Macroeconomics 3 (MAE/PhD)

Approximating nonlinear models: an introduction

Hernán D. Seoane

UC3M

October 29, 2018

Today's lecture

Approximating DSGE models

Taylor series

Introduction to approximation techniques

Log-linear approximation

Algebraic implementation and undetermined coefficients

Application to the Neoclassical Growth Model

Taylor series

Taylor's theorem

Let's consider a function f with derivatives of all orders in (a, b) and $M > 0$ such that $|f^k(x)| < M$ for all k and $x \in (a, b)$. If $x_0 \in (a, b)$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(x_0)}{k!}(x - x_0)^k$$

or

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ...$$

Notice how powerful this theorem is: any function (with derivatives) can be re-written (exactly!) in this way

Linearization...

Various ways and shortcuts for approximating a function

Consider a dynamic equation: $x_{t+1} = f(x_t)$

The linear approximation around x, by directly applying the theorem is

$$x_{t+1} \approx f(x) + f'(x)(x_t - x)$$

Note $x_{t+1} = b + ax_t$

$|a| < 1$ implies local stability

... and Log-linearization

Do the same procedure to log-linearize

Define the log deviation from a steady state by $\hat{x}_t = ln\left(\frac{x_t}{x}\right)$

Log-deviation is approximately a percentage deviation

Again, consider $x_{t+1} = f(x_t)$

Note $xe^{\hat{x}_{t+1}} = f\left(xe^{\hat{x}_t}\right)$

Now approximate with respect to $\hat{x}_{t+1}$

$$\hat{x}_{t+1} \approx f'(x)\hat{x}_t$$

$|f'(x)| < 1$ implies local stability

Log-linearization

Alternative way

We can do it in a slightly different way

Consider $f(x_t) = \frac{g(x_t)}{h(x_t)}$

Take logs and then approximate with respect to x_t around a point x

Log-linearization

Uhlig tricks

Suppose we have our function $f(x_t)$ back

Notice that $x_t = xe^{\hat{x}_t}$

Here: x is an approximation point

$\hat{x}_t = ln(x_t/x) = ln(x_t) - ln(x)$

The following rules hold:

1. $e^{\hat{x}_t + a\hat{y}_t} \approx 1 + \hat{x}_t + a\hat{y}_t$
2. $\hat{x}_t \times \hat{y}_t \approx 0$
3. If $\hat{x}_t$ is a random variable, $E_t\left[ae^{\hat{x}_t}\right] \approx aE_t[\hat{x}_t]$, up to a constant

The Uhlig toolkit (1998)

Overview

So far so good, but in macro we work with dynamic models: their solution is characterized by a system of dynamic nonlinear equations

We usually need to solve dynamic systems (instead of single equations)... and in many cases we need to do it super-fast

Our first step is to study the toolkit in Uhlig (1998)

The toolkit builds on log linearization and the method of undetermined coefficients

The Uhlig toolkit (1998)

The NGM

Modern macro models need to specify:

1. preferences
2. technology
3. endowments
4. information.

Object to study:

1. Planner's problem (specify Planner's objective function)
2. Competitive Equilibrium (specify markets)

We will do this for the NGM

The Uhlig toolkit (1998)

The NGM: Preferences

Representative agent economy

$$U = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta} - 1}{1-\eta}\right]$$

$0 < \beta < 1$ is the discount factor

$\eta > 0$ is the coefficient of relative risk aversion

The Uhlig tookit (1998)

The NGM: Technology

Cobb-Douglass production function

$$Y_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho}$$

$0 < \rho < 1$ is the K share; $0 < \delta < 1$ is the K depreciation rate

Z_t is the technology shock that evolves as

$$\log Z_t = (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t$$

where $\epsilon_t \sim N(0, \sigma^2)$

The Uhlig tookit (1998)

The NGM: Endowments and information

Endowments: the representative agent has 1 unit of labor every period, $N_t = 1$ and endowed with K_{-1} given before period 0.

Information: choice variables are set with based on all available information I_t up to time t.

