^3 - (4+\epsilon)x + 2\epsilon = 0$$

Here we are using $\epsilon \equiv 0.1$

A simple example

Now, solve for the new problem defining $x = g(\epsilon)$

Note that for $\epsilon = 0$, the equation is

$$x^3 - 4x = 0$$

That has roots for $x = -2, 0, 2$. Take $x = -2$. Then $g(0) = -2$

A simple example

Build the approximate solution around $\epsilon = 0$

Using Taylor's theorem

$$x = g(0) + \sum_{n=1}^{\infty} \frac{g^n(0)}{n!} \epsilon^n$$

A simple example

- The approximation of order 0
- Recall $g(0) = -2$

$$x^3 - 4.1x + 0.2 = 0$$

$$-8 + 8.2 + 0.2 = 0.4$$

A simple example

- The approximation of order 1

$$g(\epsilon)^3 - (4+\epsilon)g(\epsilon) + 2\epsilon = 0$$

- Take derivatives with respect to ϵ

$$3g(\epsilon)^2 g'(\epsilon) - g(\epsilon) - (4+\epsilon)g'(\epsilon) + 2 = 0$$

- Recall we approximate around $\epsilon = 0$
- Recall we just found that $g(0) = -2$

$$8g'(0) + 4 = 0$$

- This implies that $g'(0) = -\frac{1}{2}$

A simple example

- The approximation of order 1 around the approximation point, by Taylor's Theorem is then

$$x = g(0) + \frac{g'(0)}{1!}\epsilon^1$$

- Recall that in our case $\epsilon = 0.1$. This implies that

$$x = -2 - 1/2 * 0.1 = -2.05$$

A simple example

- Let's plug this value in the equation to see how good the approximation is

$$x^3 - 4.1x + 0.2 = 0$$

$$-8.615125 - 8.405 + 0.2 = -0.010125$$

- Much better than the approximation of order 0

A simple example

- The approximation of order 2
- Take a second derivative of the equation

$$3g(\epsilon)^2 g'(\epsilon) - g(\epsilon) - (4+\epsilon)g'(\epsilon) + 2 = 0$$

- with respect to ϵ

$$6g(\epsilon)(g'(\epsilon))^2 + 3g(\epsilon)^2 g''(\epsilon) - g'(\epsilon) - g'(\epsilon) - (4+\epsilon)g''(\epsilon) = 0$$

A simple example

- Recall we focus in $\epsilon = 0$

$$6g(0)(g'(0))^2 + 3g(0)^2 g''(0) - g'(0) - g'(0) - 4g''(0) = 0$$

- Recall we just found that $g(0) = -2$ and $g'(\epsilon) = -1/2$

$$8g''(0) - 2 = 0$$

- This implies that $g''(0) = \frac{1}{4}$

A simple example

- The approximation of order 2 around the approximation point, by Taylor's Theorem is then

$$x = g(0) + \frac{g'(0)}{1!}\epsilon^1 + \frac{g''(0)}{2!}\epsilon^2$$

$$x = -2 - 1/2\epsilon + 1/8\epsilon^2$$

- Recall that in our case $\epsilon = 0.1$. This implies that

$$x = -2 - 1/2 * 0.1 + 1/4 * 0.01 = -2.04875$$

A simple example

- Plugging this value here

$$x^3 - 4.1x + 0.2 = 0$$

- it is $4.997e(-004)$
- Much better than the approximation of order 0

A simple example: Important

- The first step was to transform the model adding a perturbation parameter
- Reason: we want to solve for the exact 0th order approximation

Nonlinear solution methods

- Remember the very basic problem

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha} + (1-\delta) k_t$$

$$z_t = \rho z_{t-1} + \sigma \epsilon_t$$

with $\epsilon_t \sim N(0,1)$

Nonlinear solution methods

- Equilibrium Conditions

$$\frac{1}{c_t} = \beta \mathbb{E}_t \frac{1}{c_{t+1}} \left(1 + \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} - \delta\right)$$

$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha} + (1-\delta) k_t$$

$$z_t = \rho z_{t-1} + \sigma \epsilon_t$$

Implicit function theorem

If $H(x, y) : R^n \times R^m \to R^m$ is C^k, $H(x_0, y_0) = 0$, and $H_y(x_0, y_0)$ is not singular, then there is a unique function $h : R^n \to R^m$ such that $y_0 = h(x_0)$ and for x near x_0, $H(x, h(x)) = 0$. Furthermore, if H is C^k, then h is C^k, and its derivatives can be computed by implicit differentiation of the identity $H(x, h(x)) = 0$

