Motivation, resources
and the organization of the school system[*]

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Abstract

We study a model where student effort and talent interact with parental and teachers' investments, as well as with school system resources. The model is rich, yet sufficiently stylized to provide novel implications. We can show, for example, that an improvement in parental outside options will reduce parental and school effort, which are partially compensated through school resources. In this way we provide a rationale for the existing ambiguous empirical evidence on the effect of school resources. We also provide a novel microfoundation for peer effects, with empirical implications on welfare and on preferences for sorting across schools.

JEL-Classification: I20, I21, I28, J24

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1 Introduction

Education policy is at the forefront of the social and political debate. The belief that education is a catalyst for a better and more equitable society ensures its role in the political agenda, in both developed and developing countries. As a consequence, a variety of policies and reforms are continuously being proposed with the objective of improving the outcomes of the education system. Surprisingly, the implementation and evaluation of these policies often overlooks the changes in behavior they can induce in the actors involved in the education process. For example, the debate about the role of education resources on student learning does not usually take into account behavioral responses from parents and school administrators. Similarly, proposals of school vouchers generally disregard how different ways to sort students into schools would affect the determination of school policies, or their influence on parental involvement in education and, crucially, the political support for such schemes. In this paper, we study a model of education where student learning effort and outcomes, parental and school behavior, and public resources devoted to education are endogenously determined in an integrated theoretical framework.

In our model, the determination of educational outcomes is a process involving five participants: children, parents, headmasters, teachers and the policymaker. Each child chooses a certain level of effort devoted to learning. More able children obtain a higher learning outcome from a unit of effort. Altruistic parents and schools affect the effort decision through motivation schemes. However, inducing effort is costly for parents as well as for schools. Both for parents and teachers, there is an opportunity cost for the time involved in setting up and executing the motivation plan, which may also include monitoring or helping children with their learning tasks (such as homework). How costly it is for schools depends on their resources (class sizes, for example), which are determined by the policymaker, as well as the talent of their teachers. This integrated framework provides an accurate description of the workings of the education process: parents, students and the education system interact in the determination of school resources, education quality, school and parent education methods and, through all these, on students results. An advantage of our framework is its tractability, which allows us to analyze many important dimensions of the education process.

We start with a case where children are homogeneous in terms of innate ability and parents' characteristics, such as talent and opportunity cost of time. We find that the strength of parental and school involvement and resources devoted to education increase with student innate ability. The results are less clear cut when we analyze the impact of an increase in the opportunity cost of time associated with parental involvement in the

learning process of their children. This introduces, in a natural way, the connection between labor market conditions and the direct involvement of parents in the education of their children. This link goes beyond the hourly wage. For example, it can also capture changes in opportunities and incentives for female participation in the labor market. In either case, the strength of parental involvement is decreasing in the opportunity cost of time.

The interaction between the school system and parental inputs is the reason why political considerations are important. As the parental opportunity cost of time increases, they would like to rely more heavily on schools for motivating their children, which triggers actions by those responsible for the education system. The policymaker anticipates participant choices and satisfies parental wishes by increasing the resources devoted to education. Interestingly, the increase in school resources may not be accompanied by an overall increase in educational attainment. A result far too familiar for those in the education policy arena.[1] Our model can predict "disappointingly" weak effects of school resources on student results even in situations where school resources do in fact affect learning *cæteris paribus*. The weak effect can be rationalized because *cæteris paribus* does not hold when resources increase. Parental involvement decreases because of a change in their opportunity costs. School resources increase to compensate for this reduction.[2] These additional resources have in fact an effect, but this is not apparent because of concomitant changes in parental involvement in the education process. This process could also explain why the increase in expenditures per student observed in many countries during the last decades has not been followed by better test scores or improvements in other measures of student performance.[3]

We then allow for children to differ in terms of ability and parental opportunity costs of time, which leads to a number of insights. First, as the school determines their motivational

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[1] The empirical findings of the class-size literature are ambiguous. For example, Angrist and Lavy (1999); Krueger (1999); Urquiola (2006) report positive results of class-size on student attainment while others (Hanushek (2003); Hoxby (2000); Leuven, Oosterbeek, and Rooonning (2008); Anghel and Cabrales (2010)) find no gains.

[2] As an early example of the connection between parents' opportunity costs and school resources, Flyer and Rosen (1997) attribute the increase in school expenditures taking place in the United States during 1960-1990 to the growth in female labor-force participation.

[3] See Hanushek (1998) for the case of the US. This result also provides a possible reason for a low cross-country correlation between education expenditures and school attainment levels results in standardized tests. See for example, Hanushek (2006). And this "anomaly" has been recognized for a long time. For example, in words of The Economist, "Glance at any league table of educational performance and you will find several Asian countries bunched near the top. The achievements of the region are a puzzle to people who think that educational success is all a matter of expenditures. Even in Japan most of the schools are ill-equipped by comparison withe their western equivalents [...] The children are driven on by intense family pressure. Parents badger their children to succeed, but they also make big financial and personal sacrifices to help them do so. Mothers help their children with their homework [...] Fathers promise fancy toys and activities in return to examination success..." Quote from The Economist, November 21st 1992.

policy for the average individual in the classroom, school involvement is positively affected by the mean ability of their students. In equilibrium, this affects the intensity of parental involvement. Thus, peer effects arise endogenously, as the choice of effort and motivational policy in equilibrium depend on the average ability of the student in the classroom. This effect is reinforced by the determination of school resources. The policy maker decides the level of resources optimally given the characteristics of the school attended by the median voter's child. Thus, the decision on school resources will be based on the average ability of this school and on the median child's ability. As a consequence, student effort depends on the mean abilities of peers at her/his school, plus the ability of the median child and his peers. Our model generates in this way a microfoundation for peer effects, rather than assuming they come from some exogenous "contagion" process, as it is more common in the literature.

This matters because the existence of peer effects has been the subject of a recent controversy.[4] What is important for us about this literature is that, as our paper suggests, the existence or not of peer effects seems to depend on the interaction between the peers and school organization. For example Abdulkadirolu, Angrist, Dynarski, Kane, and Pathak (2011) show that pilot schools, which operate like public schools but are more selective produce no measurable outcomes, whereas charter schools, which change organization at the same time that students are more strongly selected do present some changes.[5]

In this context, an increase in the opportunity cost of the median parent raises similar issues to those identified for the homogeneous case. However, the link between median child characteristics and individual effort generates a channel through which changes in the distribution of income (or talent) can affect the educational choices of households and schools. For example, an increase in the income of the median child's household will generate an increase of resources in the system. This will induce a positive effect on households in lower parts of the income distribution even if their incomes do not change. And the other way around, if the income of the median does not change (or it changes very little) in an environment where mean income is increasing markedly, there will be few changes in school outcomes (or even a regression) at a time when income appears to be fast increasing.

We finally incorporate private schools in our setting. We show first that a mixed education system with public and private schools generates endogenous sorting by either parental

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[4]Some papers (such as Imberman, Kugler, and Sacerdote (2012), using data from a natural experiment) find strong evidence of their presence, whereas other papers do not (Abdulkadiroglu, Angrist, and Pathak (2014) and Dobbie and Jr. (2011)), using evidence from lotteries or entry exams into selective school).

[5]Also, Duflo, Dupas, and Kremer (2011) find that strong peers have an effect only in some kind of schools, those that are non-tracking.

opportunity cost of time or student talent. That private schools imply stratification is a well established insight of the literature. For example, Epple and Romano (1998) show that the existence of positive peer-effects induces schools to be perfectly ordered according to peer-group ability and may lead to stratification by income. Our model also displays an assignment equilibrium with top-down sorting.[6] Yet, the distinctive feature of our model is the focus on the endogenous determination of peer-effects, and hence school quality and policy choices, as a result of the interaction of parents and the school system. This allows us to analyze in detail the effect of policies increasing school choice for parents, like a voucher scheme. We show that, even if the median voter is favored (and hence the voucher policy approved), the reaction of schools to the changes in classroom composition, will increase inequality in student achievement. This is because the worsening of peer-effects in the schools where the less able students stay (the "cream-skimming effect") is magnified by the responses of other actors. The headmasters will decrease motivational strength at those schools, and policymakers will decrease the resources devoted to them. Hence, our framework allows us to understand in a simple way the effects of students' quality, and the reaction of other actors to this quality, on the incentives for school sorting.[7] Our result that sorting feeds back on school quality and classroom peer-effects has important implications for the analysis of school choice programs. For example, Altonji, Huang, and Taber (2010) find that the cream-skimming effect on high school graduation was modest. However, their estimates did not consider how parents, teachers and headmasters would react to different classroom compositions. Our contribution is to show that these reactions should be incorporated in the empirical literature.[8]

Stratification by income may also emerge if the labor market considered school reputation as a signal of students' skills. MacLeod and Urquiola (2009) show that this is indeed the case. In their model, individual skills are partially observed by standardized tests and the reputation of the school attended by each student. As reputation depends on classroom

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[6]We can also obtain stratification by income by simply assuming a positive correlation between ability and income.

[7]Previous papers focus on just one of these elements. While Urquiola and Verhoogen (2009) and Epple, Romano, and Sieg (2006) focus on educational quality, Epple and Romano (1998, 2008) and Epple, Figlio, and Romano (2004) concentrate purely on peer-effects.

[8]Teachers also react to changes in classroom composition. For example, teachers may be more efficient in fairly homogenous classrooms (Duflo, Dupas, and Kremer, 2011) or with students of the same race (Dee, 2004). They may also react to the educational goals established through the implementation of new education policies. For example, Aucejo (2011) find that teachers in North Caroline have refocused their teaching efforts to relatively low performing students after the implementation of the No Child Left Behind program. Our model summarizes these concerns in a single statistic capturing the strength of motivation at the school, which we endogenously derive.

composition, school reputation reflects the quality of peers. As the reputation of the attended school increases expected wages, school quality would induce lower individual effort. In our model, the effect of peers on individual effort is the opposite: better peers encourage schools to provide incentives and induce higher student effort. While the empirical literature has not yet established which one prevails, we see both effects as complements: school motivation schemes are arguably more relevant in early and primary education while the school reputation effect on student effort may be more prevalent for older students.

The relevance of behavioral responses to education policy changes is attracting growing interest in the empirical literature. Das, Dercon, Habyarimana, Krishnan, Muralidharan, and Sundararaman (2013) emphasize behavioral aspects associated with higher school inputs. In particular, as school and household educational spending are substitutes, an unanticipated increase in school funding reduces parental school expenditures. Pop-Eleches and Urquiola (2013) uncover the effect of access in Romania to better (high) schools on student outcomes and parental behavior. They identify, for example, the positive effect of school quality on students' achievement at the national standardized examinations. Importantly for our paper, they find that parental effort and quality-improving school activities are substitutes for each other. Additional evidence of the substitution between parental effort and school resources is provided by Houtenville and Smith Conway (2008).

Stinebrickner and Stinebrickner (2008) and De Fraja, Oliveira, and Zanchi (2010) provide empirical evidence on the positive impact of parental and student effort on educational achievement. De Fraja, Oliveira, and Zanchi (2010) adds the dimension of school effort which is positively associated with examination scores. They also provide tentative evidence of feedback effects between parental, student and school efforts. They find, for example, that parental involvement induces more effort from their children which leads to higher involvement from schools.[9] In real life, parental involvement may take different forms. Doepke and Zilibotti (2014) study a model where different parenting styles (authoritarian, permissive and authoritative) are endogenously determined by the interaction of parental preferences and the socioeconomic environment. Although we do not distinguish between forms of parenting, our approach is complementary insofar as our endogenous determination of parental involvement intensity.

Our work is also related to the literature that studies the endogenous determination of class size. In Lazear (2001) class size is decided by schools according to student behavior.

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[9]Sahin (2004) provides another example of how parent and student responses affect the impact of education policies for the case of higher education tuition subsidies. Evidence of the interaction between parents and the school system mediated by monitoring of schools is offered by Liang and Ferreyra (2012).

For example, when students have a shorter attention span (i.e., they can be distracted more easily) students should be sorted in smaller classrooms as they require closer attention. In Urquiola and Verhoogen (2009), schools differ in productivity and offer different quality levels (school size). As parents differ in earnings, sorting between schools with different class sizes arises naturally. Our model offers a complementary mechanism behind the determination of class size, which relies upon the interaction of parental and school motivation, which is partly determined by the government through the (strategic) choice of school resources.

We organize the paper as follows: In section 2, we set up the model. We characterize the equilibrium in section 3, where we discuss the interdependence between parental and school motivation systems, school resources and student performance. Section 4 analyzes a case with private schools and endogenous segregation. We conclude in section 5.

2 The model

Our model has four participants: children, parents, headmasters, and the policymaker.[10] There are different ways to specify the production function of human capital (H). As a general level, we assume that learning requires innate talent (v) and effort (e). Parents and schools can affect this process through different interventions and actions. It is not clear however whether parental and school involvement induce learning through a direct impact in H or via inducing changes in effort. In the core of the paper, we emphasize that parental and school interventions affect effort and therefore learning in a indirect way. We leave for the Appendix A.1 the discussion of actions that directly impact in the accumulation of human capital such as parents hiring private tuition for their children.