The Uhlig tookit (1998)

The Planner's problem

$$\max \quad U = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta} - 1}{1-\eta}\right]$$

s.t.

$$C_t + K_t = Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1}$$

$$\log Z_t = (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t,$$

$$\epsilon_t \sim N(0, \sigma^2)$$

$$K_{-1}; \qquad Z_0$$

The Uhlig tookit (1998)

The Planner's problem

To solve it, Lagrangian: $\beta^t \lambda_t$ is the Lagrange multiplier

$$L = \max \quad \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\eta} - 1}{1-\eta} - \lambda_t \left(C_t + K_t - Z_t K_{t-1}^{\rho} - (1-\delta) K_{t-1} \right) \right) \right]$$

The set of FOC that characterize the solution is given by

$$\frac{\partial L}{\partial \lambda_t} : 0 = C_t + K_t - Z_t K_{t-1}^{\rho} - (1-\delta) K_{t-1}$$

$$\frac{\partial L}{\partial C_t} : 0 = C_t^{-\eta} - \lambda_t$$

$$\frac{\partial L}{\partial K_t} : 0 = -\lambda_t + \beta \mathbb{E}_t \left[\lambda_{t+1} \left(\rho Z_{t+1} K_t^{\rho-1} + (1-\delta) \right) \right]$$

The Uhlig tookit (1998)

The Planner's problem

Rewrite the FOC and solve for the Steady State

$$C_t = Z_t K_{t-1}^{\rho} + (1-\delta) K_{t-1} - K_t$$

$$R_t = \rho Z_t K_{t-1}^{\rho-1} + (1-\delta)$$

$$1 = \mathbb{E}_t \left[\beta \left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right]$$

$$\log Z_t = (1-\psi) \log Z + \psi \log Z_{t-1} + \epsilon_t$$

Look for a solution to the system where $C_t = \bar{C}$, $R_t = \bar{R}$, $K_t = \bar{K}$, $Z_t = \bar{Z}, \forall t$

The Uhlig tookit (1998)

The Planner's problem

Rewrite the FOC and solve for the Steady State

$$\bar{R} = \frac{1}{\beta}$$

$$\bar{K} = \left(\frac{\rho \bar{Z}}{\bar{R} - 1 + \delta} \right)^{1/(1-\rho)}$$

$$\bar{C} = \bar{Z}\bar{K}^{\rho} - \delta \bar{K}$$

The Uhlig tookit (1998)

The Competitive Equilibrium

For the competitive equilibrium we need to consider a representative firm and a representative household

The representative firm solves

$$\max Z_t \left(K_{t-1}^{(d)}\right)^{\rho} \left(N_t^{(d)}\right)^{1-\rho} + (1-\rho)K_{t-1}^{(d)} - W_t N_t^{(d)} - R_t K_{t-1}^{(d)}$$

FOC for the firm ("Demand Curves")

$$W_t = (1-\rho) Z_t \left(K_{t-1}^{(d)}\right)^{\rho} \left(N_t^{(d)}\right)^{-\rho}$$

$$R_t = \rho Z_t \left(K_{t-1}^{(d)}\right)^{\rho-1} \left(N_t^{(d)}\right)^{1-\rho} + (1-\delta)$$

The Uhlig tookit (1998)

The Competitive Equilibrium

The representative household solves

$$\max \quad U = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta} - 1}{1-\eta}\right]$$

s.t.

$$C_t + K_t^{(s)} = W_t N^{(s)} + R_t K_{t-1}^{(s)}$$

$$N_t^{(s)} = 1$$

and the no-Ponzi game condition: $0 = \lim_{t\to\infty} \mathbb{E}_0 \prod_{s=1}^{t} R_t^{-1} K_t$

The representative household FOC

$$\frac{\partial L}{\partial \lambda_t} : 0 = C_t + K_t^{(s)} - W_t - R_t K_{t-1}^{(s)}$$

$$\frac{\partial L}{\partial C_t} : 0 = C_t^{-\eta} - \lambda_t$$

$$\frac{\partial L}{\partial K_t} : 0 = -\lambda_t + \beta \mathbb{E}_t \left[\lambda_{t+1} R_{t+1} \right]$$