The techniques used in perturbation will use these theorems

Together they allow us to implicitly compute derivatives of h with respect to x at x_0

we will be able to consider implicitly defined functions

This is very powerful

Approximation + IFT

Keep in mind we want to solve a functional system (our unknowns are functions)

$$H(d) = \mathbb{E}_t[f(\cdot)] = 0$$

Where the unknowns are the decision rules d

Taylor's theorem allow us to use perturbation that solves the problem by specifying

$$d^n(x, \theta) = \sum_{i=0}^{n} \theta_i (x - x_0)^i$$

Then we use the implicit function theorem to find θ_i's

Nonlinear solution methods

- Equilibrium Conditions

$$\frac{1}{c(k_t, z_t)} = \beta \mathbb{E}_t \frac{\left(1 + \alpha e^{\rho z_t + \sigma \epsilon_{t+1}} (k(k_t, z_t))^{\alpha - 1} - \delta\right)}{c(k(k_t, z_t), \rho z_t + \sigma \epsilon_{t+1})}$$

$$c(k_t, z_t) + k(k_t, z_t) = e^{z_t} k_t^{\alpha} + (1 - \delta) k_t$$

Nonlinear solution methods

- We transform the problem introducing a small perturbation parameter
- The perturbation parameter is the standard deviation of the innovation σ
- If we set this parameter to 0, we are in the known world of the non-stochastic steady state and we know how to solve it
- Then we will look for: $c_t = c(k_t, z_t; \sigma)$ and $k_{t+1} = k(k_t, z_t; \sigma)$

Nonlinear solution methods

- Equilibrium Conditions

$$\frac{1}{c(k_t, z_t; \sigma)} = \beta \mathbb{E}_t \frac{\left(1 + \alpha e^{\rho z_t + \sigma \epsilon_{t+1}} (k(k_t, z_t; \sigma))^{\alpha - 1} - \delta\right)}{c(k(k_t, z_t; \sigma), \rho z_t + \sigma \epsilon_{t+1}; \sigma)}$$

$$c(k_t, z_t; \sigma) + k(k_t, z_t; \sigma) = e^{z_t} k_t^{\alpha} + (1 - \delta) k_t$$

Nonlinear solution methods

- Equilibrium Conditions

$$\mathbb{E}_t\left[\frac{1}{c(k_t,z_t;\sigma)}-\beta\frac{\left(1+\alpha e^{\rho z_t+\sigma\epsilon_{t+1}}(k(k_t,z_t;\sigma))^{\alpha-1}-\delta\right)}{c(k(k_t,z_t;\sigma),\rho z_t+\sigma\epsilon_{t+1};\sigma)}\right]=0$$

$$c(k_t,z_t;\sigma)+k(k_t,z_t;\sigma)-e^{z_t}k_t^{\alpha}-(1-\delta)k_t=0$$

Nonlinear solution methods

- We will take derivatives with respect to k_t, z_t and σ
- Apply Taylor's theorem and build a solution around the deterministic steady state

Nonlinear solution methods

- Asymptotic expansion of the consumption function

$$
\begin{aligned}
c_t = {} & c(k,0;0) + c_k(k,0;0)(k_t - k) + c_z(k,0;0)z_t + c_\sigma(k,0;0)\sigma \\
& + \frac{1}{2}c_{kk}(k,0;0)(k_t - k) + \frac{1}{2}c_{kz}(k,0;0)(k_t - k)z_t + \frac{1}{2}c_{k\sigma}(k,0;0)(k_t - k)\sigma \\
& + \frac{1}{2}c_{zz}(k,0;0)z_t^2 + \frac{1}{2}c_{zk}(k,0;0)(k_t - k)z_t + \frac{1}{2}c_{z\sigma}(k,0;0)z_t\sigma \\
& + \frac{1}{2}c_{\sigma\sigma}(k,0;0)\sigma^2 + \frac{1}{2}c_{\sigma k}(k,0;0)(k_t - k)\sigma + \frac{1}{2}c_{\sigma z}(k,0;0)z_t\sigma \\
& \ldots \qquad (1)
\end{aligned}
$$

Nonlinear solution methods

- Asymptotic expansion of the capital function

$$
\begin{aligned}
k_{t+1} &= k(k,0;0) + k_k(k,0;0)(k_t - k) + k_z(k,0;0)z_t + k_\sigma(k,0;0)\sigma \\
&+ \frac{1}{2}k_{kk}(k,0;0)(k_t - k) + \frac{1}{2}k_{kz}(k,0;0)(k_t - k)z_t + \frac{1}{2}k_{k\sigma}(k,0;0)(k_t - k)\sigma \\
&+ \frac{1}{2}k_{zz}(k,0;0)z_t^2 + \frac{1}{2}k_{zk}(k,0;0)(k_t - k)z_t + \frac{1}{2}k_{z\sigma}(k,0;0)z_t\sigma \\
&+ \frac{1}{2}k_{\sigma\sigma}(k,0;0)\sigma^2 + \frac{1}{2}k_{\sigma k}(k,0;0)(k_t - k)\sigma + \frac{1}{2}k_{\sigma z}(k,0;0)z_t\sigma \\
&\ldots \qquad (2)
\end{aligned}
$$