Assuming that parents and schools can affect the learning effort is consistent with what psychologists call "Achievement Goal Theory" (see, for example, Covington (2000)). According to this perspective, achievement goals influence the quality, timing and appropriateness of the students' engagement in their own learning (e.g., analyzing the demands of school tasks, planning and allocating resources to meet these demands, etc.). This effort together with innate ability affect the student's accomplishments. In our model, parents and teachers play a key role in influencing the student's achievement goals and, in turn, their effort.

Two kinds of goals have been predominantly studied in achievement goal theory: learning goals and performance goals. Learning goals refer to increasing one's competency, understanding and appreciation of what is learned. Performance goals involve outperforming

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[10]For simplicity, we refer to students as she and to teachers as he.

others in tests or other achievement measures. In our model, parents and teachers can be thought as focusing on affecting learning goals.

Crucially, inducing effort is costly for parents as well as for schools. Both for parents and teachers, the main cost is the opportunity cost of the time involved encouraging students in the pursuit of learning goals. The cost for schools depends as well on the level of resources (for example, class size), which are determined by a policymaker.

Student performance and children's short-term utility   Human capital for child i, H_i, (measured in monetary terms) is a linear function of her effort, e_i, and a productivity parameter, v_i. In particular, we assume,

$$H_i = v_i e_i. \tag{1}$$

In the core of the paper, we assume v_i to be exogenous, which can therefore be interpreted as a measure of innate talent. We discuss the implications of different specifications of H_i in Appendices A.1, A.2 and A.3.

Children internalize only imperfectly the effect of their effort on human capital. The role of parents and teachers is to induce students to exert costly effort through different motivational tools. In particular, we postulate that children's short-term utility is given by:

$$U_{S_i} = c_{1j} e_i + c_{2j} e_i - \frac{1}{2} e_i^2. \tag{2}$$

Exerting effort implies a cost that takes a quadratic form. Parent and teacher's involvement enters positively in the utility of the child, $c_{1j} e_i + c_{2j} e_i$, with the sub-index j denoting the school attended by the child i. c_{1j} (c_{2j}) is a summary of the strength of parental (teacher) incentives for every unit of child's effort. Therefore, $c_{1j} e_i + c_{2j} e_i$ is the child's total reward for her effort.[11]

We are agnostic about the real life characterization of c_{1j} (c_{2j}). We think about it as a reduced form of a complex incentive structure that arises within families (schools). For example, rewards can be extrinsic such as buying presents or taking a child to the park, but c_{1j} may also capture intrinsic forms of motivation that require direct parental involvement

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[11]This linear structure of rewards is analogous to the linear contracts that are widespread in reality. Bose, Pal, and Sappington (2011) justify the use of linear contracts by showing that, in a broad class of environments, the optimal linear contract always secures for the principal at least 90% of the expected profit secured under a fully optimal contract. Furthermore, Carroll (2013) considers an environment in which the principal is uncertain about what the agent can and cannot do and wants to write a contract that is robust to this uncertainty. He finds that, under very general circumstances, the optimal contract is linear.

in the learning process and that affect the value of acquiring knowledge. In other words, the role of parents (teachers) is to help children intenalize the value of e_i (and ultimately H_i) in their utility function.

Pomerantz, Moorman, and Litwack (2007) argue that parental involvement[12] in school can enhance children's school performance through either skill development (e.g., by instructing children) or motivational development (e.g., by providing intrinsic reasons for learning). Some motivational activities clearly play both roles. Take for example, helping with homework. On the one hand, it increases the productivity of the children effort (i.e., is tantamount to an increase in v_i). On the other hand, it increases the intrinsic value of studying for the child because parental involvement and concerns help children internalize the benefits of their effort. As discussed above, we focus on the motivational channel and leave for the appendix the discussion of parental interventions that directly affects productivity of learning effort (Appendix A.1).

The additive specification for parent and teacher incentives implies that parents and school involvement are substitutes.[13] This assumption is supported by the empirical evidence suggesting that parental and school involvement are substitutes (Houtenville and Smith Conway, 2008; Pop-Eleches and Urquiola, 2013), but carries no qualitative implications as we discuss below and in Appendix A.3.

Of course, effort may also be obtained through negative reinforcement (punishment). In this case, we could have written the utility in the alternative way:

$$U_{S_i} = -c_{ij}\left(1 - e_i\right) - \frac{1}{2}e_i^2 = c_{ij}e_i - c_{ij} - \frac{1}{2}e_i^2.$$

As will be clear below, this utility induces the same optimal action from her as the one we examine. Thus, provided the costs of the two incentive systems can be written in the same way, there will be no difference in any equilibrium value.[14]

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[12]Following Hoover-Dempsey and Sandler (1997), parental involvement includes home-based activities related to children's learning (e.g., helping with homework, discussing school events) and school-based involvement (e.g., volunteering at school, attending school functions).

[13]This formulation also assumes that effort is perfectly correlated with human capital. Nothing major would change in the model if the relationship between effort and output were noisy, provided effort were observable and contractible. We do not think this is an unrealistic assumption in the case of children. Non-observability of effort with noisy output would be, of course, more complicated but we do not think any new insights would be gained by studying that case.

[14]In our setting, this is the case because we assume that parents care about H but not about her short term utility.

The parents' utility   We assume that every parent has one child and that their utility is influenced by the sum of her performance and their own welfare, denoted by W_i (also measured in monetary terms, as H_i).[15] Hence,

$$U_{P_i} = H_i + W_i.$$

Parental welfare depends on the time spent at work or pursuing leisure activities. This is the total time (T) available minus the time spent with the child as a consequence of the reward scheme (t_i). The effective reward $(c_{1i}e_i)$ depends on time (t_i) and parental efficiency (v_{P_i}) of parent i to generate a given effective reward, so that $c_{1i}e_i = 2t_i v_{P_i}$. The parameter v_{P_i} captures the fact that some parents are better at motivating than others and consequently they generate a larger reward for any given amount of time devoted to their children. That is, we associate productivity of parental involvement with some innate (exogenous) attribute, such as talent or persuasiveness. Implicitly, we are imposing that the time devoted inducing effort, for a given effective reward, is decreasing in the ability of the parent to generate it. In Appendix A.4, we study the effect of parental attention, where v_{P_i} is derived endogenously and shown to depend on the opportunity cost of time.

Thus, letting ψ_i be the opportunity cost of parent i yields,

$$W_i = (T - t_i)\,\psi_i = \left(T - \frac{c_{1i}e_i}{2v_{P_i}}\right)\psi_i,$$

and, therefore,

$$U_{P_i} = v_i e_i + \left(T - \frac{c_{1i}e_i}{2v_{P_i}}\right)\psi_i. \tag{3}$$

Since ψ_i is an opportunity cost of time for parents, it can be interpreted as a wage rate, although it can also be the value of leisure or something else.[16] Hence, in the remainder of the paper we often refer to this parameter as parental income.[17]

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[15]The utility function below does not internalize the child's cost of effort, so it is not purely "sympathetic". On the one hand this is reasonable, since this cost of effort is not observable. But we have also done the computations with strictly sympathetic parents' utility and there are no significant changes.

[16]In our context, the marginal utility of money earned by parents is linear. Hence, the value of time for parents with high wages is larger. Things may be different with concave utility for money. In that case, low wage earners may have a higher opportunity cost of time. We are agnostic about which parents have the higher opportunity cost of time in reality.

[17]Notice we are not considering parental teaching time in this specification of parents' welfare. However, this would be easily incorporated into the analysis by assuming $H_i = T_i + W_i$, and $W_i = \left(T - \frac{c_{1i}e_i}{2v_{P_i}} - \frac{1}{2}\frac{T_i^2}{v_{P_i}}\right)\psi_i$ where $t_i' = T_i^2/2v_{P_i}$ represents the actual units of time a parent devotes to teaching and T_i represents effective teaching time. Notice as well that this component is independent of student

REMARK **1** *The parents also value (negatively) the taxes that the government will need to levy in order to pay for school resources. We do not include them here explicitly in order to avoid an excess of notation. Given the quasi-linearity in income of utility and that taxation is already decided at the time parents choose their effort, the amount of those taxes do not affect the parental effort decision.*

The school objective function According to Covington (2000), every classroom reflects rules that determine the basis on which students will be evaluated and how rewards will be distributed. We assume that teachers reward students based on individual learning expectations and not in a competition for one or a few prizes. The school rewards (and therefore the learning goals) are determined by the teachers' effort.[18]

We consider first the case of public (state) schools. In this case, we assume that the headmaster chooses the intensity of motivation (summarized in the parameter c_{2j}) and therefore, teachers' effort, in order to maximize the sum of the average student performance and the welfare of the average teacher in the school. Of course, a headmaster might have different goals and statistics to measure school's performance. In Appendices A.5 and A.6, we change the goal of maximizing average student performance in two crucial ways. In Appendix A.5, we study a case in which the headmaster cares more about a group of students according to their place in the distribution of human capital.[19]. We also study the case where headmasters are imposed an exogenous standards (Appendix A.6), such as, for example, that the average score be higher than a particular threshold (reference). In both cases, we show that the main messages of our analysis are robust to these variations in the headmaster's goals.

The average welfare of a teacher is determined by the difference between the total time available to him and the average time he devotes to his students, which we assume is a linear function of $c_{2j}e_i$, and inversely related to v_{T_j}, the motivational ability of a representative teacher at school j. Thus, letting γ be the opportunity cost of teachers' time,[20]

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effort and has no interaction with c_{1i} and therefore it will carry no effect on our results.

[18]For simplicity we assume that the costs of creating the rewards for the teachers are unrelated to student ability.

[19]More specifically, we study an alternative formulation where the headmaster cares about a weighted average of the mean and any order statistic of the distribution of human capital within the school. See Aucejo (2011) for an example of how teachers adjust their population targets within a classroom to changes in the goals imposed by the authorities.

[20]We assume that the opportunity cost of the teacher γ is unrelated to her talent v_{T_j} . This is done to simplify notation. Little is changed if γ depends on v_{T_j}. It is worth pointing out that this is not the teacher's wage but rather the value of his alternative use of time which can either be leisure or the compensation he will get from, say, private tutoring.

$$U_{HMj} = \frac{1}{N_j} \sum_{i \in j} v_i e_i + \left(T - \frac{n_j}{N_j} \sum_{i \in j} \frac{c_{2j} e_i}{2 v_{T_j}} \right) \gamma, \tag{4}$$

where N_j is the total number of students in school j and n_j is the number of students per classroom.

The policymaker objective function   The policymaker maximizes the complete utility of the (median-voter) parent (denoted by $\bar{P}_i$) which, as discussed in remark 1, requires adding the cost of school resources $(1/n)$. The decision about resources is taken and announced before parents and headmasters simultaneously decide their actions (c_{1i} and c_{2j}). Therefore, the cost of resources does not appear in U_{P_i} or U_{HMj} because parents and headmasters take them as given when making decisions about their involvement. These costs are paid by parents through general taxation, which parents care about, and are internalized by the policymaker when deciding n.

The cost depends on the number of classes to be manned. That is, the ratio of total number of students in the system, N, to the number of students per class, n. For simplicity, we assume that all public schools operate at full capacity and with the same class size so that $n_j = n$ for all j. Manning costs are assumed to be quadratic in the number of classrooms N/n. This can be justified by taking into consideration that the state has monopsony power in the market for teachers and faces a marginal cost function that increases in the number of teachers hired. This is so, for example, because to attract one more teacher the monopsonist has to pay an extra cost, since the marginal potential teacher needs a higher reward to be attracted to the profession.

Thus, we can represent the policymaker's preferences as,

$$U_{PM} = U_{\bar{P}_i} - \frac{1}{N} \frac{\omega'}{2} \left(\frac{N}{n} \right)^2, \tag{5}$$

where ω' is a constant parameter summarizing the cost of the chosen class size and $\frac{1}{N} \frac{\omega'}{2} \left(\frac{N}{n} \right)^2$ is the *per capita* cost of that class size.[21] Our formulation assumes that schools are financed out of lump sum taxation and the government keeps a balanced budget. For ease of notation,

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[21]Alternatively, the formulation can be reinterpreted by assuming that the average quality of teachers when hiring k people is $1/\sqrt{k}$. Hence, in order to man N/n classrooms and keep the quality of teaching per classroom constant, $(N/n)^2$ teachers need to be hired. If $\omega'/2$ is the wage per employed person the total cost of N/n classrooms is $\frac{\omega'}{2} \left(\frac{N}{n} \right)^2$ and the cost per student $\frac{1}{N} \frac{\omega'}{2} \left(\frac{N}{n} \right)^2$.

in the remainder we denote $\omega = \omega' N$, so that

$$U_{PM} = U_{\bar{P}_i} - \frac{\omega}{2} \frac{1}{n^2}, \tag{6}$$

The structure of the game   Summarizing, the policymaker announces first the policy variable (n). After this announcement, parents and headmasters simultaneously decide their optimal levels of rewards per unit of effort c_{1i} and c_{2j}, respectively. After observing parents' and schools' announcements, the children decide their optimal level of effort, e_i.