The Uhlig tookit (1998)

The Competitive Equilibrium. Definition

Given $K_{-1}^{(s)}$ and market wages and returns, W_t and R_t, the representative agents solves its problem

Given market wages and returns, W_t and R_t $\forall t$, the representative firm solves its problem taking TFP given

Markets clear:

$$N_t^{(d)} = N_t^{(s)} = N_t$$

$$K_{t-1}^{(d)} = K_{t-1}^{(s)} = K_{t-1}$$

$$C_t + K_t = Z_t K_{t-1}^{\rho} + (1-\delta) K_{t-1}$$

The Uhlig tookit (1998)

Solve for the dynamics around the Steady State

Define the problem

Define the object we are looking for (the concept of equilibrium)

Characterize the solution system

Then, make a choice: are we looking for a global or local solution?

If local, we need to define the Non-stochastic Steady State and approximate the solution around that point.

We now apply the Uhlig's toolbox tricks to log-linearize our set of conditions that characterize the solution to our problem

The Uhlig tookit (1998)

Log-linearized conditions

The conditions that characterize the log-linear competitive equilibrium

$$c_t = \frac{Y}{C} z_t + \frac{K}{C} R k_{t-1} - \frac{K}{C} k_t$$

$$r_t = (1 - \beta(1 - \delta))(z_t - (1 - \rho) k_{t-1})$$

$$0 = \mathbb{E}_t \left[\eta (c_t - c_{t+1}) + r_{t+1} \right]$$

$$z_t = \psi z_{t-1} + \epsilon_t$$

This is, yet, not a solution... for this simple problem, we can proceed by pencil and paper

The Uhlig tookit (1998)

The Undetermined coefficients method

Postulate recursive linear laws of motion:

$$k_t = \nu_{kk} k_{t-1} + \nu_{kz} z_t$$

$$r_t = \nu_{rk} k_{t-1} + \nu_{rz} z_t$$

$$c_t = \nu_{ck} k_{t-1} + \nu_{cz} z_t$$

We need to find the ν_{ij} coefficients (the undetermined coefficients).

Strategy: plug these rules in the log-linearized system of equations and solve for the unknowns

The Uhlig tookit (1998)

Numerical implementation

Calibrate the model

Parameters	Definition	Value
η	Risk aversion coefficient	1
β	Discount factor	$1/1.01 \approx 0.99$
δ	Depreciation rate	0.025
ρ	Capital share	0.36
$\bar{Z}$	Average TFP	1

The Uhlig tookit (1998)

With this calibration

Now it is straightforward to analyze the quantitative implications of this model

$$k_t = 0.965k_{t-1} + 0.075z_t$$

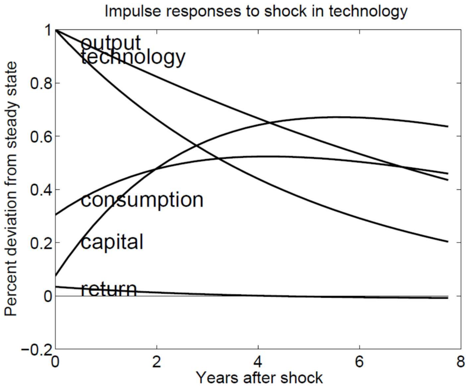
$$r_t = -0.022k_{t-1} + 0.035z_t$$

$$c_t = 0.618k_{t-1} + 0.305z_t$$

recall that $z_t = \psi z_{t-1} + \epsilon_t$. Suppose $\epsilon_0 = 1$ and $\epsilon_t = 0$ for all other $t > 0$ and compute the trajectory of variables for $t > 1$

The Uhlig tookit (1998)

Impulse Reponse



Note: From Uhlig (1998)

The Uhlig tookit (1998)

Alvaro's class this week

- quick review of this class
- go over the matlab codes for simulation, irf, etc, with the Uhlig's toolkit

UHLIG, H. (1998): “A toolkit for analyzing nonlinear dynamic stochastic models easily manuscript,” *University of Tilburg*.