Nonlinear solution methods

Equilibrium Conditions

$$F(k_t, z_t; \sigma) = \mathbb{E}_t \left[\begin{array}{c} \frac{1}{c(k_t, z_t; \sigma)} - \beta \frac{\left(1 + \alpha e^{\rho z_t + \sigma \epsilon_{t+1}} (k(k_t, z_t; \sigma))^{\alpha - 1} - \delta\right)}{c(k(k_t, z_t; \sigma), \rho z_t + \sigma \epsilon_{t+1}; \sigma)} \\ c(k_t, z_t; \sigma) + k(k_t, z_t; \sigma) - e^{z_t} k_t^{\alpha} - (1 - \delta) k_t \end{array} \right] = 0$$

Note (to simplify notation later)

$$F(k_t, z_t; \sigma) = \mathbb{E}_t H(c_t, c_{t+1}, k_t, k_{t+1}, z_t, z_{t+1}; \sigma)$$

Zero order approximation

First, evaluate $\sigma = 0$

$$F(k, 0; 0) = 0$$

Then we are studying the steady state

$$c = c(k, 0; 0) = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}$$

$$k = k(k, 0; 0) = (\alpha\beta)^{\frac{1}{1-\alpha}}$$

First order approximation

Take derivative of $F(k_t, z_t; \sigma)$ evaluated at k, 0, 0

$$F_k(k, 0; 0) = 0$$

$$F_z(k, 0; 0) = 0$$

$$F_\sigma(k, 0; 0) = 0$$

First order approximation

Using our previous notation

$$F(k_t, z_t; \sigma) = \mathbb{E}_t \left[\begin{array}{c} H(c(k_t, z_t; \sigma), c(k(k_t, z_t; \sigma), \rho z_t + \sigma \epsilon_{t+1}; \\ \sigma), k_t, k(k_t, z_t; \sigma), z_t, \rho z_t + \sigma \epsilon_{t+1}) \end{array} \right] = 0$$

$$F_k(k, 0; 0) = H_1 c_k + H_2 k_k c_k + H_3 + H_4 k_k = 0$$

$$F_z(k, 0; 0) = H_1 c_z + H_2 (k_k c_k + c_k \rho) + H_4 k_z + H_5 + H_6 \rho = 0$$

$$F_\sigma(k, 0; 0) = H_1 c_\sigma + \mathbb{E}_t H_2 (k_\sigma c_k + c_z \epsilon_{t+1} + c_\sigma) + H_4 k_\sigma + \mathbb{E}_t H_6 \epsilon_{t+1} = 0$$

First order approximation

The first 2 expressions define a quadratic system (in 4 unknowns: c_k, c_z, k_k, k_z)

$$F_k(k, 0; 0) = H_1 c_k + H_2 k_k c_k + H_3 + H_4 k_k = 0$$

$$F_z(k, 0; 0) = H_1 c_z + H_2 (k_k c_k + c_k \rho) + H_4 k_z + H_5 + H_6 \rho = 0$$

we will learn how to deal with them computationally (although we just saw how to do it algebraically)

First order approximation

- The last equation is homogeneous linear system in c_σ and k_σ.

$$F_\sigma(k,0;0) = H_1 c_\sigma + \mathbb{E}_t H_2 (k_\sigma c_k + c_z \epsilon_{t+1} + c_\sigma) + H_4 k_\sigma + \mathbb{E}_t H_6 \epsilon_{t+1} = 0$$

- Hence, we have the certainty equivalence result we discuss before
- Now we want to deal with higher terms

Second order approximation

- Take second order derivatives of $F(k_t, z_t; \sigma)$ evaluated at k, 0 and 0

$$F_{kk}(k, 0; 0) = 0$$

$$F_{kz}(k, 0; 0) = 0$$

$$F_{k\sigma}(k, 0; 0) = 0$$

$$F_{zz}(k, 0; 0) = 0$$

$$F_{z\sigma}(k, 0; 0) = 0$$

$$F_{\sigma\sigma}(k, 0; 0) = 0$$

Second order approximation

- We substitute the coefficients we already know
- Notice the system is one of 12 equations in 12 unknowns
- Notice that the crossed terms with $k\sigma$ and $z\sigma$ are zero
- Now we want to deal with higher terms

Second order approximation

- Important: we have a term in σ^2
- This is a correction for risk
- We don't have certainty equivalence anymore!

Higher orders

- We can take higher order derivatives for the whole system
- The procedure is recursive
- Accuracy gains: in simple models Arouba et al shows that 5th order terms have the magnitude of computer precision (that is very small), higher terms will not improve significantly the solution
- Burden: take analytical derivatives of large models
- We can do it in the computer! Symbolic toolbox in matlab, dynare, mathematica

References

- Fernandez-Villaverde, J., et al. “Risk matters: The real effects of volatility shocks."” The American Economic Review 101.6 (2011)

- Aruoba, S. Boragan, Jesus Fernandez-Villaverde, and Juan F. Rubio-Ramirez. “Comparing solution methods for dynamic equilibrium economies.” Journal of Economic Dynamics and Control 30.12 (2006): 2477-2508

- Companion webpage: matlab and mathematica codes for non-linear perturbation