3   Equilibrium

We solve the game by backward induction.

3.1   Students', parents', and school choices

From equation (2), it follows that the optimal student action is

$$e_i = c_{1i} + c_{2j}. \tag{7}$$

Substituting this expression into the parents' utility, equation (3), we obtain

$$U_{P_i} = (c_{1i} + c_{2j}) \, v_i + \left(T - \frac{1}{2v_{P_i}} c_{1i} \, (c_{1i} + c_{2j}) \right) \psi_i$$

The first-order conditions for the parents's problem is then

$$v_i - \left(c_{1i} + \frac{c_{2j}}{2} \right) \frac{\psi_i}{v_{P_i}} = 0.$$

Given that this condition is sufficient, the optimal choice of the parent is

$$c_{1i} = \max \left\{ \frac{v_{P_i} v_i}{\psi_i} - \frac{c_{2j}}{2}, 0 \right\} \tag{8}$$

which is always non-negative given that motivating requires investing parental time.

It is clear from the expression for c_{1i} that the strength of parental involvement is increasing in the abilities of the child (v_i) and the parent (v_{P_i}), and decreasing in the parental opportunity cost of time (ψ_i). Also, equation (8) shows the negative relationship between c_{1i} and c_{2j}. When motivation in the school is high, the gains from additional effort induced

by parental motivation are smaller. We shall discuss below how both incentive schemes may compensate each other in responding to changes in ψ_i, υ_{P_i} and υ_i.

At this point, the following clarification is in order:

REMARK **2** *The assumption of substitute rewards is not essential for the negative relationship between c_{1i} and c_{2j}. A similar result is obtained with other specifications where parental and school efforts are complements. For example, when $H_i = \upsilon_i e_i^\alpha$, with $\alpha < 1$ and $c_{ij} = c_{1i} c_{2j}$ (see Appendix A.3). The driving force in our result is that greater school incentives will reduce the marginal benefit of parental effort.*

By substituting the optimal choice of children's effort into the utility function of the headmaster (4) we obtain:

$$U_{HMj} = \frac{1}{N_j} \sum_{i \in j} (c_{1i} + c_{2j}) \upsilon_i + \left(T - \frac{n_j}{N_j} \sum_{i \in j} \frac{c_{2j} (c_{1i} + c_{2j})}{2\upsilon_{T_j}} \right) \gamma.$$

It follows that an interior solution for the headmaster's optimization problem implies

$$c_{2j} = \frac{\upsilon_{T_j} \overline{\upsilon}_j}{\gamma n_j} - \frac{\overline{c}_{1j}}{2}, \tag{9}$$

where $\overline{\upsilon}_j$ is the mean student ability and $\overline{c}_{1j}$ is the mean parental reward for the students attending school j.

We can interpret $\upsilon_{T_j} \overline{\upsilon}_j$ as the school quality. Clearly, talent of teachers plays a key role in defining school's quality. Also, as in Epple and Romano (1998), quality of schools depends on the average of peer's talent. A higher average peer talent is associated with greater classroom motivation. This association, which will tend to amplify school differences in student performance, will be important in the emergence of peer effects and will provide the motivation for a segregated school system, as we shall discuss below. Also, raising opportunity costs for teachers and larger class sizes lead to lower levels of classroom motivation. It is, of course, through this latter channel that public resources affect student performance. Finally, a high level of parental involvement is associated with lower school incentives.

3.2 Equilibrium values for c_1, c_2 and n

3.2.1 Homogeneous children

In order to solve for the first stage of the game, let us first assume that children, parents and teachers are all identical, so that $\upsilon_i = \upsilon$, $\upsilon_{P_i} = \upsilon_P$ and $\psi_i = \psi$ for all $i \in \{1, .., N\}$ and

$\upsilon_{T_j} = \upsilon_T$ for all j. In this case we have that $c_{1i} = c_1$ for all i and and $c_{2j} = c_2$ for all j. Therefore, from (8) and (9), an interior solution for c_1 and c_2 implies

$$c_1 = \frac{\upsilon_P \upsilon}{\psi} - \frac{c_2}{2} \tag{10}$$

and

$$c_2 = \frac{\upsilon_T \upsilon}{\gamma n} - \frac{c_1}{2}. \tag{11}$$

In the first stage, the policymaker considers the optimal level of school and parental involvement. After substituting (11) in (10) and plugging the resulting expression, together with (7), into (6) we obtain

$$U_{PM} = \frac{2\upsilon^2}{3}\left(\frac{\upsilon_P}{\psi} + \frac{\upsilon_T}{\gamma n}\right) + \left(T - \frac{2\upsilon^2}{9\upsilon_P}\left(\frac{2\upsilon_P^2}{\psi^2} + \frac{\upsilon_P \upsilon_T}{\psi \gamma n} - \frac{\upsilon_T^2}{(\gamma n)^2}\right)\right)\psi - \frac{\omega}{2}\frac{1}{n^2}$$

As we show for the general heterogeneous case in the next section, preferences are unimodal in the policy parameter n and we can apply the median-voter theorem to obtain the optimal level of n. The interior solution that results from maximizing the above expression with respect to n is (assuming $\omega > (2\upsilon\upsilon_T/3\gamma)^2 (\psi/\upsilon_P)$)

$$n = \frac{\omega - \left(\frac{2\upsilon\upsilon_T}{3\gamma}\right)^2 \frac{\psi}{\upsilon_P}}{\frac{\upsilon_T}{\gamma}\left(\frac{2\upsilon}{3}\right)^2}. \tag{12}$$

Therefore, class size is increasing in the cost of manning classes, ω, parental motivation ability υ_P and the opportunity cost of time of teachers, γ, and decreasing on student ability, υ, teacher talent υ_T and parental opportunity cost of time, ψ. The post-war decades were characterized by a large increase in female labor-force participation, arguably compatible with an increase in ψ. This would suggest a decrease in n (probably moderated by a parallel increase in γ). Indeed, Flyer and Rosen (1997) find a connection between the rapid growth in public expenditures associated with elementary and secondary education in the US and the rising value of women's time. Between 1960 and 1990 the real costs of elementary and secondary education increased by 300 percent. The main cause of the increase in expenditures per student was rising school staff, with a halving of the number of the student-teacher ratios between 1950 and 1990. The paper studies further the connection between female labor force participation and student-teacher ratios using a panel of 50 US states during 4 decades. They found that increases in female labor force participation rates explain a

significant part of both the level and growth in states' student-teacher ratios. Moreover, the findings are robust to other plausible explanations such as increases in unionization and changes in fertility patterns.

We substitute the optimal level of n in equation (11) and then into equations (7) and (10) to obtain the equilibrium values of c_1, c_2 and e. These are:

$$c_1 = \frac{2\upsilon_P\upsilon}{3\psi}\left(\frac{2\omega - 3\left(\frac{2\upsilon\upsilon_T}{3\gamma}\right)^2\frac{\psi}{\upsilon_P}}{\omega - \left(\frac{2\upsilon\upsilon_T}{3\gamma}\right)^2\frac{\psi}{\upsilon_P}}\right),$$

$$c_2 = \frac{2\upsilon_P\upsilon}{3\psi}\left(\frac{3\left(\frac{2\upsilon\upsilon_T}{3\gamma}\right)^2\frac{\psi}{\upsilon_P} - \omega}{\omega - \left(\frac{2\upsilon\upsilon_T}{3\gamma}\right)^2\frac{\psi}{\upsilon_P}}\right),$$

and

$$e = \frac{\frac{2\upsilon_P\upsilon}{3\psi}\omega}{\omega - \left(\frac{2\upsilon\upsilon_T}{3\gamma}\right)^2\frac{\psi}{\upsilon_P}}. \tag{13}$$

From inspecting the above expressions it becomes clear that a necessary condition for a positive c_1 is

$$\omega > \frac{3}{2}\left(\frac{2\upsilon\upsilon_T}{3\gamma}\right)^2\frac{\psi}{\upsilon_P}. \tag{14}$$

This condition is sufficient for n and e to be positive as well. A necessary condition for a positive c_2 is

$$\omega < 3\left(\frac{2\upsilon\upsilon_T}{3\gamma}\right)^2\frac{\psi}{\upsilon_P}. \tag{15}$$

The comparative static results of school and parental involvement with respect to υ and υ_T are simple. An increase in student ability (υ, which one can think of as innate or the result of early parental stimulation) leads to stronger school and parental motivation. The same effect is associated with higher parental motivation ability. These factors in turn induce higher student effort.

The effect on effort of an increase in ψ is ambiguous. First, a higher ψ imposes a higher opportunity cost for parents to engage in motivational activities. Hence, c_1 is decreasing in ψ. The school system reacts to this by reducing n and therefore c_2 is increasing in ψ.

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The driving force for this result is that the policymaker devotes more resources to classroom education, which lowers the cost of inducing effort by the school. Conversely, lower class size and the consequent stronger school ethos, reduces the gain from staying at home inducing children's effort.

The optimal effort decision is equal to the sum of parental and school involvement. When the opportunity costs for parents increases the resulting fall in parental involvement is not always fully compensated by the school system. Therefore, the net effect of an increase of ψ on student performance may be negative. To see this:

$$\frac{\partial e}{\partial \psi} = -2v_P v \omega \frac{\omega - 2\left(\frac{2vv_T}{3\gamma}\right)^2 \psi}{\left[\omega\psi - \left(\frac{2vv_T}{3\gamma}\right)^2 \frac{\psi^2}{v_P}\right]^2}$$

which implies that effort is decreasing in ψ when

$$\omega > 2\left(\frac{2vv_T}{3\gamma}\right)^2 \frac{\psi}{v_P}.$$

Notice that from (14) and (15)

$$3\left(\frac{2vv_T}{3\gamma}\right)^2 \frac{\psi}{v_P} > \omega > \frac{3}{2}\left(\frac{2vv_T}{3\gamma}\right)^2 \frac{\psi}{v_P}$$

so that effort and school performance can be both increasing or decreasing within our parametric range.

The same ambiguity, but in the reverse direction, is generated by differences in parental motivational talent v_P. This is the case since in all expressions we have ψ/v_P.

The model highlights an important issue on empirical estimates of the effect of class-size (or educational resources in general) on educational outcomes. Following Todd and Wolpin (2003), the typical paper considers an educational production technology which is a function of a set of school inputs, like class-size, and household inputs. Like in our model, optimizing agents (see, for example, Glewwe (2002)) determine their input choice based on a set of market and shadow prices. Isolating the effect of class size, or any other input, on educational outcomes in empirical work is hard because it essentially requires conditioning on all inputs. In general, what we can do is to find an exogenous source of variation in a market or shadow price -excluded by definition from the production function- which affects directly the input of interest and indirectly other inputs of the production function. Thus, we estimate what is called in the jargon a policy impact which contemplates the direct effect

of the input of interest, say class-size, and changes in other inputs -behavioral changes. A key contribution of our model is to highlight that different sources of variation for class-size result in different behavioral responses and therefore different policy parameter estimates.

For example, our model has identified two sources of variation for class size, ω and ψ, that may lead to different policy estimates of the impact of class size on student performance. As we pointed out above, an increase in the opportunity cost of parents and a fall in the cost of manning classes both lead to lower class sizes. However, a fall in the cost of manning classes leads to a unambiguous increase in effort (as can be seen from (13)) and an improvement in student performance. The fact that an increase in class size may lead to different effects depending on the source of variability that generates these changes provides an interesting lens through which we can interpret the findings in the empirical literature. When the source of variability for class size comes from exogenous changes in the costs of manning classes, like in most randomized experiments (see, for example, Krueger (1999)) we should expect positive impacts on student performance. However, in cross-country or cross-state panel studies, where differences in resources, like class-size, may be the consequence of increases in opportunity costs for the median parents, we may find it more difficult to observe improvement in educational performance.

3.2.2 Heterogeneous children

We relax now the assumption of identical children. For expositional clarity, we define the following parameter:

$$\overline{\Omega}_j \equiv \frac{1}{N_j} \sum_{i \in j} \frac{\upsilon_{P_i} \upsilon_i}{\psi_i}.$$

In words, $\overline{\Omega}_j$ is the average at the school level of student ability times the ratio between parental motivational ability and their opportunity cost of time. Thus, each school j is associated with a particular $\overline{\Omega}_j$.[22]

To obtain the utility of the policymaker, we substitute (9) in (8) and plug the resulting expression into (7). This yields

$$e_i^* = \frac{\upsilon_{P_i} \upsilon_i}{\psi_i} + \frac{1}{3} \left(\frac{2}{\gamma n} \upsilon_{T_j} \overline{\upsilon}_j - \overline{\Omega}_j \right), \tag{16}$$

where $\overline{\upsilon}_j$ is the average student ability in school j. Thus, equation (6) becomes:

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[22]Notice that in the plausible cases where υ_i, υ_{P_i} and ψ_i are correlated, the ranking of schools would be invariant to whether the ranking is based on υ, υ_P, ψ or $\overline{\Omega}$ (clearly, in different directions for each one).

$$U_{PM} = \left(\frac{\upsilon_{P_{iM}} \upsilon_{i_M}}{\psi_{i_M}} + \frac{1}{3} \left(\frac{2}{\gamma n} \upsilon_{T_{jM}} \overline{\upsilon}_{jM} - \overline{\Omega}_{jM} \right) \right) \upsilon_{i_M} +$$

$$+ \left(T - \frac{1}{2\upsilon_{P_{iM}}} \left(\left(\frac{\upsilon_{P_{iM}} \upsilon_{i_M}}{\psi_{i_M}} \right)^2 - \frac{1}{9} \left(\frac{2}{\gamma n} \upsilon_{T_{jM}} \overline{\upsilon}_{jM} - \overline{\Omega}_{jM} \right)^2 \right) \right) \psi_{i_M} - \frac{\omega}{2n^2},$$

where M stands for the median voter and, therefore, $\overline{\upsilon}_{jM}$ and $\overline{\Omega}_{jM}$ express the characteristics in the school the median voter attends.

The following lemma states that preferences are unimodal in the policy parameter, n, and proves the validity of the median-voter theorem to determine school resources (n) in our framework.

LEMMA 1 *Preferences of parents with respect to n are unimodal for all parents if $\frac{1}{N_j} \sum_{i \in j} \frac{\upsilon_{P_i} \upsilon_i}{\psi_i} <$*
$3 \min_{i \in \{1,\dots,N\}} \frac{\upsilon_{P_i} \upsilon_i}{\psi_i}$.

Proof Please see Appendix B. ∎

The sufficient condition for this result means that the minimal value of $\upsilon_{P_i} \upsilon_i / \psi_i$ (the ratio of parental teaching talent to her opportunity cost, times the child's talent) is not lower than a third of the average value of $\upsilon_{P_i} \upsilon_i / \psi_i$ in the population.

The first-order condition for the policymaker's maximization problem, provided an interior solution exists, is:

$$\frac{\partial U_{PM}}{\partial n} = -\frac{2}{3} \frac{\upsilon_{i_M} \upsilon_{T_{jM}} \overline{\upsilon}_{jM}}{\gamma n^2} - \frac{2}{9} \frac{\psi_{i_M} \upsilon_{T_{jM}} \overline{\upsilon}_{jM}}{\gamma \upsilon_{P_{iM}} n^2} \left(\frac{2}{\gamma n} \upsilon_{T_{jM}} \overline{\upsilon}_{jM} - \overline{\Omega}_{jM} \right) + \frac{\omega}{n^3} = 0. \quad (17)$$

We can make the following assumption to reduce notational complexity:

ASSUMPTION 1
$$\frac{\upsilon_{P_{iM}} \upsilon_{i_M}}{\psi_{i_M}} = \overline{\Omega}_{jM}.$$

This means that the ratio of parental and individual parameters for the median child is equal to the average ratio in her school. Using this assumption, equation (17) simplifies to:

$$\frac{\partial U_{PM}}{\partial n} = -\frac{2}{3} \frac{\upsilon_{i_M} \upsilon_{T_{jM}} \overline{\upsilon}_{jM}}{\gamma n^2} + \frac{2}{9} \frac{\upsilon_{T_{jM}} \overline{\upsilon}_{jM} \upsilon_{i_M}}{\gamma n^2} - \left(\frac{2\upsilon_{T_{jM}} \overline{\upsilon}_{jM}}{3\gamma} \right)^2 \frac{\psi_{i_M}}{\upsilon_{P_{iM}} n^3} + \frac{\omega}{n^3} = 0$$

and thus we obtain:

$$n = \frac{\omega - \left(\frac{2\upsilon_{T_{jM}}\overline{\upsilon}_{jM}}{3\gamma}\right)^2 \frac{\psi_{iM}}{\upsilon_{P_{iM}}}}{\frac{4}{9}\frac{\upsilon_{iM}\upsilon_{T_{jM}}\overline{\upsilon}_{jM}}{\gamma}}. \tag{18}$$

A positive class size requires $\omega - \frac{4}{9\gamma^2_{jm_M}}\left(\overline{\upsilon}_{jM}\right)^2\frac{\psi_{iM}}{\upsilon_{P_i}} > 0$. Like in the homogeneous case, class size increases with the opportunity cost of manning classes. School resources are increasing in the opportunity cost of the median parent and the ability of the median child as well as the quality, $\upsilon_{T_{jM}}\overline{\upsilon}_{jM}$, of the school she attends.

From the derivation of (18) it is clear that parents of children with υ_i above υ_{iM} would like the level of school resources to be higher (e.g., smaller class sizes). So it would make sense for them to supply the school with extra resources, in the form of their own time and material resources. As we explore in the next section, this has strong implications for segregation. But even within public schools, they can choose, if allowed, to do so. This could explain why parents choose to organize activities in schools, which as Anghel and Cabrales (2010) document for the case of Spain have a sizable effect on student achievement.

For ease of exposition, it is convenient to define:

$$\theta_j \equiv \frac{\overline{\Omega}_j}{\overline{\Omega}_{jM}}$$

which under assumption (1) implies that

$$\overline{\Omega}_j = \theta_j\overline{\Omega}_{jM} = \theta_j\frac{\upsilon_{P_{iM}}\upsilon_{iM}}{\psi_{iM}}.$$

Using equation (18), the equilibrium values for c_{1i}, c_{2j} and e_i follow:

$$c_{1i} = \frac{\upsilon_{P_i}\upsilon_i}{\psi_i} - \frac{2}{3}\left(\frac{\upsilon_{T_j}\overline{\upsilon}_j}{\gamma n} - \frac{\overline{\Omega}_j}{2}\right) \tag{19}$$

$$c_{2j} = \frac{4}{3}\left(\frac{\upsilon_{T_j}\overline{\upsilon}_j}{\gamma n} - \frac{\overline{\Omega}_j}{2}\right) \tag{20}$$

$$e_i = \frac{\upsilon_{P_i}\upsilon_i}{\psi_i} + \frac{1}{3}\frac{\upsilon_{P_{iM}}\upsilon_{iM}}{\psi_{iM}}\left(\frac{\left(\frac{2\upsilon_{T_j}\overline{\upsilon}_j}{\upsilon_{T_{jM}}\overline{\upsilon}_{jM}} + \theta_j\right)\left(\frac{2\upsilon_{T_{jM}}\overline{\upsilon}_{jM}}{3\gamma}\right)^2\frac{\psi_{iM}}{\upsilon_{P_{iM}}} - \theta_j\omega}{\left(\omega - \left(\frac{2\upsilon_{T_{jM}}\overline{\upsilon}_{jM}}{3\gamma}\right)^2\frac{\psi_{iM}}{\upsilon_{P_{iM}}}\right)}\right) \tag{21}$$

REMARK **3** *The interaction between parents, schools and education policy generates peer-effects.*

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The expression for (21) reveals that, in equilibrium, the performance of student i depends on the ability of her peers at different levels. Therefore, the model provides a micro-foundation for the emergence of peer effects in the classroom without technological assumptions.[23]

First, we obtain peer-group effects in the sense that the student's own performance increases in $\overline{v}_j$. The driving force is the reward scheme at the school level, which depends on the mean ability of her peers. To our knowledge, this is new in the literature.

Thus, our model predicts that child that attends a school or class with better peers would improve his effort and educational outcomes. The following are the mechanisms operating in this situation. First, higher quality students increase school effort. However, parents do react to the increase in school efforts in the expected direction. Parents of children in schools with better (worse) peers reduce (increase) their involvement. Under the parametric assumptions of our model the parental reactions are not strong enough to compensate for the higher school effort.[24] Thus, our model will predict a positive effect for a child that enters a higher quality school but this will be attenuated by parental behavioral responses. Interestingly, Cullen, Jacob, and Levitt (2006) and Pop-Eleches and Urquiola (2013) find that the parents of children that enter elite tracks or schools with better peers spend less time helping their children with homework than the counterfactual. Pop-Eleches and Urquiola (2013) find evidence that students benefit academically from access to schools and tracks with better peers. However, Cullen, Jacob, and Levitt (2006) and Clark (2010) find scant evidence of improvement in educational outcomes.

Second, performance is affected by the ability of the median student, the characteristics of her peers (ability and parental involvement) and her teacher's ability. This sort of peer-cohort and teacher effects result from the determination of the school resources that affect classroom motivation in public schools. Overall, we view these results as a cautionary note regarding empirical analyses which aim at measuring the effect of different education policies as if they were exogenous to the political process.[25]

The evidence on the existence of peer effects has been the object of some recent controversy. Some papers, such as Imberman, Kugler, and Sacerdote (2012) using data from a natural experiment find strong evidence of their presence, whereas other papers, mostly using evidence from lotteries or entry exams into selective school do not (e.g. Abdulkadiroglu,

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[23]In Appendix A.2, we discuss an extension with standard exogenous peer effects, where the human capital of the child is multiplied by $1 + \lambda \overline{e_j}$.

[24]This is straightforward to check by looking at the derivatives of c_{1i} and c_{2j} with respect to $\overline{v}_j$ in equations 19 and 20.

[25]A related point appears in Besley and Case (2000).

Angrist, and Pathak (2014) and Dobbie and Jr. (2011)).[26] The most important aspect of this controversy, from our point of view, is the fact that the peer effects are most likely happening, as our paper suggests, from a subtle interaction between the peers and school organization. This can be seen, for example, in the fact that Abdulkadirolu, Angrist, Dynarski, Kane, and Pathak (2011) pilot schools, which operate like public schools but are more selective produce no measurable outcomes, whereas charter schools, which change organization at the same time that students are more strongly selected to present some changes. Similarly, Duflo, Dupas, and Kremer (2011) find that strong peers have an effect only in some kind of schools, those that are non-tracking.

As in the case of homogeneous parents and children, an economy-wide increase in parental opportunity costs induces a reduction of involvement by the parent (the effect of ψ_i on c_{1j} is negative) which is partially compensated by the increased involvement of the school system (the effect of ψ_{i_M} on c_{2j} is positive). But because of the link between the median child and individual effort, changes in the distribution of income (or talent) can affect outcomes as well. For example, an increase in ψ_{i_M} will generate (for fixed θ_j) an increase of resources (a decrease in n) which will have a positive effect on lower-income household even if their incomes do not change. Thus, a rising tide lifts all votes in this case.[27] And the other way around, if ψ_{i_M} is unchanged (or almost, again for fixed θ_j) in an environment where mean income is increasing markedly, there will be few changes in school outcomes (or even a regression, because of the negative reaction in c_{1i} of very high income households) at a time when GDP is increasing.

Equation (21) underlines an important issue regarding the generalization of partial equilibrium studies of stratification to the entire education system. In general, reduced form work (e.g., Duflo, Dupas, and Kremer (2011)) will focus on isolating the direct effect of $\overline{v}_j$ on H_i. However, a generalized system for tracking students affects both $\overline{v}_j$ and $\overline{v}_{j_M}$. Thus, the general equilibrium implications may differ from the partial equilibrium ones through the political determination of the resources in the school system. We explore this issue and others in the next section.

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[26]Clearly, there is not a consensus yet, but the fact that such schools are so heavily oversubscribed make us doubt that the effect is nonexistent. It is possible that the schools do something to the students that the standardized exams fail to pick up, but which later earnings data would show, as Chetty, Friedman, Hilger, Saez, Schanzenbach, and Yagan (2011) have shown for the STAR experiment.

[27]Corcoran and Evans (2010) find that 12 to 22 percent of the increase in local school spending in the U.S. over the period 1970-2000 is attributable to rising inequality.

4   Endogenous segregation

Our model emphasizes the interdependencies established between (parents and school) motivational schemes, class resources and children effort. As a consequence, student performance depends on group and cohort peer effects. The implications of this finding run deeply into the different variables of the system. We introduce private schools and investigate sorting in the school market and the effect of policies inducing segregation, like school vouchers.

REMARK 4 *Notice that in all equilibrium values parental characteristics always enter as the ratio ψ_i/υ_{P_i} and teacher characteristics as the ratio γ/υ_{T_j}. Hence, from now on we normalize and let $\widehat{\psi}_i = \psi_i/\upsilon_{P_i}$, $\widehat{T}_i = T/\upsilon_{P_i}$ and $\widehat{\gamma}_j = \gamma/\upsilon_{T_j}$.*

The differential sensitivity of different types of parents to the composition of classrooms has segregation-inducing effects that we now analyze. The analysis will gain in clarity if we identify first a generic condition for an assignment equilibrium with sorting. Once a condition for this type of equilibrium is identified, we can then verify if it is satisfied for specific attributes, like talent or income, and a mixed system with public and private schools. Formally, consider a generic attribute ξ. Take two schools $\bar{\xi}_1$ and $\bar{\xi}_2$, where $\bar{\xi}$ is an average of that attribute in the school. Parents are allowed to send their children to a private school paying a fee. The differential willingness to pay of two parents, with types $\xi_i, \xi_{i'}$ and $\xi_i > \xi_{i'}$ is:

$$\int_{\bar{\xi}_1}^{\bar{\xi}_2} \frac{\partial U_{P_i}}{\partial \bar{\xi}_j} d\bar{\xi}_j - \int_{\bar{\xi}_1}^{\bar{\xi}_2} \frac{\partial U_{P_{i'}}}{\partial \bar{\xi}_j} d\bar{\xi}_j = \int_{\bar{\xi}_1}^{\bar{\xi}_2} \left(\frac{\partial U_{P_i}}{\partial \bar{\xi}_j} - \frac{\partial U_{P_{i'}}}{\partial \bar{\xi}_j} \right) d\bar{\xi}_j$$

which implies that so long as $\partial U_{P_i}/\partial \bar{\xi}_j$ is monotonic in ξ_i the willingness to pay is monotonic. This type-monotonicity in relative gains is what leads to an equilibrium with school segregation by types. More precisely, consider a finite set of schools $l \in \{1, ..., L\}$, each with n_l slots. Assume as well that each n_l is high enough so that the compositional impact of changing one child's type on the $\bar{\xi}_l$ of school l is small. Order arbitrarily the available schools. We denote by *top-down sorting* the following assignment of children into schools according to their type. School 1 gets assigned the n_1-highest type children, school 2 the n_2-highest type children among the remaining ones, and so on until all children are assigned to one (and only one) school. The top-down sorting leads to a segregated school structure with types stratified from higher to lower. Namely, given two schools $l > l'$ and two children i, i' that are assigned to either school by top-down sorting, then, $\xi_i > \xi_{i'}$ and $\bar{\xi}_l \geq \bar{\xi}_{l'}$. To ensure that this inequality is strict for at least one pair of players in two different schools,

we assume that two successive schools cannot be fully occupied by players of the same type. To join a school l, parents must pay a fee p_l to the owner of the school l. The last school (or set of schools) in the list is public, free and has enough capacity for N students (the full group). We say that an assignment of children to schools and a vector of school prices forms an equilibrium when, given the prices, no individual prefers to change schools and either a school is full or its associated fee is zero.

PROPOSITION 1 *There exists an assignment equilibrium with top-down sorting if whenever P_i and $P_{i'}$ are such that $\xi_i > \xi_{i'}$ we have that*

$$\frac{\partial U_{Pi}}{\partial \bar{\xi}_j} - \frac{\partial U_{Pi'}}{\partial \bar{\xi}_j} > 0 \tag{22}$$

Letting $\xi_{i^*(l)}$ be the type of the lowest type parent in school l, the fee for a full school l is defined recursively as:

$$p_l = \int_{\bar{\xi}_{l+1}}^{\bar{\xi}_l} \frac{\partial U_{P_{i^*(l)}}}{\partial \bar{\xi}_j} d\bar{\xi}_j + p_{l+1}, l = 1, ..., L-1, \tag{23}$$

and $p_L = 0$.

Proof Please see Appendix B. ∎

This condition provides a test for the existence of endogenous segregation in different settings and considering different attributes like income or student talent.

4.1   Segregation with private schools

We explore the emergence of sorting in a framework with private schools. To this end, we need first to describe the governance structure of these schools. Once private schools' behavior is discussed, it has to be shown that this structure satisfies the condition for segregation given by equation (22).

School behavior Private schools announce fees and allow parents to run schools as clubs.[28]

More precisely, once school l is formed, the headmaster chooses education policies (in our case, c_{2l} and n_j) to maximize the utility of the median parent. For simplicity we assume

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[28]A similar school governance is assumed, for example, by Ferreyra (2007).

in this section that there is heterogeneity in only one dimension v_i and $\widehat{\psi}_i = \widehat{\psi}_j$ for all i, j. Parents cover the running costs of the school in addition to paying the entry fee p_l. Hence, after school is formed the headmaster maximizes:

$$U_{P_{i_M}} = \left(\frac{v_{i_M}}{\widehat{\psi}} + \frac{c_{2l}}{2} \right) v_{i_M} + \left(\widehat{T}_i - \frac{1}{2} \left(\left(\frac{v_{i_M}}{\widehat{\psi}} \right)^2 - \left(\frac{c_{2l}}{2} \right)^2 \right) \right) \widehat{\psi}_{i_M}$$
$$- \frac{\omega}{2n_l^2} + \left(\widehat{T}_l - \frac{1}{2} n_l c_{2l} \left(\overline{\Omega}_{l_M} + \frac{c_{2l}}{2} \right) \right) \widehat{\gamma}_l, \qquad (24)$$

which represents the utility of the median parent in the school l. Notice that the optimal education policy depends on the characteristics of the median student. This feature differs from the case of a public school where both the resources determined by the policymaker and the incentives decided by the headmaster are based on the mean student abilities. The utility of any parent in school l is:

$$U_{P_i} = \left(\frac{v_i}{\widehat{\psi}} + \frac{c_{2l}}{2} \right) v_i + \left(\widehat{T}_i - \frac{1}{2} \left(\left(\frac{v_i}{\widehat{\psi}} \right)^2 - \left(\frac{c_{2l}}{2} \right)^2 \right) \right) \widehat{\psi} - \frac{\omega}{2n_l^2} + \left(\widehat{T}_l - \frac{1}{2} n_l c_{2l} \left(\overline{\Omega}_{l_M} + \frac{c_{2l}}{2} \right) \right) \widehat{\gamma}_l,$$

which can be expressed in terms of $U_{P_{i_M}}$ as

$$U_{P_i} = U_{P_{i_M}} + \frac{1}{2} \left(\frac{v_i^2}{\widehat{\psi}} - \frac{v_{i_M}^2}{\widehat{\psi}} \right) + (v_i - v_{i_M}) \frac{c_{2l}}{2} + \left(\widehat{T}_i + \frac{1}{2} \left(\frac{c_{2l}}{2} \right)^2 \right) \left(\widehat{\psi}_i - \widehat{\psi}_{i_M} \right). \qquad (25)$$

At this point, we impose the following assumption for ease of computations:

PROPOSITION 2 *Let P_i and $P_{i'}$ be such that $v_i > v_{i'}$, then:*

$$\frac{\partial U_{Pi}}{\partial \overline{v}_l} - \frac{\partial U_{Pi'}}{\partial \overline{v}_l} > 0 \qquad (26)$$

Proof Please see Appendix B. ∎

REMARK 5 *This result establishes (22) and hence, by proposition (1), it demonstrates the existence of an assignment equilibrium with top-down sorting.*

In our model, a private school attracting students from the public system affects the policy variables in a predictable way. Higher student talent induces an increase in school system resources and the power of school incentives. Students enrolling in a private school

(and hence leaving the public school) tend to come from the upper parts of the talent distribution. From equation (18) one can see that these children leaving the public schools would entail automatically an increase in n. Similarly from equation (9) one can see that c_{2j} (school incentives) are directly reduced through the effect of the increase in n.[29] This effect is relevant for evaluating the effect of vouchers, to which we now turn.[30]

4.1.1 Discussion

The presence of private schools may lead to sorting.[31] In line with the literature, we can discuss the effect of increasing the school choice through vouchers (see e.g. Epple and Romano (1998), Urquiola and Verhoogen (2009)). Our distinctive feature is our focus on the endogenous determination of peer-effects, and hence school quality and policy choices, as a result of the interaction of parents and the school system.[32] Taking the interaction between parents and the school system seriously also has ramifications for estimating the effect of vouchers on sorting outcomes. [33]

Consider the implementation of a voucher scheme subsidizing private schools. The voucher will induce peers with higher talent to leave the public school.[34] As discussed above, this implies a decline in incentives and resources received by the students who stay in the public school.[35] This result would follow mechanically from assuming the existence of peer effects as in most of the literature (e.g., Benabou (1993)). In our model, where peer-effects are endogenously generated by the interaction of parents and the school system, there is an amplification effect via the reaction of school resources and incentives to changes in the average ability of the peers. So, the negative effect of sorting on students left behind is greater than the estimates of a model with exogenous peer-effects would suggest.[36]

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[29]McMillan (2004) in an otherwise quite different model, also finds that public schools can decrease their performance in the presence of vouchers.

[30]In our model, the authorities invest more in public education the higher its marginal productivity, and this is why a lower median ability level in the class would decrease public funds. It is conceivable, though, that in the short run, public funds may be fixed. In that case, the fact that some children moved to a private school would increase *per capita* resources in the public one.

[31]We have described an equilibrium with private schools and sorting. Clearly, there are other equilibria were parents do not expect sorting, and as a result, it does not happen, and there are no fees due to ability sorting,.

[32]Integrating in a analytically tractable framework peer-effects, school quality and education policies, both in terms of education incentives and resources, is, to the best of our knowledge, novel in the literature.

[33]In particular, the estimation of computational/structural general equilibrium models have become a common tool for policymakers to understand the impacts of various educational policies.

[34]This effect has been empirically uncovered by many studies. See for example, Howell and Peterson (2002) for the case of the US, Hsieh and Urquiola (2006), for Chile or Ladd (2002) for New Zealand.

[35]Altonji, Huang, and Taber (2004, 2010) provides evidence of this effect for the case of the U.S.

[36]Ferreyra (2007) finds that a generalized voucher scheme even if positive in terms of welfare generates a

Also, when students can afford moving to better schools after receiving a voucher, the positive effect might also be smaller than the pre-voucher situation would suggest. This is because the receiving school would enroll on average less talented students, and its best students would like to move to a better private school. The new composition of the private school would entail a new median student and thus affect the level of resources and school incentives. Our effect hinges critically on the reactions of the actors involved in the education process and the consequence of neglecting the implied feedback effects would overstate both the gains by those favored by the voucher policy and understate the losses suffered by those who stay in the public school.

5   Concluding remarks

In this paper, we study a model of education where student learning effort and outcomes, parental and school behavior, and public resources devoted to education are endogenously determined in an integrated and tractable framework. Our model, provides a rationale for why the evaluation of educational interventions often provides mixed results. Beyond rationalizing this phenomenon ex-post, the paper serves as a warning: many evaluation exercises in education may be seriously compromised by issues of external validity. Indeed, we show that the effects of changing educational inputs on educational outcomes depend crucially on the sources of variation that cause this change. For this reason it may be hard to formulate education policy based on a menu of evaluation results.

The model also provides a microfoundation for *peer effects*. Groups of children with higher average ability are more "profitable" to manage by teachers, who as a consequence exert more effort in them. Then, any child will benefit from their presence in the school. *Peer effects*, as in other models, produce an incentive for sorting. We show that in some circumstances (e.g., when teaching technology favors low variance classrooms) sorting can be Pareto improving. Even in this case, the welfare gain from sorting is not evenly distributed, which can explain the ambiguous empirical evidence on sorting. The potentially ambiguous effect of changes in classroom composition carries implications on different education policies. In Albornoz, Berlinski, and Cabrales (2010), we extend our model to investigate the effects of sorting according to talent, cultural preferences or religious beliefs. We show that the resulting changes in classroom composition may reduce parents' and school involvement and resources in disadvantaged classes. As a consequence, the existing evaluations of these

negative effect on poor students.

policies may need to be re-evaluated upon this light.

It is clear that there are circumstances when higher ability children are not necessarily those in which parents want to invest more effort. For example, if the objective of a parent is to have her child get into an Ivy league school and she is so talented that even without effort the goal would be achieved, there is no point in making the investment in incentives. On the other hand, a slightly less talented child may be on the verge of achieving the goal and some investment in incentives could be indeed profitable. Thus, in reality the relationship between talent and parental investment may be nonmonotonic. Introducing explicitly these nonmonotonicities would not change the relationship between parental opportunity costs and investments, as well as the reaction to this by the education system. Our specific results on segregation would indeed change. But the most important message is that heterogeneity in talent or opportunity cost create incentives for segregation through the responses of both politicians and educators to the school composition. And this message would remain unaffected by a potential nonmonotonic relationship between talent and parental investment.

The richness of the model allows it to be used in further research. The political aspects of school choice, for example, are barely scratched in this paper. Since the political authorities have a single instrument, school resources, and preferences over this instrument are single-peaked, we can resort to the median voter theorem in discussing the policymaker's choice. If there were more instruments (say, because the level of funding of charter schools is an electoral issue) more challenging (and more interesting) political interactions involving education could be studied (as in, for example, Boldrin and Montes (2005) or Levy (2005)). Another aspect we have not explored is that of teacher sorting and teacher peer effects (something that Jackson (2009), Jackson and Bruegmann (2009), and Pop-Eleches and Urquiola (2013) have documented). We leave this sort of work for future research.

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A Appendix A: Extensions and robustness checks

A crucial insight of this paper is that the empirical analysis on the effect of school resources should control for changes in parental involvement to derive meaningful estimates. In this appendix, we discuss variations of some assumptions made throughout the analysis to clarify whether these are critical to obtain our results.

A.1 Monetary investments in learning

If school performance is determined alone by effort and talent, then it is natural to consider, as we did so far, that parents induce more effort through time-consuming motivational and monitoring activities. However, student outcomes may also be enhanced by other activities with a direct effect on learning, such as paying for private tuition. Dealing with this concern

requires changes to the production function of human capital (H_i). Therefore, we assume now that children performance depends on a function of monetary resources paid by their parents.[37] To study how parental (monetary) education investments interplay with school (monetary) resources, schools are also allowed to allocate monetary resources to enhance learning. School (monetary) resources can be used, for example, to hire personal tutors or teaching assistants. For simplicity, we assume that school monetary resources are chosen by the policy maker. We also assume that monetary education resources are substitute to time effort in order to focus on their direct effect. As an additional advantage, this assumption greatly simplifies the analysis since we can abstract from the determination of student effort. For a formal analysis, let H_i, U_{Pi} and U_{PM} be specified in the following way:[38]

$$H_i = (g_i + g_{PM})^{1/2}$$

$$U_{P_i} = 2 (g_i + g_{PM})^{1/2} - \frac{\eta_i}{2} g_i^2$$

$$U_{PM} = 2 \sum_{i=1}^{n} (g_i + g_{PM})^{1/2} - \frac{\varphi}{2} g_{PM}^2$$

Where g_i g_{PM} represent monetary investment in learning by parents and the school, respectively. The first order conditions of this problem are:

$$
\begin{aligned}
(g_i + g_{PM})^{-1/2} &= \eta_i g_i \\
\sum_{i=1}^{n} (g_i + g_{PM})^{-1/2} &= \varphi g_{PM}
\end{aligned}
$$

Interestingly, it readily follows that the policy maker increases g_{PM} as a response of parents lowering g_i, and vice versa. If we take the analysis further, we obtain that in equilibrium:

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[37]Clearly, monetary resources do not mean that children receive cash for their performance.
[38]For simplicity in this discussion we subsume school decisions with those of the policymaker.

$$g = (\frac{1}{\eta})^{\frac{2}{3}} \left(\frac{1}{\frac{n\eta}{\varphi} + 1} \right)^{\frac{1}{3}}$$

$$g_{PM} = \frac{n}{n\varphi^{\frac{2}{3}} + \frac{\varphi}{\eta^{\frac{1}{3}}}}$$

An important implication behind these expressions is that the impact evaluation of public education expenditure requires controlling for parental investments, even if we consider that parental monetary involvement as a substitute for learning effort; a similar insight we obtained when focusing on the effect of motivation on student learning effort.

A.2 Direct peer effects

Peer effects in our model arise so far only because a class with better (worse) students induces the teacher to make a higher (lower) effort. But in reality it is plausible that students benefit (or harm) one another directly, and not only through their effect on the effort level of teachers, or the investment of policymakers. We now also add this kind of *direct* peer effects, where the human capital of a child is affected by the average effort in the school. Specifically, let us assume that

$$H_i = v_i e_i \left(1 + \lambda \bar{e}_{1j} \right)$$

This immediately implies, following the same steps as before that

$$c_{1i} = \max \left\{ \frac{v_{P_i} v_i}{\psi_i} \left(1 + \lambda \bar{e}_{1j} \right) - \frac{c_{2j}}{2}, 0 \right\}$$

$$c_{2j} = \frac{v_{T_j} \overline{v}_j}{\gamma n_j} \left(1 + \lambda \bar{e}_{1j} \right) - \frac{\overline{c}_{1j}}{2}$$

and hence

$$\overline{c}_{1j} = \frac{\overline{v_{P_i} v_i}}{\psi_i} \left(1 + \lambda \bar{e}_{1j} \right) - \frac{c_{2j}}{2}$$

$$\overline{e}_{1j} = \frac{1}{\frac{3}{2} - \frac{\overline{v_{P_i} v_i}}{\psi_i} \lambda - \frac{v_{T_j} \overline{v}_j}{\gamma n_j} \lambda} \left(\frac{\overline{v_{P_i} v_i}}{\psi_i} + \frac{v_{T_j} \overline{v}_j}{\gamma n_j} \right)$$

$$e_i^* = \frac{v_{P_i} v_i}{\psi_i} + \frac{v_{T_j} \overline{v}_j}{\gamma n_j} + \frac{\lambda \frac{v_{P_i} v_i}{\psi_i} + \lambda \frac{v_{T_j} \overline{v}_j}{\gamma n_j} - \frac{1}{2}}{\frac{3}{2} - \frac{\overline{v_{P_i} v_i}}{\psi_i} \lambda - \frac{v_{T_j} \overline{v}_j}{\gamma n_j} \lambda} \left(\frac{\overline{v_{P_i} v_i}}{\psi_i} + \frac{v_{T_j} \overline{v}_j}{\gamma n_j} \right)$$

As can be readily seen by inspecting the previous expressions, the *direct* peer effects make the impact of all the exogenous variables larger, but there are no additional qualitative effects.

A.3 Parent and teacher complements

For tractability reasons, let us assume that human capital is characterized by $H_i = v_i e_i^\alpha$, with $\alpha < 1$.

If parents' and teachers' efforts are complements so that

$$c = c_1 c_2$$

the optimal student action is

$$e = c_1 c_2. \tag{27}$$

Substituting this expression into the parents' utility we obtain

$$U_P = (c_1 c_2)^\alpha v + \left(T - \frac{c_1 (c_1 c_2)}{2 v_P} \right) \psi.$$

The first-order condition for the parents's problem is then

$$\alpha c_1^{\alpha-1} c_2^\alpha v - c_1 c_2 \frac{\psi}{v_P} = 0.$$

$$\alpha c_2^\alpha v - c_1^{2-\alpha} c_2 \frac{\psi}{v_P} = 0$$

Given that this condition is sufficient, the optimal choice of the parent is

$$c_1 = \left(\frac{\alpha v_P v}{\psi c_2^{1-\alpha}} \right)^{\frac{1}{2-\alpha}}, \tag{28}$$

which is always non-negative given that producing the rewards requires investing parental time. By substituting the optimal choice of children's effort into the utility function of the headmaster we obtain:

$$U_{HM} = (c_1 c_2)^\alpha v + \left(T - \frac{n c_2 (c_1 c_2)}{2 v_T} \right) \gamma$$

It follows that an interior solution for the headmaster's optimization problem implies

$$c_2 = \left(\frac{\alpha \upsilon_T \upsilon}{\gamma n c_1^{1-\alpha}} \right)^{\frac{1}{2-\alpha}},$$
(29)

From (28) and (29) it is clear that the equilibrium values of c_1 and c_2 move in opposite directions even though in this version of the model they are technological complements.

A.4   Endogenous quality of parental involvement

In the core of the paper, we associated productivity of parental motivation with something innate (exogenous), like talent or capacity of persuasion. However, for a given unit of time investment (and a given talent), parental motivation may also differ according to different levels of attention or commitment to the task of motivating student effort. In our model, this dimension can be captured by treating υ_{P_i} as an endogenous variable. Consider that the utility of parents is now defined by:

$$U_{P_i} = \upsilon_i e_i + \left(T - \frac{c_{1i} e_i}{2 \upsilon_{P_i}} \right) \psi_i - \frac{1}{2} \upsilon_{P_i}^2,$$
(30)

where υ_{P_i} resumes the quality of parental effort (attention) and involves quadratic costs. For concreteness, we abstract from motivation provided by the school system. This implies that effort is solely determined by c_{1i}. From the first order condition with respect to υ_{P_i}, we obtain:

$$\upsilon_{P_i} = \sqrt[3]{\frac{c_{1i}^2}{2} \psi_i}$$

Notice that, for a given c_{1i}, attention increases with the opportunity cost of time. Now, if we solve for the optimal level of c_{1i}, we obtain:

$$c_{1i} = \frac{\upsilon_i^3}{2 \psi_i^2}$$

The variation explored in this section delivers an interesting new insight: when the quality of parental involvement is endogenous, the strength of rewards becomes more sensitive to student talent and (in the opposite direction) to parental opportunity cost of time, although in a structural way attention and the opportunity cost of time are naturally positively associated.[39]

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[39]A similar analysis would carried over with similar insights if we dealt with endogenous quality of effort

A.5 Other population targets

Headmasters care about the learning outcomes of a whole distribution of students attending their school. In our main analysis, we assumed that the relevant moment for the headmasters was the mean of the outcome distribution. This is, of course, a crude description of the reality and headmaster (as well as policy makers) target different measures of education achievements. In order to obtain more generality, suppose that the headmaster has a utility such that he cares about a weighted average of the average student and the n-th order statistic of the human capital distribution (H_i) at school j, denoted $H_j^{(n)}$. Then, let $v_j^{(n)}$ be the talent corresponding to $H_j^{(n)}$ and assume

$$U_{HMj} = \lambda \frac{1}{N_j} \sum_{i \in j} H_i + (1-\lambda) H_j^{(n)} + \left(T - \frac{n_j}{N_j} \sum_{i \in j} \frac{c_{2j}(c_{1i} + c_{2j})}{2v_{T_j}} \right) \gamma.$$

Observe that $\lambda \in [0,1]$ parameterizes the weight given to each goal. Following the previous analysis, we obtain that the equilibrium individual effort is given by:

$$e_i = \frac{v_{P_i} v_i}{\psi_i} + \frac{1}{3} \frac{v_{P_{iM}} v_{iM}}{\psi_{iM}} \left(\frac{\left(\frac{2v_{T_j}\left(\lambda \bar{v}_j + (1-\lambda)v_j^{(n)}\right)}{v_{T_{jM}}\left(\lambda \bar{v}_{jM}+(1-\lambda)v_{jM}^{(n)}\right)} + \theta_j \right) \left(\frac{2v_{T_{jM}}\left(\lambda \bar{v}_{jM}+(1-\lambda)v_{jM}^{(n)}\right)}{3\gamma} \right)^2 \frac{\psi_{iM}}{v_{P_{iM}}} - \theta_j \omega}{\left(\omega - \left(\frac{2v_{T_{jM}}\left(\lambda \bar{v}_{jM}+(1-\lambda)v_{jM}^{(n)}\right)}{3\gamma} \right)^2 \frac{\psi_{iM}}{v_{P_{iM}}} \right)} \right)$$

This expression is similar to 21 since it relates individual student effort to the achievement of the targeted goal at the school level and at the school attended by the median child. Importantly, this means that any measure of inequality that is captured by an order statistic may be easily accommodated within our formulation, and the nature of the endogenous peer-effects are qualitatively unchanged, although they would depend on different features of the classroom composition. Notice, however, that other measures of human capital inequality based on different moments of the distribution would be more difficult to handle analytically, although this result suggests that it is unlikely they would yield qualitatively different implications.

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for teachers.

A.6 Relative performance evaluation as the target

What would happen to school motivation effort if headmasters had an imposed goal to achieve a certain average student outcome? Let H_R be the human capital of average student in the reference (R) school and assume that:

$$U_{HMj} = \kappa I_{\frac{1}{N_j} \sum_{i \in j} (c_{1i} + c_{2j}) v_i \geq H_R} + \left(T - \frac{n_j}{N_j} \sum_{i \in j} \frac{c_{2j}(c_{1i} + c_{2j})}{2v_{T_j}} \right) \gamma.$$

under these circumstances in equilibrium the effort of the c_{2j} is

$$c_{2j} = \frac{H_R - \frac{1}{N_j} \sum_{i \in j} c_{1i} v_i}{\overline{v}_j}$$

iff

$$\kappa \geq \gamma \frac{n_j}{N_j} \sum_{i \in j} \frac{\frac{H_R - \frac{1}{N_j} \sum_{i \in j} c_{1i} v_i}{\overline{v}_j} \left(c_{1i} + \frac{H_R - \frac{1}{N_j} \sum_{i \in j} c_{1i} v_i}{\overline{v}_j} \right)}{2v_{T_j}}$$

Since in equilibrium

$$c_{1i} v_i = \frac{v_{P_i} v_i^2}{\psi_i} - \frac{H_R - \frac{1}{N_j} \sum_{k \in j} c_{1k} v_k}{2\overline{v}_j} v_i$$

we have (after some algebra) that

$$\frac{1}{N_j} \sum_{k \in j} c_{1k} v_k = \frac{2}{3N_j} \sum_{k \in j} \frac{v_{P_k} v_k^2}{\psi_k} - \frac{1}{3} H_R$$

so c_{2j} is positive iff

$$\kappa \geq \gamma \frac{n_j}{N_j} \sum_{i \in j} \frac{\frac{\frac{2}{3} H_R - \frac{2}{3N_j} \sum_{k \in j} \frac{v_{P_k} v_k^2}{\psi_k}}{\overline{v}_j} \left(c_{1i} + \frac{\frac{2}{3} H_R - \frac{2}{3N_j} \sum_{k \in j} \frac{v_{P_k} v_k^2}{\psi_k}}{\overline{v}_j} \right)}{2v_{T_j}}$$

Notice that the effect of increasing the target is to increase effort for those schools which are in a position to achieve it, but it decreases the effort to zero for those with low types.

Online Appendix: Proofs

Proof of lemma 1

Proof Notice that the preferences of an arbitrary parent i with respect to a class size level n (once he takes into account the taxes that the policymaker will have to levy to pay for the costs of such class size) is:

$$U_{P_i} = \left(\frac{\upsilon_{P_i} \upsilon_i}{\psi_i} + \frac{1}{3} \left(\frac{2}{\gamma n} \upsilon_{T_{jM}} \overline{\upsilon}_j - \overline{\Omega}_j \right) \right) \upsilon_i +$$

$$+ \left(T - \frac{1}{2\upsilon_{P_i}} \left(\left(\frac{\upsilon_{P_i} \upsilon_i}{\psi_i} \right)^2 - \frac{1}{9} \left(\frac{2}{\gamma n} \upsilon_{T_j} \overline{\upsilon}_j - \overline{\Omega}_j \right)^2 \right) \right) \psi_i - \frac{\omega}{2n^2},$$

so that

$$sign \left(\frac{\partial U_{P_i}}{\partial n} \right) = sign \left(-\frac{\upsilon_i}{3} \left(\frac{2\upsilon_{T_j} \overline{\upsilon}_j}{\gamma n^2} \right) - \frac{2\upsilon_{T_j} \overline{\upsilon}_j}{9\upsilon_{P_i} \gamma n^2} \left(\frac{2}{\gamma n} \upsilon_{T_j} \overline{\upsilon}_j - \overline{\Omega}_j \right) \psi_i + \frac{\omega}{n^3} \right)$$

$$= sign \left(\left(-\frac{2\upsilon_i \upsilon_{T_j} \overline{\upsilon}_j}{3\gamma} + \overline{\Omega}_j \frac{2\upsilon_{T_j} \overline{\upsilon}_j \psi_i}{9\upsilon_{P_i} \gamma} \right) n - \frac{4\upsilon_{T_j}^2 \overline{\upsilon}_j^2 \psi_i}{9\upsilon_{P_i} \gamma^2} + \omega \right)$$

and therefore the sign of the derivative of U_{P_i} with respect to n can change sign only once. This means that there is at most one interior critical point. For this critical point to be a maximum it is sufficient that

$$-\frac{2\upsilon_i \upsilon_{T_j} \overline{\upsilon}_j}{3\gamma} + \overline{\Omega}_j \frac{2\upsilon_{T_j} \overline{\upsilon}_j \psi_i}{9\upsilon_{P_i} \gamma} < 0$$

$$\overline{\Omega}_j = \frac{1}{N_j} \sum_{i \in j} \frac{\upsilon_{P_i} \upsilon_i}{\psi_i} < 3 \frac{\upsilon_{P_i} \upsilon_i}{\psi_i}$$

And this is verified if

$$\frac{1}{N_j} \sum_{i \in j} \frac{\upsilon_{P_i} \upsilon_i}{\psi_i} < 3 \min_{i \in \{1, \dots, N\}} \frac{\upsilon_{P_i} \upsilon_i}{\psi_i}.$$

∎

Proof of proposition 1

Proof A parent of a child in school l with type ξ_i does not want to move the child to school $l + 1$ provided that:

$$U_{P_i} \left(\overline{\xi}_l \right) - p_l \geq U_{P_i} \left(\overline{\xi}_{l+1} \right) - p_{l+1}$$

$$U_{P_i} \left(\overline{\xi}_l \right) - U_{P_i} \left(\overline{\xi}_{l+1} \right) \geq p_l - p_{l+1}.$$

Such parent will have a type such that $\xi_{i^*(l-1)} \geq \xi_i \geq \xi_{i^*(l)}$. Then we have that:

$$
\begin{aligned}
U_{P_i}\left(\bar{\xi}_l\right) - U_{P_i}\left(\bar{\xi}_{l+1}\right) &= \int_{\bar{\xi}_{l+1}}^{\bar{\xi}_l} \frac{\partial U_{P_i}}{\partial \bar{\xi}_j} d\bar{\xi}_j \\
&\geq \int_{\bar{\xi}_{l+1}}^{\bar{\xi}_l} \frac{\partial U_{P_{i^*(l)}}}{\partial \bar{\xi}_j} d\bar{\xi}_j \\
&= p_l - p_{l+1},
\end{aligned}
$$

where the inequality is true by (22). Similarly a parent of a child in school l with type ξ_i does not want to move the child to school $l-1$ provided that:

$$
\begin{aligned}
U_{P_i}\left(\bar{\xi}_l\right) - p_l &\geq U_{P_i}\left(\bar{\xi}_{l-1}\right) - p_{l-1} \\
p_{l-1} - p_l &\geq U_{P_i}\left(\bar{\xi}_{l-1}\right) - U_{P_i}\left(\bar{\xi}_l\right)
\end{aligned}
$$

Remember that $\xi_{i^*(l-1)} \geq \xi_i \geq \xi_{i^*(l)}$. Thus:

$$
\begin{aligned}
p_{l-1} - p_l &= \int_{\bar{\xi}_l}^{\bar{\xi}_{l-1}} \frac{\partial U_{P_{i^*(l-1)}}}{\partial \bar{\xi}_j} d\bar{\xi}_j \\
&\geq \int_{\bar{\xi}_l}^{\bar{\xi}_{l-1}} \frac{\partial U_{P_i}}{\partial \bar{\xi}_j} d\bar{\xi}_j \\
&= U_{P_i}\left(\bar{\xi}_{l-1}\right) - U_{P_i}\left(\bar{\xi}_l\right),
\end{aligned}
$$

where, again, the inequality is true by (22). $\blacksquare$

Proof of proposition 2

Proof

The first order conditions associated with (24) are:

$$
\begin{aligned}
\frac{1}{2}v_{iM} + \frac{c_{2l}}{4}\widehat{\psi} - \frac{\widehat{\gamma_l}}{2}n_l\left(\overline{\Omega}_{l_M} + c_{2l}\right)c_{2l} &= 0, \\
\frac{2\omega}{n_l^3} - \widehat{\gamma_l}\left(\overline{\Omega}_{l_M} + \frac{c_{2l}}{2}\right)c_{2l} &= 0.
\end{aligned}
$$

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These conditions imply that:

$$n_l = \frac{2v_{i_M} + c_{2l}\widehat{\psi}}{2\widehat{\gamma}_l\left(\overline{\Omega}_{l_M} + c_{2l}\right)c_{2l}},$$

$$2\omega\left(\frac{2\widehat{\gamma}_l\left(\overline{\Omega}_{l_M} + c_{2l}\right)c_{2l}}{2v_{i_M} + c_{2l}\widehat{\psi}}\right)^3 - \widehat{\gamma}_l\left(\overline{\Omega}_{l_M} + \frac{c_{2l}}{2}\right)c_{2l} = 0.$$

Using these conditions, by equation (25), we can calculate the following derivatives:

$$\frac{\partial U_{Pi}}{\partial v_{i_M}} = \frac{\partial U_{Pi_M}}{\partial v_{i_M}} - \frac{v_{i_M}}{\widehat{\psi}} - \frac{c_{2l}}{2} + \frac{(v_i - v_{i_M})}{2}\frac{\partial c_{2l}}{\partial v_{i_M}} \tag{31}$$

$$\frac{\partial c_{2l}}{\partial v_{i_M}} = -\frac{6\omega\left(\frac{2\widehat{\gamma}_l\left(\frac{v_{i_M}}{\widehat{\psi}}+c_{2l}\right)c_{2l}}{2v_{i_M}+c_{2l}\widehat{\psi}}\right)^2\left(\frac{-2\widehat{\gamma}_l c_{2l}^2}{\left(2v_{i_M}+c_{2l}\widehat{\psi}\right)^2}\right)}{6\omega\left(\frac{2\widehat{\gamma}_l\left(\frac{v_{i_M}}{\widehat{\psi}}+c_{2l}\right)c_{2l}}{2v_{i_M}+c_{2l}\widehat{\psi}}\right)^2\left(\frac{4\widehat{\gamma}_l\left(\frac{v_{i_M}^2}{\widehat{\psi}}+2c_{2l}v_{i_M}\right)+2\widehat{\gamma}_l c_{2l}^2\widehat{\psi}}{\left(2v_{i_M}+c_{2l}\widehat{\psi}\right)^2}\right) - \widehat{\gamma}_l\left(\frac{v_i}{\widehat{\psi}}+c_{2l}\right)} > 0 \tag{32}$$

Let P_i and $P_{i'}$ be such that $v_i > v_{i'}$ then, from (31), we obtain:

$$\frac{\partial U_{Pi}}{\partial v_{i_M}} - \frac{\partial U_{Pi'}}{\partial v_{i_M}} = \frac{(v_i - v_{i'})}{2}\frac{\partial c_{2l_M}}{\partial v_{i_M}}$$

The result then follows from (32). ■

Appendix C: Algebra (not intended for publication)

n derivation

$$U_{PM} = (tg)^{1/2} + \frac{2v^2}{3}\left(\frac{v_P}{\psi} + \frac{v_T}{\gamma n}\right) + \left(T - \frac{2v^2}{9v_P}\left(\frac{2v_P^2}{\psi^2} + \frac{v_P v_T}{\psi\gamma n} - \frac{v_T^2}{(\gamma n)^2}\right)\right)\psi - \frac{\omega}{2}\frac{1}{n^2} - t$$

$$-\frac{2v^2}{3}\left(\frac{v_T}{\gamma n^2}\right) - \frac{2v^2}{9v_P}\left(-\frac{v_P v_T}{\psi \gamma n^2} + 2\frac{v_T^2}{\gamma^2 n^3}\right)\psi + \omega\frac{1}{n^3} = 0$$

$$-\frac{2v^2}{3}\left(\frac{v_T}{\gamma}\right) - \frac{2v^2}{9v_P}\left(-\frac{v_P v_T}{\psi \gamma} + 2\frac{v_T^2}{\gamma^2 n}\right)\psi + \omega\frac{1}{n} = 0$$

$$-\frac{2v^2}{3}\left(\frac{v_T}{\gamma}\right) + \frac{2v^2}{9}\frac{v_T}{\gamma} - \frac{2v^2}{9v_P}\left(2\frac{v_T^2}{\gamma^2 n}\right)\psi + \omega\frac{1}{n} = 0$$

$$-\frac{4v^2}{9}\left(\frac{v_T}{\gamma}\right) - \frac{2v^2}{9v_P}\left(2\frac{v_T^2}{\gamma^2 n}\right)\psi + \omega\frac{1}{n} = 0$$

$$n = \frac{\omega - \frac{2v^2}{9v_P}\left(2\frac{v_T^2}{\gamma^2}\right)\psi}{\frac{4v^2}{9}\left(\frac{v_T}{\gamma}\right)}$$

$$\frac{2v^2}{3}\left(\frac{v_P}{\psi} + \frac{v_T}{\gamma x}\right) + \left(T - \frac{2v^2}{9v_P}\left(\frac{2v_P^2}{\psi^2} + \frac{v_P v_T}{\psi \gamma x} - \frac{v_T^2}{(\gamma x)^2}\right)\right)\psi - \frac{\omega}{2}\frac{1}{x^2}$$

Gradient is

$$\frac{1}{x^3}\omega - \frac{2}{9}v^2\frac{\psi}{v_P}\left(\frac{2}{x^3\gamma^2}v_T^2 - \frac{1}{x^2\gamma\psi}v_P v_T\right) - \frac{2}{3x^2\gamma}v^2 v_T = 0$$

Solution is: $\left\{\frac{1}{4\gamma v^2 v_P v_T}\left(9\gamma^2\omega v_P - 4v^2\psi v_T^2\right)\right\}$ if $\gamma \neq 0 \wedge v \neq 0 \wedge v_P \neq 0 \wedge v_T \neq 0 \wedge \frac{9}{4}\frac{\gamma}{v^2}\frac{\omega}{v_T} - \frac{1}{\gamma}\frac{\psi}{v_P}v_T \neq 0$

$$n = \frac{\omega - \frac{2v^2}{9v_P}\left(2\frac{v_T^2}{\gamma^2}\right)\psi}{\frac{4v^2}{9}\left(\frac{v_T}{\gamma}\right)}$$

$$= \frac{\omega - \frac{\psi}{v_P}\left(\frac{2v_T v}{3\gamma}\right)^2}{\left(\frac{2v}{3}\right)^2\frac{v_T}{\gamma}}$$

Comparative statics homogeneous case

Comparative statics for n

$$n = \frac{1}{\gamma}\left[\frac{\omega}{\left(\frac{2v}{3\gamma}\right)^2} - \psi\right]$$

$$= \frac{\gamma\omega}{\left(\frac{2v}{3}\right)^2} - \frac{\psi}{\gamma}$$

$$\frac{\partial n}{\partial \omega} = \frac{1}{\gamma} \left[\frac{1}{\left(\frac{2v}{3\gamma}\right)^2} \right] > 0$$

$$\frac{\partial n}{\partial \gamma} = \frac{\omega}{\left(\frac{2v}{3}\right)^2} + \frac{\psi}{\gamma^2} > 0$$

$$\frac{\partial n}{\partial \psi} = -\frac{1}{\gamma} < 0$$

$$\frac{\partial n}{\partial v} = -\frac{\gamma \omega 2 v \left(\frac{2}{3}\right)^2}{\left(\frac{2v}{3}\right)^4} < 0$$

Comparative statics for c_1, c_2, and e with respect to v

$$c_1 = \frac{2v}{3\psi} \left(\frac{2\omega - 3\left(\frac{2v}{3\gamma}\right)^2 \psi}{\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi} \right)$$

$$= \frac{2}{3\psi} \left(\frac{2\omega v - 3\left(\frac{2}{3\gamma}\right)^2 \psi v^3}{\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi} \right)$$

$$\frac{\partial c_1}{\partial v} = \frac{2}{3\psi} \frac{\left[2\omega - 9\left(\frac{2v}{3\gamma}\right)^2 \psi\right]\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right] + \left[2\omega v - 3\left(\frac{2v}{3\gamma}\right)^2 \psi v\right]\left(\frac{2}{3\gamma}\right)^2 2v\psi}{\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]^2}$$

$$= \frac{2}{3\psi} \frac{\left[2\omega - 9\left(\frac{2v}{3\gamma}\right)^2 \psi\right]\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right] + \left[4\omega - 6\left(\frac{2v}{3\gamma}\right)^2 \psi\right]\left(\frac{2v}{3\gamma}\right)^2 \psi}{\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]^2}$$

$$= \frac{2}{3\psi} \frac{\left(\frac{2v}{3\gamma}\right)^2 \psi\left[\left(4\omega - 6\left(\frac{2v}{3\gamma}\right)^2 \psi\right) - \left(2\omega - 9\left(\frac{2v}{3\gamma}\right)^2 \psi\right)\right] + \omega\left[2\omega - 9\left(\frac{2v}{3\gamma}\right)^2 \psi\right]}{\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]^2}$$

$$= \frac{2}{3\psi} \frac{\left(\frac{2v}{3\gamma}\right)^2 \psi\left[2\omega + 3\left(\frac{2v}{3\gamma}\right)^2 \psi\right] + \omega\left[2\omega - 9\left(\frac{2v}{3\gamma}\right)^2 \psi\right]}{\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]^2}$$

$$= \frac{2}{3\psi} \frac{3\left(\frac{2v}{3\gamma}\right)^4 \psi^2 + \omega\left[2\omega - 7\left(\frac{2v}{3\gamma}\right)^2 \psi\right]}{\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]^2}$$

and $c_2 > 0$ implies that:

$$2\omega < 6\left(\frac{2v}{3\gamma}\right)^2 \psi$$

and hence

$$2\omega < 6\left(\frac{2v}{3\gamma}\right)^2 \psi + \left(\frac{2v}{3\gamma}\right)^2 \psi$$

which leaves $\frac{\partial c_1}{\partial v} > 0$.

$$c_2 = \frac{2v}{3\psi}\left(\frac{3\left(\frac{2v}{3\gamma}\right)^2 \psi - \omega}{\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi}\right)$$

$$= \frac{2}{3\psi}\left(\frac{3\left(\frac{2}{3\gamma}\right)^2 \psi v^3 - \omega v}{\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi}\right)$$

$$\frac{\partial c_2}{\partial v} = \frac{2}{3\psi} \frac{\left[9\left(\frac{2}{3\gamma}\right)^2 \psi v^2 - \omega\right]\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right] + \left[3\left(\frac{2}{3\gamma}\right)^2 \psi v^3 - \omega v\right]\left(\frac{2}{3\gamma}\right)^2 2v\psi}{\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]^2}$$

$$= \frac{2}{3\psi} \frac{\left[9\left(\frac{2v}{3\gamma}\right)^2 \psi - \omega\right]\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right] + \left[3\left(\frac{2v}{3\gamma}\right)^2 \psi - \omega\right]\left(\frac{2v}{3\gamma}\right)^2 2\psi}{\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]^2}$$

but $c_1 > 0$ implies

$$\omega > \left(\frac{2v}{3\gamma}\right)^2 \psi$$

and $c_2 > 0$ implies

$$3\left(\frac{2v}{3\gamma}\right)^2 \psi > \omega$$

and therefore that

$$9\left(\frac{2v}{3\gamma}\right)^2 \psi > \omega.$$

Then, $\frac{\partial c_2}{\partial v} > 0$.

$$e = \frac{\frac{2v\omega}{3\psi}}{\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi}$$

$$\frac{\partial e}{\partial v} = \frac{\frac{2\omega}{3\psi}\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right] + 2\left(\frac{2v}{3\gamma}\right)^2 \psi \frac{2\omega}{3\psi}}{\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]^2}$$

$$= \frac{\frac{2\omega}{3\psi}\left[\omega + \left(\frac{2v}{3\gamma}\right)^2 \psi\right]}{\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]^2} > 0.$$

Comparative statics for c_1, c_2, and e with respect to ψ

$$c_1 = \frac{2v}{3\psi}\left(\frac{2\omega - 3\left(\frac{2v}{3\gamma}\right)^2\psi}{\omega - \left(\frac{2v}{3\gamma}\right)^2\psi}\right)$$

$$= \frac{2v}{3}\left(\frac{2\omega - 3\left(\frac{2v}{3\gamma}\right)^2\psi}{\psi\omega - \left(\frac{2v}{3\gamma}\right)^2\psi^2}\right)$$

$$\frac{\partial c_1}{\partial \psi} = \frac{2v}{3}\frac{-3\left(\frac{2v}{3\gamma}\right)^2\left[\psi\omega - \left(\frac{2v}{3\gamma}\right)^2\psi^2\right] - \left[\omega - 2\left(\frac{2v}{3\gamma}\right)^2\psi\right]\left[2\omega - 3\left(\frac{2v}{3\gamma}\right)^2\psi\right]}{\left[\psi\omega - \left(\frac{2v}{3\gamma}\right)^2\psi^2\right]^2}$$

$$= \frac{2v}{3}\frac{-3\psi\left(\frac{2v}{3\gamma}\right)^2\left[\omega - \left(\frac{2v}{3\gamma}\right)^2\psi\right] - \left[\omega - 2\left(\frac{2v}{3\gamma}\right)^2\psi\right]\left[2\omega - 3\left(\frac{2v}{3\gamma}\right)^2\psi\right]}{\left[\psi\omega - \left(\frac{2v}{3\gamma}\right)^2\psi^2\right]^2}$$

$$= \frac{2v}{3}\frac{-\psi\left(\frac{2v}{3\gamma}\right)^2\left[3\omega - 3\left(\frac{2v}{3\gamma}\right)^2\psi\right] - \left[\omega - \left(\frac{2v}{3\gamma}\right)^2\psi\right]\left[2\omega - 3\left(\frac{2v}{3\gamma}\right)^2\psi\right] + \psi\left(\frac{2v}{3\gamma}\right)^2\left[2\omega - 3\left(\frac{2v}{3\gamma}\right)^2\psi\right]}{\left[\psi\omega - \left(\frac{2v}{3\gamma}\right)^2\psi^2\right]^2}$$

$$= \frac{2v}{3}\frac{-\left[\omega - \left(\frac{2v}{3\gamma}\right)^2\psi\right]\left[2\omega - 3\left(\frac{2v}{3\gamma}\right)^2\psi\right] - \omega\psi\left(\frac{2v}{3\gamma}\right)^2}{\left[\psi\omega - \left(\frac{2v}{3\gamma}\right)^2\psi^2\right]^2} < 0$$

$$c_2 = \frac{2v}{3\psi}\left(\frac{3\left(\frac{2v}{3\gamma}\right)^2\psi - \omega}{\omega - \left(\frac{2v}{3\gamma}\right)^2\psi}\right)$$

$$= \frac{2v}{3}\left(\frac{3\left(\frac{2v}{3\gamma}\right)^2\psi - \omega}{\psi\omega - \left(\frac{2v}{3\gamma}\right)^2\psi^2}\right)$$

$$\frac{\partial c_2}{\partial \psi} = \frac{2v}{3} \frac{3\psi \left(\frac{2v}{3\gamma}\right)^2 \left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right] - \left[\omega - 2\left(\frac{2v}{3\gamma}\right)^2 \psi\right]\left[3\left(\frac{2v}{3\gamma}\right)^2 \psi - \omega\right]}{\left[\psi\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi^2\right]^2}$$

$$= \frac{2v}{3} \frac{3\psi \left(\frac{2v}{3\gamma}\right)^2 \left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right] - 3\psi \left(\frac{2v}{3\gamma}\right)^2 \left[\omega - 2\left(\frac{2v}{3\gamma}\right)^2 \psi\right] + \omega\left[\omega - 2\left(\frac{2v}{3\gamma}\right)^2 \psi\right]}{\left[\psi\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi^2\right]^2}$$

$$= \frac{2v}{3} \frac{3\psi \left(\frac{2v}{3\gamma}\right)^2 \left(\frac{2v}{3\gamma}\right)^2 \psi + \omega\left[\omega - 2\left(\frac{2v}{3\gamma}\right)^2 \psi\right]}{\left[\psi\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi^2\right]^2}$$

$$= \frac{2v}{3} \frac{3\psi \left(\frac{2v}{3\gamma}\right)^2 \left(\frac{2v}{3\gamma}\right)^2 \psi - \omega\left(\frac{2v}{3\gamma}\right)^2 \psi + \omega\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]}{\left[\psi\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi^2\right]^2}$$

$$= \frac{2v}{3} \frac{\left(3\psi \left(\frac{2v}{3\gamma}\right)^2 - \omega\right)\left(\frac{2v}{3\gamma}\right)^2 \psi + \omega\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]}{\left[\psi\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi^2\right]^2}$$

but $c_1 > 0$ implies

$$\omega > \left(\frac{2v}{3\gamma}\right)^2 \psi$$

and $c_2 > 0$ implies

$$3\left(\frac{2v}{3\gamma}\right)^2 \psi > \omega$$

thus

$$\frac{\partial c_2}{\partial \psi} = \frac{2v}{3} \frac{\left(3\psi \left(\frac{2v}{3\gamma}\right)^2 - \omega\right)\left(\frac{2v}{3\gamma}\right)^2 \psi + \omega\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]}{\left[\psi\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi^2\right]^2} > 0$$

$$e = \frac{\frac{2v\omega}{3\psi}}{\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi}$$

$$e = \frac{2\upsilon\omega}{3\omega\psi - 3\left(\frac{2\upsilon}{3\gamma}\right)^2 \psi^2}$$

$$\frac{\partial e}{\partial \psi} = \frac{-\left[\omega - 2\left(\frac{2\upsilon}{3\gamma}\right)^2 \psi\right] 6\upsilon\omega}{\left[3\omega\psi - 3\left(\frac{2\upsilon}{3\gamma}\right)^2 \psi^2\right]^2} \lesseqgtr 0$$

The effect to is negative if $\omega > 2\left(\frac{2\upsilon}{3\gamma}\right)^2 \psi$.

Comparative statics for c_1, c_2, and e with respect to ϖ

$$c_1 = \frac{2\upsilon}{3\psi}\left(\frac{2\omega - 3\left(\frac{2\upsilon}{3\gamma}\right)^2 \psi}{\omega - \left(\frac{2\upsilon}{3\gamma}\right)^2 \psi}\right)$$

$$\frac{\partial c_1}{\partial \omega} = \frac{2\upsilon}{3\psi}\frac{2\left[\omega - \left(\frac{2\upsilon}{3\gamma}\right)^2 \psi\right] - \left[2\omega - 3\left(\frac{2\upsilon}{3\gamma}\right)^2 \psi\right]}{\left[\omega - \left(\frac{2\upsilon}{3\gamma}\right)^2 \psi\right]^2}$$

$$= \frac{2\upsilon}{3\psi}\frac{\left(\frac{2\upsilon}{3\gamma}\right)^2 \psi}{\left[\omega - \left(\frac{2\upsilon}{3\gamma}\right)^2 \psi\right]^2} > 0$$

$$c_2 = \frac{2\upsilon}{3\psi}\left(\frac{3\left(\frac{2\upsilon}{3\gamma}\right)^2 \psi - \omega}{\omega - \left(\frac{2\upsilon}{3\gamma}\right)^2 \psi}\right)$$

$$\frac{\partial c_2}{\partial \omega} = \frac{2\upsilon}{3\psi}\frac{-\left[\omega - \left(\frac{2\upsilon}{3\gamma}\right)^2 \psi\right] - \left[3\left(\frac{2\upsilon}{3\gamma}\right)^2 \psi - \omega\right]}{\left[\omega - \left(\frac{2\upsilon}{3\gamma}\right)^2 \psi\right]^2}$$

$$= -\frac{2\upsilon}{3\psi}\frac{2\left(\frac{2\upsilon}{3\gamma}\right)^2 \psi}{\left[\omega - \left(\frac{2\upsilon}{3\gamma}\right)^2 \psi\right]^2} < 0$$

$$e = \frac{\frac{2v\omega}{3\psi}}{\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi}$$

$$
\begin{aligned}
\frac{\partial e}{\partial \omega} &= \frac{\frac{2v}{3\psi}\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right] - \frac{2v\omega}{3\psi}}{\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]^2} \\
&= \frac{-\left(\frac{2v}{3\gamma}\right)^2 \psi}{\left[\omega - \left(\frac{2v}{3\gamma}\right)^2 \psi\right]^2} < 0
\end{aligned}
$$